Remarks on axiomatizable classes closed under subdirect products

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As ussially, $\mathbf{P}_u(\mathcal{R})$ is ultraproduct closure of class $\mathcal{R}$.

Lemma

Let $\mathcal{R}$ be a class of algebras. Then $\mathbf{P}_u\mathbf{P}_{fs}\mathbf{P}_u(\mathcal{R}) = \mathbf{P}_{fs}\mathbf{P}_u(\mathcal{R})$. In particular, if $\mathcal{R}$ is closed under ultraproducts then $\mathbf{P}_{fs}(\mathcal{R})$ is closed under ultraproducts.

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Let $\mathcal{R}, \mathcal{K}$ be two classes and $\mathcal{R} \subseteq \mathcal{K}$. We say that $\mathcal{R}$ is (finitely) axiomatizable relative $\mathcal{K}$ if there is a (finite) set of first-order sentences Σ such that $\mathcal{R} = \mathcal{K} \cap Mod(\Sigma)$.

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Denote by $\mathcal{R}_{FSR}$ and $\mathcal{R}_{FSI}$ the classes of finitely subdirectly $\mathcal{R}$-reducible and finitely subdirectly $\mathcal{R}$-irreducible algebras of class $\mathcal{R}$, respectively.

Corollary

Let $\mathcal{R}$ be axiomatizable class of algebras closed under subdirect products. If $\mathcal{R}_{FSI}$ is closed under ultraproducts then $\mathcal{R}_{FSI}$ and $\mathcal{R}_{FSR}$ are finitely axiomatizable relative $\mathcal{R}$.

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In particular,

Corollary

Let $\mathcal{V}$ be a variety of finite signature. If $\mathcal{V}$ is congruence distributive and $\mathcal{V}_{FSI}$ is infinitely axiomatizable then $\mathcal{V}$ is not finitely based.

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1. Variety of all lattices is finitely axiomatizable and has no axiomatizable $\mathcal{R}_{FSI}$.
2. K. Baker have found residually small infinitely axiomatizable variety of lattices.

Thank you very much for your attention.