Covering a plane convex body by its negative homothets

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Abstract. For every positive integer k, let λ_k denote the smallest positive number such that every plane convex body can be covered by k homothetic copies of itself with homothety ratio $-\lambda_k$. In this note, we verify a conjecture of Januszewski and Lassak that $\lambda_7 = \frac{10}{17}$. Furthermore, we give an estimate for λ_6.

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Neumann [6] proved that every convex body, in the Euclidean plane $\mathbb{E}^2$, can be covered by a homothetic copy of itself of homothety ratio -2, and that 2 may not be replaced by any smaller positive number. In [5], Lassak and Vásárhelyi asked what happens if we cover a plane convex body by more than one homothetic copy of itself with the same negative homothety ratio. More specifically, let us define λ_k, where k is a positive integer, as the smallest positive number such that every plane convex body C can be covered by k translates of $-\lambda_k C$. The problem of finding the values of λ_k for small values of k is mentioned also in [2].

From [6] it follows that $\lambda_1 = 2$. By [5], we have that $\lambda_2 \leq \sqrt{2}$, $\lambda_3 = 1$ and $\lambda_4 < 1$. The authors of [5] conjectured that $\lambda_2 = \frac{4}{3}$ and $\lambda_4 = \frac{4}{5}$. The first conjecture was verified in [4], whereas the second one is still open. Januszewski and Lassak in [4] gave a short and simple proof also for $\lambda_7 \leq \frac{2}{3}$. They made the conjecture $\lambda_7 = \frac{10}{17}$, and remarked that $\lambda_7 \geq \frac{10}{17}$ follows from the example of a triangle. For other results in the plane and in higher dimensions, the interested reader is referred to the papers [3], [7] and [8].

In the first part of this paper we prove, by a fairly simple consideration, that $\lambda_6 \leq \frac{2}{3}$. We show also that $\lambda_5 \geq \frac{5}{7}$ and that $\lambda_6 \geq \frac{20}{31}$. In the second part, using the

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same approach as in the first part, we give a more complicated proof for $\lambda_7 \leq \frac{10}{17}$, thus verifying the conjecture of Januszewski and Lassak mentioned above.

For simplicity, we denote points by small Latin letters, sets by capital Latin letters, and real numbers by small Greek letters. We identify a point of the plane $\mathbb{E}^2$ with its position vector. We denote the convex hull, the boundary and the interior of the set $A \subset \mathbb{E}^2$ by $\operatorname{conv} A$, by $\operatorname{bd} A$ and by $\operatorname{int} A$, respectively, and, for $p, q \in \mathbb{E}^2$, the closed segment with endpoints p and q by $[p, q]$. The Euclidean distance of $p, q \in \mathbb{E}^2$ is denoted by $\operatorname{dist}(p, q)$ and, for $A, B \subset \mathbb{E}^2$, we set $\operatorname{dist}(A, B) = \inf\{\operatorname{dist}(a, b) : a \in A \text{ and } b \in B\}$.

Theorem 1. *Every plane convex body C can be covered by six translates of $-\frac{2}{3}C$.*

Proof. By [1], there is an affine regular hexagon H inscribed in C. Note that if C can be covered (respectively, cannot be covered) by six translates of $-\frac{2}{3}C$, then any affine image C' of C can be covered (respectively, cannot be covered) by six translates of $-\frac{2}{3}C'$. Hence, we may assume that H is a regular hexagon of unit side length, with the origin o as its centre.

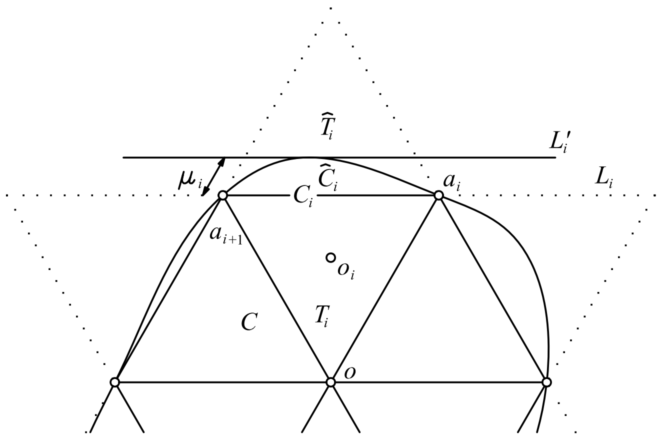

FIGURE 1

Let $a_1, a_2, \ldots, a_6 = a_0$ denote the vertices of H in counterclockwise cyclic order. For $i = 1, 2, \ldots, 6$, let L_i denote the sideline of H containing $[a_i, a_{i+1}]$, and let $\hat{T}_i$ denote the regular triangle with sidelines L_{i-1}, L_i and L_{i+1}. Let $\hat{C}_i = C \cap \hat{T}_i$, and let $\hat{\mu}_i$ be the distance between L_i and a point of $\hat{C}_i$ farthest from L_i. For simplicity, we set $\mu_i = \frac{2}{3}\sqrt{3}\hat{\mu}_i$ and $\mu_0 = \mu_6$. Observe that the existence of a supporting line of C at a_i yields $\mu_{i-1} + \mu_i \leq 1$ for every value of i. Let $T_i = \operatorname{conv}\{a_i, a_{i+1}, o\}$ and $C_i = \hat{C}_i \cup T_i$, and let o_i be the centroid of the triangle T_i (cf. Figure 1). In the following, we define six translates $D_1, D_2, \ldots, D_6$ of $D_0 = -\frac{2}{3}C$, and show that they cover C.

If $\mu_i < \frac{1}{3}$ (equivalently, if $\hat{\mu}_i \leq \frac{\sqrt{3}}{6}$), we let $D_i = o_i + D_0$. Note that $o_i + \frac{2}{3}H$ is circumscribed about T_i. Thus D_i contains T_i and every point of $\hat{T}_i$ that is not farther from $[a_i, a_{i+1}]$ than $\frac{\sqrt{3}}{6}$, and hence, $C_i \subset D_i$.

Assume that $\frac{1}{3} < \mu_i < \frac{2}{3}$. Let L_i' denote the supporting line of $\hat{C}_i$, different from L_i, that is parallel to L_i. Observe that L_i' supports C and its distance from L_i is $\hat{\mu}_i$. The strip bounded by L_i and L_i' intersects $\hat{T}_i$ in a trapezoid Q_i. Let q_i denote the intersection point of the two diagonals of Q_i. Observe that $q_i \in C$, and thus $\operatorname{conv}\big(H \cup \{q_i\}\big) \subset C$. We define $D_i = \frac{2}{3}q_i + D_0$ and note that $q_i \in C$ implies $-\frac{2}{3}q_i \in D_0$ and $o \in D_i$. We show that L_i' separates o and the line $L_i^* = \frac{2}{3}q_i + \frac{2}{3}L_i$.

An easy computation yields that

$$\begin{aligned} \frac{2}{3}\sqrt{3}\,\operatorname{dist}(L_i, q_i) &= \frac{\mu_i}{2-\mu_i}, \\ \frac{2}{3}\sqrt{3}\,\operatorname{dist}(o, L_i^*) &= \frac{2}{3}\sqrt{3}\,\operatorname{dist}\left(o, \frac{2}{3}q_i\right) + \frac{2}{3} = \frac{2}{3}\cdot\frac{4-\mu_i}{2-\mu_i}, \text{ and} \\ \frac{2}{3}\sqrt{3}\,\operatorname{dist}(o, L_i') &= 1 + \mu_i. \end{aligned}$$

From these inequalities and $\frac{1}{3} < \mu_i < \frac{2}{3}$ it follows that

$$\frac{2}{3}\sqrt{3}\big(\operatorname{dist}(o, L_i^*) - \operatorname{dist}(o, L_i')\big) = \frac{(1-\mu_i)(2/3-\mu_i)}{2-\mu_i} > 0.$$

Hence, $C_i \subset D_i$.

Finally, assume that $\mu_i \geq \frac{2}{3}$. Put $\mu_7 = \mu_1$. From $\mu_{i-1} + \mu_i \leq 1$ and $\mu_i + \mu_{i+1} \leq 1$ we obtain that $\mu_{i-1}, \mu_{i+1} \leq \frac{1}{3}$. Thus, by its definition, D_{i-1} contains every point of T_i not farther from $[o, a_i]$ than o_i, and D_{i+1} contains every point of T_i not farther from $[o, a_{i+1}]$ than o_i. Note that the remaining points of T_i are closer to $[a_i, a_{i+1}]$ than o_i. We set $D_i = 2o_i + D_0$, and observe that $C_i \subset D_{i-1} \cup D_i \cup D_{i+1}$. □

Proposition. *Let T be a triangle.* **.1.** *T can be covered by five homothetic copies of ratio $-\frac{5}{7}$, and cannot be covered by five negative homothetic copies of a smaller ratio.*
.2. *T can be covered by six homothetic copies of ratio $-\frac{20}{31}$, and cannot be covered by six negative homothetic copies of a smaller ratio.*

Proof. Note that it is sufficient to prove the assertion for a regular triangle of unit side length.

We start with proving the first statement. Let λ be the smallest positive number such that T can be covered by five negative copies of itself of ratio $-\lambda$, and let $T_i = x_i - \lambda T$, where $i = 1, \ldots, 5$, cover T. By the configuration in Figure 2, we may assume that $\lambda \leq \frac{5}{7}$. Then it follows that no T_i contains more than one vertex of T. Let the vertices of T be a_1, a_2 and a_3 in counterclockwise cyclic order, and let the triangles be labeled in a way such that $a_i \in T_i$ for $i = 1, 2, 3$. Without loss of generality, we may assume that $a_i \in \operatorname{bd} T_i$ for $i = 1, 2, 3$.

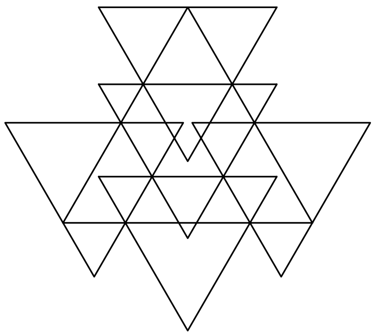

Figure 2

Suppose that no side of T is covered by two of the triangles T_i. It is easy to see that T_4 or T_5 does not intersect each side of T. Hence, we may assume that $[a_1, a_2] \subset T_1 \cup T_2 \cup T_4$.

Assume that T_4 intersects only one side of T. Then $[a_1, a_3] \subset T_1 \cup T_3 \cup T_5$ and $[a_2, a_3] \subset T_2 \cup T_3 \cup T_5$. Thus we may assume that a_3 is the midpoint of a side of T_3, that $\operatorname{bd} T_5$ contains the other two points of $\operatorname{bd} T \cap \operatorname{bd} T_3$, and that no point of $\operatorname{bd} T$ lies in $\operatorname{int} T_5 \cap \operatorname{int} T_i$ for $i = 1, 2$. Note that, under these assumptions, the length of the open segment $S = [a_1, a_2] \setminus (T_1 \cup T_2)$ is $3 - \frac{7}{2}\lambda$, independently of the position of T_5.

If $S \not\subset T_4$, then we may assume that a_i is the midpont of a side of T_i for some $i \in \{1, 2\}$, say, a_1 is the midpont of a side of T_1. Then $T_4 \cup T_2$ covers a translate of $(2 - 2\lambda)T$. Since, by [5], T is not covered by two translates of $-\tau T$ if $\tau < \frac{4}{3}$, we have $\frac{4}{3}(2 - 2\lambda) \leq \lambda$, which yields $\lambda \geq \frac{8}{11} > \frac{5}{7}$. If $S \subset T_4$, then T_4 contains the point of $\operatorname{bd} T_5 \cap \operatorname{bd} T_1$ and the point of $\operatorname{bd} T_5 \cap \operatorname{bd} T_2$, closest to $[a_1, a_2]$. Thus we may assume that these two points are at the same distance from $[a_1, a_2]$. From this, we obtain that $\lambda \geq \left(3 - \frac{7}{2}\lambda\right) + \left(2 - \frac{5}{2}\lambda\right) = 5 - 6\lambda$, from which it follows that $\lambda \geq \frac{5}{7}$.

If T_4 intersects exactly two sides of T, then a simple estimate, regarding how much part of the perimeter of T is covered, yields the assertion. Hence, we may assume that T_4 intersects each side of T. If the intersection of T_5 with $[a_1, a_3]$ or $[a_2, a_3]$ is farther from a_3 than the intersection of T_4 and the corresponding side of T, then we may change the configuration such that the new configuration also covers T whereas we decrease the homothety ratio. In the opposite case, we may assume that $T_1 \cap T$ and $T_2 \cap T$ are rhombi, and $T \setminus (T_1 \cup T_2 \cup T_4)$ is a regular triangle of side length $2 - 2\lambda$. Thus two translates of λT cover a translate of $(2 - 2\lambda)T$, which, by [5], yields that $\frac{4}{3}(2 - 2\lambda) \leq \lambda$, and hence, it follows that $\lambda \geq \frac{8}{11}$. If there is a side of T covered by two of the triangles T_i, we may apply a similar consideration.

We only sketch the proof of the second statement. Let the vertices of T be a_1, a_2 and $a_3 = a_0$ in counterclockwise cyclic order, and let the six translates covering T be $T_1, T_2, \ldots, T_6$. We may assume that $a_i \in \operatorname{bd} T_i$ for $i = 1, 2, 3$. Note that, for $i = 1, 2, 3$, the triangle T_i intersects $\operatorname{bd} T$ in the union of two segments, the sum of whose lengths is λ. Let α_i denote the length of the segment $T_i \cap [a_i, a_{i+1}]$.

We prove the assertion in various cases depending on the positions of the triangles $T_1, T_2, \ldots, T_6$. The positions of these translates are given by linear inequalities with variables α_1, α_2, α_3 and λ. In each case, the fact that the translates cover T may be translated into linear inequalities with the same variables. The minimal value of λ such that the corresponding inequalities have a solution is provided by the simplex method. This minimal value is at least $\frac{20}{31}$ in each case, and $\lambda = \frac{20}{31}$ is attained for two configurations (cf. Figure 3). □

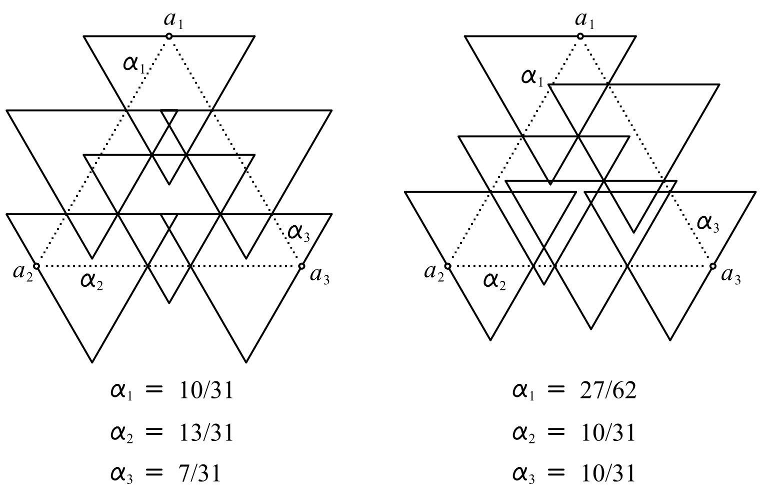

FIGURE 3

We conjecture the following.

Conjecture. *Every plane convex body C can be covered by five translates of $-\frac{5}{7}C$.*

In the last part of the paper, we prove the following theorem.

Theorem 2. *Every plane convex body C can be covered by seven translates of $-\frac{10}{17}C$.*

Proof. For simplicity, we use the notations, assumptions and observations of the first two paragraphs of the proof of Theorem 1. Note that if $\mu_i \le \frac{13}{17}$ for every value of i, then there are seven translates of $-\frac{10}{17}H$ covering C (cf. Figure 4).

Let us assume that $\mu_i > \frac{13}{17}$ for some values of i. We find seven translates of $-\frac{10}{17}C$, denoted by $D_0, D_1, \ldots, D_6$, that cover C. We define $D_0 = -\frac{10}{17}C$ and $H_0 = -\frac{10}{17}H = \frac{10}{17}H$, and denote the vertex of H_0 closest to a_i by $\bar{a}_i$. We construct $D_1, D_2, \ldots, D_6$ in a way that $D_{i-1} \cup D_i \cup D_{i+1}$ contains $C_i \setminus H_0$ for every i. We choose D_i depending on the values of μ_{i-1}, μ_i and μ_{i+1}.

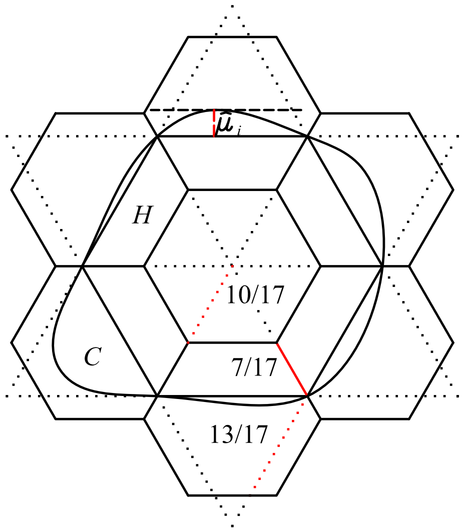

Figure 4

Case 1, if $\mu_i > \frac{13}{17}$. We set $D_i = (\frac{21}{34} + \sqrt{3}\hat{\mu}_i)o_i + D_0$, and note that $(\frac{21}{34} + \sqrt{3}\hat{\mu}_i)o_i + H_0$ contains exactly those points of $\hat{T}_i$ whose distance from $[a_i, a_{i+1}]$ is at most $\hat{\mu}_i$ (cf. Figure 5). For later use, we introduce some notations and investigate what part of $C_i \setminus H_0$ is not covered by D_i.

Let b_i be the vertex of $\hat{T}_i$ different from a_i and a_{i+1}, and let t_i and $\bar{t}_i$ be the points of $[a_i, b_i]$ and $[a_{i+1}, b_i]$, respectively, at a distance $\hat{\mu}_i$ from $[a_i, a_{i+1}]$. Let q_i be the intersection point of the segments $[a_i, \bar{t}_i]$ and $[a_{i+1}, t_i]$, and observe that $\operatorname{conv}\left(H \cup \{q_i\}\right) \subset C$.

It is not difficult to show that

$$\frac{2}{3}\sqrt{3} \cdot \operatorname{dist}(q_i, L_i) = \frac{\mu_i}{2 - \mu_i},$$

and that the image of $\operatorname{conv}\left(H \cup \{q_i\}\right)$ under the homothety that maps C into D_i intersects $[\bar{a}_i, \bar{a}_{i+1}]$ in a closed segment. Let $\bar{x}_i$ denote the endpoint of this segment closer to $\bar{a}_i$. A simple calculation yields that

$$\kappa_i = \operatorname{dist}(\bar{x}_i, \bar{a}_i) = \frac{(\mu_i - 13/17)\,(2 - \mu_i)}{2\mu_i}. \tag{1}$$

Thus κ_i is a strictly increasing function of μ_i on the interval $\left(\frac{13}{17}, 1\right]$, and we have $0 < \kappa_i \leq \frac{2}{17}$.

Let x_i be the intersection point of $[a_i, b_i]$ with the line, parallel to $[\bar{a}_i, a_i]$, that passes through $\bar{x}_i$ (cf. Figure 5). Observe that the trapezoid $\mathrm{conv}\{a_i, x_i, \bar{x}_i, \bar{a}_i\}$ contains the points of $C_i \setminus (H_0 \cup D_i)$ closer to a_i than to a_{i+1}.

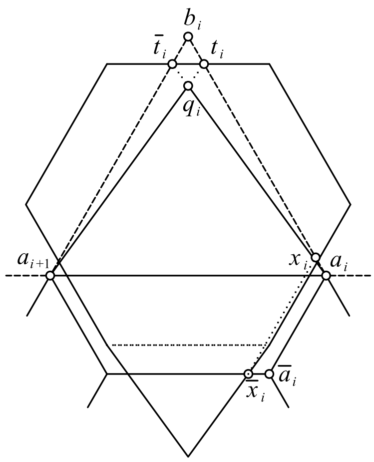

FIGURE 5

Case 2, each of μ_{i-1}, μ_i and μ_{i+1} is at most $\frac{13}{17}$. We let $D_i = \frac{30}{17} o_i - \frac{10}{17} C$. Note that $\frac{30}{17} o_i$ is the centre of a translate of $-\frac{10}{17} H$ in Figure 4.

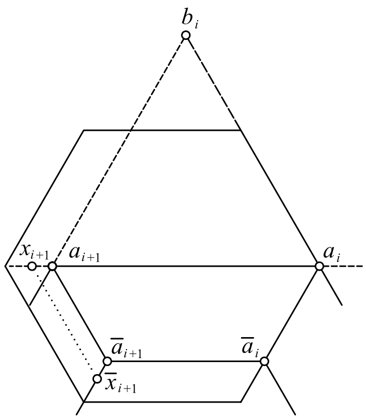

FIGURE 6

Case 3, $\mu_i \leq \frac{13}{17}$, and either $\mu_{i-1} > \frac{13}{17}$ or $\mu_{i+1} > \frac{13}{17}$. We may assume that $\mu_{i+1} > \frac{13}{17}$ and that $\mu_{i-1} \leq \frac{13}{17}$. We define $D_i = a_i - \bar{a}_{i-1} + D_0$ (cf. Figure 6). Since $\mu_i \leq 1 - \mu_{i+1} \leq \frac{4}{17}$, we have $D_i \subset C_i \setminus H_0$. Furthermore, from $\kappa_{i+1} \leq \frac{2}{17}$, we obtain $\operatorname{conv}\{a_{i+1}, x_{i+1}, \bar{x}_{i+1}, \bar{a}_{i+1}\} \subset D_i$.

Case 4, $\mu_i \leq \frac{13}{17}$, $\mu_{i-1} > \frac{13}{17}$ and $\mu_{i+1} > \frac{13}{17}$. Let y_i denote the point of $[b_{i-1}, a_i]$ with $\operatorname{dist}(y_i, a_i) = \kappa_{i-1}$. We define $\bar{y}_i$ similarly, and note that the trapezoid $\operatorname{conv}\{a_i, \bar{a}_i, \bar{y}_i, y_i\}$ contains the points of $C_{i-1} \setminus (H_0 \cup D_{i-1})$ that are closer to a_i than to a_{i-1}. To finish the proof, we need only show that there is a translate of D_0 that contains C_i, $\operatorname{conv}\{a_{i+1}, x_{i+1}, \bar{x}_{i+1}, \bar{a}_{i+1}\}$ and $\operatorname{conv}\{a_i, \bar{a}_i, \bar{y}_i, y_i\}$. Then we may define D_i as such a translate.

Let $P = -\frac{10}{17} \operatorname{conv}(H \cup \{q_{i-1}, q_{i+1}\})$. Since $P \subset D_0$, it is sufficient to find a translate of P that satisfies the conditions above. If $\kappa_{i-1} + \kappa_{i+1} \leq \frac{3}{17}$, then $\operatorname{dist}(y_i, x_{i+1}) \leq \frac{20}{17}$, which clearly yields the existence of a suitable translate of P (or even that of $H_0 \subset P$). Thus we may assume that $\kappa_{i-1} + \kappa_{i+1} > \frac{3}{17}$.

Let $P_i = u_i + P$ be the translate of P such that $y_i, \bar{y}_i \in \operatorname{bd} P_i$ and $a_i, \bar{a}_i \in P_i$ (cf. Figure 7). Note that $u_i \in \hat{T}_i$ and that y_i is on the translated image of the side $[\bar{a}_{i-1}, -\frac{10}{17} q_{i+1}]$ of P. Let $s_i = u_i + \bar{a}_{i-1}$, and let w_i denote the intersection point of $[a_i, a_{i+1}]$ with the line passing through u_i and parallel to $[s_i, y_i]$. Since $\operatorname{conv}\{y_i, s_i, u_i, w_i\}$ is a parallelogram, we have $\operatorname{dist}(a_i, w_i) = \frac{10}{17} - \kappa_{i-1}$. Furthermore,

$$\delta_i = \frac{2}{3}\sqrt{3}\operatorname{dist}(u_i, L_i) = \frac{3/17 - \kappa_{i-1}}{2 - \mu_{i-1}}. \tag{2}$$

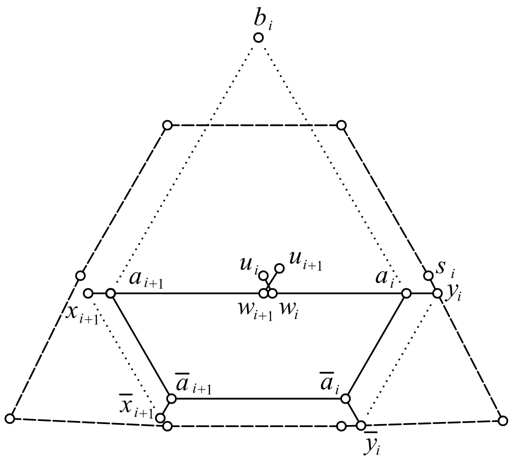

FIGURE 7

Let $P_{i+1} = u_{i+1} + P$ be the translate of P such that $x_{i+1}, \bar{x}_{i+1} \in \operatorname{bd} P_{i+1}$ and $a_{i+1}, \bar{a}_{i+1} \in P_{i+1}$. We define s_{i+1} and w_{i+1} similarly like s_i and w_i. Note that $\operatorname{dist}(a_{i+1}, w_{i+1}) = \frac{10}{17} - \kappa_{i+1}$ and, as $\kappa_{i-1} + \kappa_{i+1} > \frac{3}{17}$, w_i is closer to a_i than w_{i+1}.

For $j = i, i+1$, let R_j be the ray emanating from w_j and passing through u_j. From $\mu_{i-1} + \mu_{i+1} \geq \frac{26}{17} > 1$, we have that R_i and R_{i+1} intersect. Let x denote the intersection point of R_i and R_{i+1}. Observe that $y_i, \bar{y}_i \in m_i + P$ for any $m_i \in [u_i, w_i]$, and $x_{i+1}, \bar{x}_{i+1} \in m_{i+1} + P$ for any $m_{i+1} \in [u_{i+1}, w_{i+1}]$. Thus to prove the assertion it is sufficient to prove that x lies on the segments $[u_i, w_i]$ and $[u_{i+1}, w_{i+1}]$. A straightforward calculation yields that

$$\delta = \frac{2}{3}\sqrt{3}\operatorname{dist}(x, L_i) = \frac{\kappa_{i-1} + \kappa_{i+1} - 3/17}{\mu_{i-1} + \mu_{i+1} - 1}. \tag{3}$$

We omit a tedious computation showing that $\delta_i - \delta$, which we may express from (1), (2) and (3), is nonnegative. From this, we have that $x \in [u_i, w_i]$. Then $x \in [u_{i+1}, w_{i+1}]$ follows by symmetry. □

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