RELATIONAL SETS AND CATEGORICAL EQUIVALENCE OF ALGEBRAS*

LÁSZLÓ ZÁDORI
Bolyai Institute, Aradi vértanúk tere 1, 6720 Szeged, Hungary

Communicated by R. Mckenzie
Received 6 December 1995
Revised 16 September 1996

AMS Mathematics Subject Classification: 08C05, 08A05, 08B25

Keywords: colored resets, obstructions, product, retract, relation varieties, irreducible resets, minimal resets, category equivalence, matrix power, invertible term, arithmetical, congruence primal algebras

We study relation varieties, i.e. classes of relational sets (resets) of the same type that are closed under the formation of products and retracts. The notions of an irreducible reset and a representation of a reset are defined similarly to the ones for partially ordered sets. We give a characterization of finite irreducible resets. We show that every finite reset has a representation by minimal resets which are certain distinguished irreducible retracts. It turns out that a representation by minimal resets is a smallest one in some sense among all representations of a reset. We prove that non-isomorphic finite irreducible resets generate different relation varieties. We characterize categorical equivalence of algebras via product and retract of certain resets associated with the algebras. In the finite case the characterization involves minimal resets. Examples are given to demonstrate how the general theorems work for particular algebras and resets.

0. Introduction

In [5] Duffus and Rival define the notions of an order variety, a representation of a poset and an irreducible poset. Their definitions carry over to arbitrary relational sets (resets). It is natural to ask whether some nice properties involving those notions remain valid for arbitrary relational sets. In the present paper we give an affirmative answer to this question by generalizing the results and developing the ideas given for posets in [15]. The main emphasis, just as in [15], is on finite structures, although some results remain valid in the infinite case.

In Sec. 1 we generalize the results in [15] to arbitrary relational sets. We give a characterization of membership in a relation variety. We also characterize irreducible relational sets. We define minimal resets of a reset and show that they

*This research was done during the author's stay at the Département de mathématiques et statistiques, Université de Montréal. The support of the department and especially of Professor I. G. Rosenberg is gratefully acknowledged. Research partially supported by OTKA grant no. T17005.

International Journal of Algebra and Computation Vol. 7, No. 5 (1997) 561–576

yield a representation of the reset. A representation by minimal resets turns out to be a smallest one in the obvious sense. We get that two nonisomorphic finite irreducible resets generate different relation varieties. The characterizations and proofs are based on the key notions: nonextendible colored reset and, in the finite case, obstruction.

In Sec. 2 we describe categorical equivalence of algebras on the relational side. In the finite case minimal resets play a crucial role in this description. We apply the general theory developed here to finite, arithmetical, congruence primal algebras. We determine the irreducible resets in a certain finite relation variety associated with any finite, arithmetical, congruence primal algebra. As a consequence we get a new proof of a theorem of Bergman and Berman [2] concerning categorical equivalence between finite, arithmetical, congruence primal algebras. Some other applications are also mentioned.

1. Relation Varieties and Irreducible Resets

Throughout the present paper we work with *relational sets*, sets equipped with (possibly infinitary) relations. For brevity, a relational set is called a *reset*. We use the same boldface and slanted capital letters to denote a reset and its base set, respectively. If $\mathbf{Q} = (Q,\ (r^s_{\mathbf{Q}})_{s\in S})$ and $\mathbf{P} = (P, (r^s_{\mathbf{P}})_{s\in S})$ are resets of a fixed type then we write $f : \mathbf{Q} \to \mathbf{P}$ if f is a morphism from $\mathbf{Q}$ to $\mathbf{P}$, meaning that f preserves each relation of $\mathbf{Q}$, i.e. if $(a_t)_{t\in T} \in r^s_{\mathbf{Q}}$ then $(f(a_t))_{t\in T} \in r^s_{\mathbf{P}}$ for each $s \in S$. A reset $\mathbf{Q} = (Q, (r^s_{\mathbf{Q}})_{s\in S})$ is a *subreset* of $\mathbf{P} = (P, (r^s_{\mathbf{P}})_{s\in S})$ if $Q \subseteq P$ and $r^s_{\mathbf{Q}} = r^s_{\mathbf{P}} |_Q$ for all $s \in S$. We say that a reset $\mathbf{Q}$ is *contained* in $\mathbf{P}$ if $Q \subseteq P$ and $r^s_{\mathbf{Q}} \subseteq r^s_{\mathbf{P}} |_Q$ for all $s \in S$. If $\mathbf{Q}$ is contained in $\mathbf{P}$ we write $\mathbf{Q} \subseteq \mathbf{P}$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be resets of the same type. A pair $(\mathbf{Q}, f)$ is called a $\mathbf{P}$*-colored reset* if f is a partially defined map from Q to P. If f can be extended to a fully defined morphism $f' : \mathbf{Q} \to \mathbf{P}$ on Q then f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$*-extendible*, otherwise f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$*-nonextendible*. For two $\mathbf{P}$-colored resets $(\mathbf{H}, f)$ and $(\mathbf{Q}, g)$ we write $(\mathbf{H}, f) \subseteq (\mathbf{Q}, g)$ and say that $(\mathbf{H}, f)$ is *contained* in $(\mathbf{Q}, g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f \subseteq g$.

A $\mathbf{P}$*-obstruction* is a $\mathbf{P}$-nonextendible, $\mathbf{P}$-colored reset $(\mathbf{H}, f)$, where H is finite and every $\mathbf{P}$-colored reset $(\mathbf{K}, g)$ properly contained in $(\mathbf{H}, f)$, is $\mathbf{P}$-extendible. Roughly speaking, the $\mathbf{P}$-obstructions are the finite, minimal $\mathbf{P}$-nonextendible $\mathbf{P}$-colored resets. The notion of an obstruction defined above is related to that of the gap[11], hole [10], obstruction [9] and zigzag [14, 15]. When it is clear what $\mathbf{P}$ is we omit it in the terms $\mathbf{P}$-obstructions, $\mathbf{P}$-extendible, etc.

We define two important constructions of resets: product and retract. Let I be an index set and let $\mathbf{P}_i$, $i \in I$, be resets of the same type. Then the *product* $\prod_{i\in I} \mathbf{P}_i$ is a reset with the base set $\prod_{i\in I} P_i$ on which the relation r^s is defined componentwise for every $s \in S$. Let $\mathbf{P}$ and $\mathbf{R}$ be two resets of the same type. We say that $\mathbf{R}$ is a *retract* of $\mathbf{P}$ if there are morphisms $r : \mathbf{P} \to \mathbf{R}$ and $e : \mathbf{R} \to \mathbf{P}$ such that re is equal to the identity function of R. The maps r and e are called *retraction* and *coretraction*, respectively. A morphism $h : \mathbf{P} \to \mathbf{P}$ is called *idempotent* if $h^2 = h$. Obviously, every retraction r is associated with an idempotent morphism, namely er is idempotent.

Let K be a class of resets of the same type. The *relation variety* generated by K is the smallest class of resets containing K and closed under the retract and product constructions. It is easy to see that the relation variety generated by K exists and is equal to $\mathcal{RP}(K)$, where $\mathcal{R}$ is the operator of taking retracts of resets and $\mathcal{P}$ is the operator of taking products of resets.

In this section we generalize the results obtained for posets in [15]. A reset whose base set is finite, and whose relations are finitary is called a *finite reset.* First we want to introduce a particular relation variety that turns out to be useful when working with finite resets. We say that a reset $\mathbf{P} = (P, (r^s_{\mathbf{P}})_{s\in S})$ is *finite-like* if each $r^s_{\mathbf{P}}$, $s \in S$, is finitary and every $\mathbf{P}$-colored reset $(\mathbf{H}, f)$ is extendible whenever every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is extendible. Since every finite nonextendible colored reset contains an obstruction we have the following simple proposition.

Proposition 1.1. *Let* $\mathbf{P}$ *be a finite-like reset. A* $\mathbf{P}$*-colored reset is* $\mathbf{P}$*-extendible if and only if it does not contain a* $\mathbf{P}$*-obstruction.*

One expects the following result.

Proposition 1.2. *Every finite reset is finite-like.*

Proof. Let $\mathbf{P}$ be a finite reset and let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored reset. Suppose that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\mathbf{P}$-extendible. We want to show that f is $\mathbf{P}$-extendible to $\mathbf{H}$. We give a compactness argument using Tikhonov's theorem, which states that a product of compact topological spaces is compact. For every $h \in H$ we define a compact topological space T_h, namely, if f is defined on h then T_h is the one element set $\{f(h)\}$ otherwise T_h is P with the discrete topology. Then $\prod_{h\in H} T_h$ can be considered to be the set of all functions from H to P which extend f to H. For each n-ary relation r^s, $s \in S$, each $(h_1, \dots, h_n) \in r^s_{\mathbf{H}}$ and $(a_1, \dots, a_n) \notin r^s_{\mathbf{P}}$, where $a_1, \dots, a_n \in P$, we define an open set of $\prod_{h\in H} T_h$ in the product topology consisting of those elements of $\prod_{h\in H} T_h$ whose h_i-component is a_i, $1 \le i \le n$. Let O denote the set of all open sets obtained in this way. Observe that every extension of f that does not preserve one of the r^s is in one of the open sets of O. Let us suppose that f has no extension to $\mathbf{H}$ that preserves all the r^s. Then the open sets of O cover $\prod_{h\in H} T_h$. But since $\prod_{h\in H} T_h$ is compact there are even finitely many members of O which cover $\prod_{h\in H} T_h$. So there is a finite set $M \subseteq H$ which contains all the elements of H which occur in the definition of this finite cover. Then in $\prod_{h\in M} T_h$ there exists no morphism from the finite subreset of $\mathbf{H}$ having base set M to $\mathbf{P}$. But this contradicts the assumption. □

Proposition 1.3. *The finite-like resets of the same type form a relation variety.*

Proof. First we show that a product $\prod_{i\in I} \mathbf{P}_i$ of finite-like resets $\mathbf{P}_i$, $i \in I$, is finite-like. Let $(\mathbf{H}, f)$ be a $\prod_{i\in I} \mathbf{P}_i$-colored reset such that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\prod_{i\in I} \mathbf{P}_i$-extendible. Looking at the ith component of f as a coloring on $\mathbf{H}$ the so obtained $\mathbf{P}_i$-colored reset has a $\mathbf{P}_i$-extension since $\mathbf{P}_i$ is finite-like. Now, the ith component of a $\prod_{i\in I}\mathbf{P}_i$-extension of f is defined by this extension for all $i \in I$.

Second we show that a retract $\mathbf{R}$ of a finite-like reset $\mathbf{P}$ is finite-like. Let $r : \mathbf{P} \to \mathbf{R}$ be an onto retraction and let $e : \mathbf{R} \to \mathbf{P}$ be a corresponding coretraction. Let $(\mathbf{H}, f)$ be an $\mathbf{R}$-colored reset such that every finite $(\mathbf{H}', f') \subseteq (\mathbf{H}, f)$ is $\mathbf{R}$-extendible. The $\mathbf{P}$-colored reset $(\mathbf{H}, ef)$ is $\mathbf{P}$-extendible since $\mathbf{P}$ is finite-like. So there exists a morphism $g : \mathbf{H} \to \mathbf{P}$ which extends ef to $\mathbf{H}$. But then $rg : \mathbf{H} \to \mathbf{R}$ is an $\mathbf{R}$-extension of f to $\mathbf{H}$ since $ref = f$. □

Corollary 1.4. *The relation variety generated by all finite resets of a fixed type is a subvariety of the relation variety of all finite-like resets of the same type.*

Proof. Apply Proposition 1.2 and 1.3. □

Next we shall prove some general theorems concerning relation varieties and, in particular, relation varieties generated by finite resets.

For a reset $\mathbf{P}$ let $E(\mathbf{P})$, $N(\mathbf{P})$ and $O(\mathbf{P})$ denote the class of all $\mathbf{P}$-extendible $\mathbf{P}$-colored resets, the class of all nonextendible $\mathbf{P}$-colored resets and the class of all $\mathbf{P}$-obstructions, respectively.

Throughout the following proofs we frequently use the fact that, if $(\mathbf{H}, f) \in O(\mathbf{P})$ and $g : \mathbf{P} \to \mathbf{Q}$ is a morphism then $(\mathbf{H}, gf) \in O(\mathbf{Q}) \cup E(\mathbf{Q})$.

Proposition 1.5. *Let $\mathbf{P}$ and $\mathbf{R}$ be two resets of the same type. Then there exists a retraction $r : \mathbf{P} \to \mathbf{R}$ if and only if there is a one-to-one morphism $e : \mathbf{R} \to \mathbf{P}$ such that for every $\mathbf{R}$-colored reset $(\mathbf{H}, f)$ in $N(\mathbf{R})$ the $\mathbf{P}$-colored reset $(\mathbf{H}, ef)$ is in $N(\mathbf{P})$.*

Proof. Let r be a retraction from $\mathbf{P}$ to $\mathbf{R}$ with a corresponding coretraction e. Let $(\mathbf{H}, f) \in N(\mathbf{R})$. Let us suppose that $(\mathbf{H}, ef)$ is $\mathbf{P}$-extendible and let g be a morphism that extends ef to $\mathbf{H}$. Then rg extends $ref = f$ to $\mathbf{H}$ which contradicts the fact that $(\mathbf{H}, f) \in N(\mathbf{R})$.

Conversely, let us suppose that there is a morphism e from $\mathbf{R}$ to $\mathbf{P}$ such that for every $\mathbf{R}$-colored reset $(\mathbf{H}, f) \in N(\mathbf{R})$ we have $(\mathbf{H}, ef) \in N(\mathbf{P})$. Let us take the $\mathbf{R}$-colored reset $(\mathbf{P}, e^{-1})$. This is $\mathbf{R}$-extendible by a morphism r; for otherwise, $(\mathbf{P}, e^{-1}) \in N(\mathbf{R})$ and then by the hypothesis, $(\mathbf{P}, ee^{-1}) = (\mathbf{P}, id_{e(\mathbf{R})}) \in N(\mathbf{P})$. This is impossible since $id_{\mathbf{P}}$ extends $id_{e(\mathbf{R})}$ to $\mathbf{P}$. Clearly, $re = id_{\mathbf{R}}$ showing that r is a retraction of $\mathbf{P}$ onto $\mathbf{R}$. □

Corollary 1.6. *Let $\mathbf{P}$ and $\mathbf{R}$ be resets of the same type. Let us suppose that $\mathbf{R}$ is finite-like. Then there exists a retraction r from $\mathbf{P}$ onto $\mathbf{R}$ if and only if there is a one-to-one morphism $e : \mathbf{R} \to \mathbf{P}$ such that for every $\mathbf{R}$-colored reset $(\mathbf{H}, f)$ in $O(\mathbf{R})$ the $\mathbf{P}$-colored reset $(\mathbf{H}, ef)$ is in $O(\mathbf{P})$.*

Proof. Use Proposition 1.1 and Proposition 1.5. □

Proposition 1.7. *Let $\mathbf{P}_i$, $i \in I$, be resets of the same type. Then a $\prod_{i\in I}\mathbf{P}_i$-colored reset $(\mathbf{H}, f)$ is in $N(\prod_{i\in I}\mathbf{P}_i)$ if and only if there exists an $i \in I$ such that $(\mathbf{H}, f_i)$ is in $N(\mathbf{P}_i)$ where f_i is the ith component of f.*

Proof. The claim is obvious. □

Corollary 1.8. *Let $\mathbf{P}_i$, $i \in I$, be resets of the same type. Then a $\prod_{i\in I}\mathbf{P}_i$-colored reset $(\mathbf{H}, f)$ is in $O(\prod_{i\in I}\mathbf{P}_i)$ if and only if there exists an $i \in I$ such that $(\mathbf{H}, f_i)$ is in $O(\mathbf{P}_i)$ where f_i is the ith component of f.*

Proof. Use Proposition 1.7 and the fact preceding Proposition 1.5. □

Let $\mathbf{P}$, $\mathbf{Q}$ be posets of the same type and $(\mathbf{H}, f) \in N(\mathbf{P})$. We say that a morphism $g : \mathbf{P} \to \mathbf{Q}$ *separates* $(\mathbf{H}, f)$, if $(\mathbf{H}, gf) \in N(\mathbf{Q})$. If $(\mathbf{H}, gf) \in E(\mathbf{Q})$ we say that g *collapses* $(\mathbf{H}, f)$. Let K and M be classes of resets of the same type. A set G of morphisms with

$$G \subseteq \{g : \mathbf{P} \to \mathbf{Q} \mid \ \mathbf{P} \in K,\ \mathbf{Q} \in M\}$$

is called a *separating set* from K to M, if for every $\mathbf{P} \in K$,

$$\bigcap_{g\in G, Dom(g)=P} Ker(g) = 0$$

and for every $(\mathbf{H}, f) \in N(\mathbf{P})$ there exist $\mathbf{Q} \in M$ and $g : \mathbf{P} \to \mathbf{Q}$ such that g separates $(\mathbf{H}, f)$. Note that, if every poset is finite-like in K then we get an equivalent definition by replacing $N(\dots)$ by $O(\dots)$. Instead of writing that G is a separating set from $\{\mathbf{P}\}$ to M or from K to $\{\mathbf{Q}\}$ we write that G is a separating set from $\mathbf{P}$ to M or from K to $\mathbf{Q}$, respectively.

Theorem 1.9. *Let $\mathbf{P}$ be a reset and let K be a class of resets of the same type as $\mathbf{P}$. Then $\mathbf{P} \in \mathcal{RP}(K)$ if and only if there exists a separating set from $\mathbf{P}$ to K.*

Proof. Let $\mathbf{P}$ be a retract of the product of $\prod_{i\in I}\mathbf{P}_i$ where $\mathbf{P}_i \in K$ for every $i \in I$. Then by Proposition 1.5 there exists a one-to-one morphism $e : \mathbf{P} \to \prod_{i\in I}\mathbf{P}_i$ which sends every $\mathbf{P}$-colored poset $(\mathbf{H}, f) \in N(\mathbf{P})$ to $(\mathbf{H}, ef) \in N(\prod_{i\in I}\mathbf{P}_i)$. Let $G = \{\pi_i e | i \in I\}$ where π_i, $i \in I$, is the ith projection map for $\prod_{i\in I}\mathbf{P}_i$. Clearly, $\cap_{g\in G} Ker(g) = 0$. Let $(\mathbf{H}, f) \in N(\mathbf{P})$. Then, by Proposition 1.7, there exists an i such that $(\mathbf{H}, \pi_i ef) \in N(\mathbf{P}_i)$. Hence $\pi_i e$ separates $(\mathbf{H}, f)$. So G is a separating set from $\mathbf{P}$ to K.

To prove the other direction, by Proposition 1.5 it suffices to show that there exist a set I, resets $\mathbf{P}_i \in K$, $i \in I$, and a one-to-one morphism $e : \mathbf{P} \to \prod_{i\in I}\mathbf{P}_i$ such that for every $(\mathbf{H}, f) \in N(\mathbf{P})$ we have $(\mathbf{H}, ef) \in N(\prod_{i\in I}\mathbf{P}_i)$. Let I be a separating set from $\mathbf{P}$ to K. For every $i \in I$ let $\mathbf{P}_i$ be the target of i that is $i : \mathbf{P} \to \mathbf{P}_i$. Let e be the map from $\mathbf{P}$ to $\prod_{i\in I}\mathbf{P}_i$ given by $e_i(a) = i(a)$, $a \in P$. Now, e is obviously a one-to-one morphism. Moreover, if $(\mathbf{H}, f) \in N(\mathbf{P})$ then $(\mathbf{H}, ef) \in N(\prod_{(i\in I}\mathbf{P}_i)$. Indeed, by the hypothesis there exists some $i \in I$ such that $(\mathbf{H}, if) \in N(\mathbf{P}_i)$ and we can apply Proposition 1.7. □

From Theorem 1.9 we easily get the following generalization of a well known result for posets in [5].

Corollary 1.10. *Let a finite reset* $\mathbf{P}$ *be in* $\mathcal{RP}(K)$*, where* K *is a finite set of finite resets of the same type as* $\mathbf{P}$*. Then* $\mathbf{P}$ *is a retract of a finite product of some members of* K*.*

Proof. The claim follows from the "only if" part of Theorem 1.9 and from the proof of the "if" part of Theorem 1.9 since I will be finite. □

The analogue of the following definitions for posets can be found in [5]. A *representation* of a reset $\mathbf{P}$ is a family $(\mathbf{P}_i \mid i \in I)$ of resets of the same type as $\mathbf{P}$ such that $\mathbf{P}_i$ is a retract of $\mathbf{P}$ for each $i \in I$, and $\mathbf{P}$ is a retract of $\prod_{i\in I} \mathbf{P}_i$. A reset $\mathbf{P}$ is *irreducible* if for every representation $(\mathbf{P}_i \mid i \in I)$ of $\mathbf{P}$, $\mathbf{P}$ is a retract of $\mathbf{P}_i$ for some $i \in I$. If $\mathbf{P}$ is not irreducible then it is called *reducible.*

Now we characterize finite reducible resets in terms of separating sets.

Proposition 1.11. *Let* $\mathbf{P}$ *be a finite reset. Let* G_1 *be the set of morphisms from* $\mathbf{P}$ *to* $\mathbf{P}$ *whose ranges are contained in the range of a non-identity idempotent morphism on* $\mathbf{P}$*. Let* G_2 *be the set of non-onto morphisms from* $\mathbf{P}$ *to* $\mathbf{P}$*. Then the following are equivalent.*

(1) $\mathbf{P}$ *is reducible.*

(2) G_1 *is separating from* $\mathbf{P}$ *to* $\mathbf{P}$*.*

(3) G_2 *is separating from* $\mathbf{P}$ *to* $\mathbf{P}$*.*

Proof. Let $\mathbf{P}$ be a finite reset. Let G_1 and G_2 be the sets as defined in the claim. Let M be the set of idempotent images on $\mathbf{P}$, different from $\mathbf{P}$. By rephrasing (1) with the help of Proposition 1.9 and by using that for every $\mathbf{R} \in M$ every $\mathbf{R}$-obstruction is a $\mathbf{P}$-obstruction, one can easily see that (1) is equivalent to (2). Since $G_1 \subseteq G_2$, (2) implies (3). Now, suppose (3). We show (2). We claim that, if for every $\mathbf{P}$-obstruction there is a morphism in G_2 that separates it then for every $\mathbf{P}$-obstruction there is a morphism in G_1 that separates it. Let $(\mathbf{H}, f)$ be an arbitrary $\mathbf{P}$-obstruction. By the assumption there exists a sequence $g_i : \mathbf{P} \to \mathbf{P}$, $1 \leq i < \omega$, of non-onto maps such that $(\mathbf{H}, g_i \dots g_1 f)$ is a $\mathbf{P}$-obstruction for every i with $1 \leq i < \omega$. Let $f_i = g_i \dots g_1$, $1 \leq i < \omega$. By finitenes s there exist i and j such that $1 \leq i < j < \omega$ and $f_i = f_j$. Now, the map f_i is non-onto and separates $(\mathbf{H}, f)$. Moreover, the range of f_i is a subset of the range of any finite power of the non-onto map $g = g_j \dots g_{i+1}$ since $g f_i = f_i$. By taking into account that g has an idempotent power because of finiteness, we have the claim. One can show similarly that $\cap_{g\in G_2} Ker(g) = 0$ implies $\cap_{g\in G_1} Ker(g) = 0$. Thus (3) implies (2). □

The following corollary might be a useful tool of proving irreducibility for particular resets.

Corollary 1.12. *Let* $\mathbf{P}$ *be a finite reset. Then* $\mathbf{P}$ *is irreducible if and only if the set of non-onto morphisms from* $\mathbf{P}$ *to* $\mathbf{P}$ *is not separating.*

When we are dealing with morphisms in a separating set from a reset $\mathbf{P}$, they have to have kernels which intersect in 0, i.e. they have to separate the pairs of points in P. Sometimes it happens that morphisms separating all $\mathbf{P}$-obstructions do not separate the pairs of points in P and so they fail to produce a representation of $\mathbf{P}$. In order to obtain a certain representation of $\mathbf{P}$, described in details in the next theorem, we need the concept of minimality for obstructions and for pairs, as well.

Let $\mathbf{P}$ be a finite reset. A $\mathbf{P}$-obstruction $(\mathbf{H}, f)$ is called a *minimal obstruction* if for every morphism $g : \mathbf{P} \to \mathbf{P}$ with $(\mathbf{H}, gf) \in O(\mathbf{P})$ there exists a morphism $g' : \mathbf{P} \to \mathbf{P}$ such that $(\mathbf{H}, g'gf) = (\mathbf{H}, f)$. The name comes from the observation that the minimal obstructions are just the minimal elements of the quasiordered set defined on the set of the $\mathbf{P}$-obstructions with a fixed base poset $\mathbf{H}$ where the quasiordering is given by $(\mathbf{H}, f') \leq (\mathbf{H}, f'')$ if and only if there exists $g : \mathbf{P} \to \mathbf{P}$ with $(\mathbf{H}, gf'') = (\mathbf{H}, f')$. A pair formed by two distinct elements a and b in P is called a *minimal pair* if for every morphism $g : \mathbf{P} \to \mathbf{P}$ with $g(a) \neq g(b)$ there exists a morphism $g' : \mathbf{P} \to \mathbf{P}$ such that $g'g(a) = a$ and $g'g(b) = b$. To justify the name minimal pair a similar quasiorder can be defined as for minimal obstructions. We call an idempotent image $\mathbf{R}$ of $\mathbf{P}$ a *minimal reset* if there exists a minimal $\mathbf{P}$-obstruction $(\mathbf{H}, f)$ (a minimal pair (a, b)) such that $\mathbf{R}$ is one of the idempotent images of $\mathbf{P}$ that contain the range of f (a and b) and have minimum cardinality.

Theorem 1.13. *For every finite reset* $\mathbf{P}$ *the following hold.*

(1) *All minimal resets of* $\mathbf{P}$ *associated with the same minimal obstruction (pair) are isomorphic.*
(2) *Every minimal reset of* $\mathbf{P}$ *is irreducible.*
(3) $\mathbf{P}$ *has a representation by minimal resets.*

Proof. Let $(\mathbf{H}, f)$ be a minimal $\mathbf{P}$-obstruction. Suppose that $\mathbf{R}$ and $\mathbf{T}$ are two minimal resets of $\mathbf{P}$ such that both are associated with $(\mathbf{H}, f)$. Let r and t be two idempotent retractions to the minimal resets $\mathbf{R}$ and $\mathbf{T}$, respectively. Observe that r is onto R when restricted to T; otherwise $\mathbf{T}$ would not be an idempotent image of minimum cardinality which contains the range of f. Then because of finiteness we have (1). (For minimal pairs the proof of (1) is similar.)

Let $\mathbf{R}$ be a minimal reset of $\mathbf{P}$. The set of non-onto morphisms from $\mathbf{R}$ to $\mathbf{R}$ is not separating since the corresponding minimal obstruction (pair) is not separated by this set. So, by Corollary 1.12, we have (2).

For every $\mathbf{P}$-obstruction $(\mathbf{H}, f)$ there exists a morphism $g : \mathbf{P} \to \mathbf{P}$ such that $(\mathbf{H}, gf)$ is a minimal $\mathbf{P}$-obstruction and for every minimal reset $\mathbf{R}$ with respect to $(\mathbf{H}, gf)$, the $\mathbf{R}$-colored reset $(\mathbf{H}, gf)$ is an $\mathbf{R}$-obstruction. Let $r : \mathbf{P} \to \mathbf{R}$ be an idempotent retraction. Clearly, $(\mathbf{H}, rgf) = (\mathbf{H}, gf)$ and so the morphism $rg : \mathbf{P} \to \mathbf{R}$ separates $(\mathbf{H}, f)$. A similar argument works for pairs in P. Then, by Theorem 1.9 and Corollary 1.10 we can conclude that for every finite reset there is finite representation by minimal resets. Thus, (3) also holds. □

Theorem 1.14. *Let* $\mathbf{R}$ *be a minimal reset of a finite reset* $\mathbf{P}$ *such that* $\mathbf{R}$ *is a retract of* $\prod_{i\in I}\mathbf{P}_i$ *where the* $\mathbf{P}_i$, $i \in I$, *are finite members of the relation variety generated by* $\mathbf{P}$. *Then there exist an* $i \in I$ *and a minimal reset* $\mathbf{T}$ *of* $\mathbf{P}_i$ *such that* $\mathbf{R}$ *is isomorphic to* $\mathbf{T}$.

Proof. Let $\mathbf{R}$ be a minimal reset of a finite reset $\mathbf{P}$. Suppose that $\mathbf{P}_1, \dots, \mathbf{P}_n$ are retracts of a finite power of $\mathbf{P}$ and $\mathbf{R}$ is a retract of $\prod_{i\in I}\mathbf{P}_i$. Since $\mathbf{R}$ is a minimal reset it is associated with a minimal $\mathbf{P}$-obstruction $(\mathbf{H}, f)$. (If $\mathbf{R}$ is associated with a minimal pair in P then the proof is similar.) By Theorem 1.9, there exist an $i \in I$ and a morphism $g : \mathbf{P} \to \mathbf{P}_i$ such that $(\mathbf{H}, gf)$ is a $\mathbf{P}_i$-obstruction. We choose g so that $(\mathbf{H}, gf)$ is a minimal $\mathbf{P}_i$-obstruction. Let $\mathbf{T}$ be a minimal reset with respect to $(\mathbf{H}, gf)$ in $\mathbf{P}_i$. Since $\mathbf{P}_i$ is in the relation variety generated by $\mathbf{P}$, by Theorem 1.9 there exists a morphism $g' : \mathbf{P}_i \to \mathbf{P}$ which separates $(\mathbf{H}, gf)$. By the minimality of $(\mathbf{H}, f)$ there exists a morphism $g'' : \mathbf{P} \to \mathbf{P}$ such that $(\mathbf{H}, g''g'gf) = (\mathbf{H}, gf)$. Let $r : \mathbf{P} \to \mathbf{R}$ and $t : \mathbf{P}_i \to \mathbf{T}$ be idempotent retractions. Now, we must have that the morphisms $tg|_R : \mathbf{R} \to \mathbf{T}$ and $rg''g'|_T : \mathbf{T} \to \mathbf{R}$ are onto; otherwise their two way compositions are non-onto and separate $(\mathbf{H}, f)$ and $(\mathbf{H}, gf)$, respectively, which contradicts the irreducibility of $\mathbf{R}$ and $\mathbf{T}$. Hence, by finiteness, $\mathbf{R}$ is isomorphic to $\mathbf{T}$. □

It is clear from Theorem 1.14 and (3) of Theorem 1.13 that for every finite reset $\mathbf{P}$ there is a smallest finite representation given by the minimal resets of $\mathbf{P}$. It is smallest in the sense that from the set of members in any other representation of $\mathbf{P}$ the set of minimal resets can be obtained via taking retracts. In the same sense there is the largest representation of $\mathbf{P}$ by irreducibles. It is given by those irreducible idempotent images of $\mathbf{P}$ that are maximal with respect to containment in P. We call these irreducible resets *maximal resets.* It is an interesting problem for particular classes of resets: Does the set of minimal resets contain the set of maximal resets? For posets, this problem is equivalent to the unique factorization problem in [5] which is still open. We remark that in [7] Kabil and Pouzet showed that unique factorization does not hold in the class of finite reflexive graphs. We do not know the answer for the following question related to the unique factorization problem. Does there exist a finite reset such that the relation variety generated by it contains infinitely many finite irreducible resets? Theorem 1.14 has the following easy corollary.

Corollary 1.15. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be two finite resets of the same type. Then* $\mathbf{P}$ *and* $\mathbf{Q}$ *generate the same relation variety if and only if* $\mathbf{P}$ *and* $\mathbf{Q}$ *have the same minimal resets up to isomorphism.*

Since every finite irreducible reset is a minimal reset of itself we get the following corollary of Corollary 1.15.

Corollary 1.16. *Non-isomorphic finite irreducible resets generate different relation varieties.*

The next theorem is a generalization of a remark by Tardos in [14]. It gives an important reason why we defined obstructions in the way as we did. Let P be a finite set. For $n \geq 3$ an operation $f : P^n \to P$ is called an n-ary *near unanimity function* if $f(a,\dots,a,\underbrace{b}_{i},a,\dots,a) = a$ for all $a, b \in P$ and for all i with $1 \leq i \leq n$. A ternary near unanimity function is usually called a *majority function.*

Theorem 1.17. *Let $n \geq 3$ and let $\mathbf{P}$ be a finite reset. Then $\mathbf{P}$ admits an n-ary near unanimity function if and only if the number of colored elements in every $\mathbf{P}$-obstruction is at most $n-1$.*

Proof. Let $n \geq 3$. Let $\mathbf{P}$ be a reset that admits an n-ary near unanimity function. Then observe that $\mathbf{P}$ admits an m-ary near unanimity function for all $m \geq n$. Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-colored reset whose number of colored elements is m. Suppose that $m \geq n$ and every $\mathbf{P}$-colored reset properly contained in $(\mathbf{H}, f)$ is extendible. We show that $(\mathbf{H}, f)$ is also extendible. By removing the color of any colored element in $(\mathbf{H}, f)$ we get an extendible $\mathbf{P}$-colored reset. Let $f_1, \dots, f_m$ be extensions wich are obtained by removing the colors of the m colored elements in $(\mathbf{H}, f)$, respectively. Let M be an m-ary near unanimity function admitted by $\mathbf{P}$. Now, the map $M(f_1, \dots, f_m)$ is a morphism from $\mathbf{H}$ to $\mathbf{P}$ and extends f.

Let $\mathbf{P}$ be a finite reset. Suppose that the number of colored elements in every $\mathbf{P}$-obstruction is at most $n-1$. Let g be a partial map from P^n to P defined by the equality $f(a,\dots,a,\underbrace{b}_{i},a,\dots,a) = a$ for all $a, b \in P$ and for all i with $1 \leq i \leq n$. We claim that g extends to a morphism from $\mathbf{P}^n$ to $\mathbf{P}$ that is the $\mathbf{P}$-colored poset $(\mathbf{P}^n, g) \in E(\mathbf{P})$. For otherwise $(\mathbf{P}^n, g)$ contains a $\mathbf{P}$-obstruction $(\mathbf{H}, f)$ with at most $n-1$ colored elements. Now, $H \subseteq P^n$ and the colored elements in $(\mathbf{H}, f)$ are of the form $(a,\dots,a,\underbrace{b}_{i},a,\dots,a)$ where $f(a,\dots,a,\underbrace{b}_{i},a,\dots,a) = a$. This is impossible because there exists a j with $1 \leq j \leq n$ such that the j-th projection extends f to $\mathbf{H}$. $\square$

2. Categorical Equivalence and Congruence Primal Algebras

We say that two categories U and W are *categorically equivalent* if there exists a functor F from U to W such that for every object B in W there is an object A in U with B isomorphic to $F(A)$ and for all objects C and D in U, F is a bijection between $\mathrm{Hom}(C, D)$ and $\mathrm{Hom}(F(C), F(D))$. In this case the functor F is called a *categorical equivalence* between U and W.

We can view a variety of algebras as a category; the objects are the algebras in the variety and the morhisms are the homomorphisms between algebras in the variety. Under a categorical equivalence between two varieties of algebras, many algebraic properties are preserved; lists of properties preserved are given in [4] and [8]. So when studying the structure of algebras it is always useful to know whether the varieties generated by the given algebras are categorically equivalent.

The following definition is from [2]. We say that two algebras A and B are *categorically equivalent*, $A \equiv_c B$ for short, if there is a categorical equivalence F

between the varieties generated by A and B such that $F(A) = B$. The next characterization of categorically equivalent algebras given in [8] involves two algebraic constructions called idempotent, invertible image and matrix power. Let A be an algebra and let r be an *idempotent unary term* for A, that is, r is a unary term in the language of A and $r^2(x) = r(x)$ is satisfied by A. The term r is called *invertible* if there exist unary terms $t_1, \dots, t_m$ and an m-ary term t in the language of A such that $t(rt_1(x), \dots, rt_m(x)) = x$ is satisfied by A. On the range of r acting on A we define the algebra $r(A)$ whose basic operations are given by restricting the term operation rt acting on A for each term t in the language of A. We call $r(A)$ the *idempotent, invertible image* of A under r. Let n be a positive integer. On the nth power of the base set of A we define the algebra $A^{[n]}$ called the nth *matrix power* of A by taking the m-ary basic operation t for all mn-ary terms $t_1, \dots, t_n$ in the language of A such that the ith component of t is t_i acting on A, $1 \le i \le n$. We say that two algebras A and B are *weakly isomorphic* if there exists an algebra C such that C is of the same type as A, C is isomorphic to A, C has the same base set as B and C and B have the same sets of term operations.

Theorem 2.1. *Let A and B be two algebras. Then $A \equiv_c B$ if and only if there exist a positive integer n and an idempotent, invertible, unary term r for $A^{[n]}$ such that B is weakly isomorphic to $r(A^{[n]})$.*

When we are studying classes of algebras, sometimes it is more convenient and natural to use relations than operations. The algebras may be given in terms of preservation. For example, *congruence primal algebras* are the algebras whose term operations are the operations preserving a set of equivalence relations on their underlying set. The relational point of view that we pursue by dealing with algebras is reflected in the next theorems.

We say that two resets $\mathbf{P}$ and $\mathbf{Q}$ of the same type are *equivalent*, $\mathbf{P} \equiv \mathbf{Q}$ for short, whenever each one of them is a retract of a finite power of the other. In the finite case, this is equivalent to saying that $\mathbf{P}$ and $\mathbf{Q}$ generate the same relation variety. A morphism $r : \mathbf{P} \to \mathbf{P}$ is called *invertible* if there exist morphisms $t_1, \dots, t_m$ from $\mathbf{P}$ to $\mathbf{P}$ and a morphism $t : \mathbf{P}^m \to \mathbf{P}$ such that $t(rt_1, \dots, rt_m) = id_{\mathbf{P}}$.

Theorem 2.2. *Let $\mathbf{P}$ and $\mathbf{Q}$ be resets of the same type. Then $\mathbf{P} \equiv \mathbf{Q}$ if and only if there exist a positive integer n and an idempotent, invertible morphism $r : \mathbf{P}^n \to \mathbf{P}^n$ such that $\mathbf{Q}$ is isomorphic to $r(\mathbf{P}^n)$.*

Proof. Suppose that $\mathbf{P} \equiv \mathbf{Q}$. Then there exist positive integers n, m and retractions $r : \mathbf{P}^n \to \mathbf{Q}$ and $s : (\mathbf{Q}^m)^n \to \mathbf{P}^n$ where the corresponding coretractions are $e : \mathbf{Q} \to \mathbf{P}^n$ and $f : \mathbf{P}^n \to (\mathbf{Q}^n)^m$, respectively. Let $u = er$. We claim that the idempotent, unary morphism $u : \mathbf{P}^n \to \mathbf{P}^n$ is invertible. Define $t_i = e\pi_i f : \mathbf{P}^n \to \mathbf{P}^n$, $i = 1, \dots, mn$. Moreover, let $t : (\mathbf{P}^n)^{mn} \to \mathbf{P}^n$ be defined by $t(x_1, \dots, x_{mn}) = s(r(x_1), \dots, r(x_{mn}))$. For an arbitrary $(a_1, \dots, a_n) \in P^n$ let $f(a_1, \dots, a_n) = (b_1, \dots, b_{mn})$. Since $sf = id_{\mathbf{P}^n}$ we have that $s(b_1, \dots, b_{mn}) = (a_1, \dots, a_n)$. Clearly, $\mathbf{Q}$ is isomorphic to $u(\mathbf{P}^n)$. Now, the invertibility of u is verified by the following lines of equalities:

$$t(ut_i(a_1,\ldots,a_n),\ldots,ut_{mn}(a_1,\ldots,a_n)) =$$
$$t(ere\pi_1 f(a_1,\ldots,a_n),\ldots,ere\pi_{mn} f(a_1,\ldots,a_n)) =$$
$$t(e(b_1),\ldots,e(b_{mn})) = s(re(b_1),\ldots,re(b_{mn})) =$$
$$s(b_1,\ldots,b_{mn}) = (a_1,\ldots,a_n).$$

Suppose that there exist a positive integer n and an idempotent, invertible morphism $r : \mathbf{P}^n \to \mathbf{P}^n$ such that $\mathbf{Q}$ is isomorphic to $r(\mathbf{P}^n)$. We assume without loss of generality that $\mathbf{Q} = r(\mathbf{P}^n)$. Observe that the morphism $t : (\mathbf{P}^n)^m \to \mathbf{P}^n$ occurring in the definition of invertibility is a retraction to $\mathbf{P}^n$ when restricted to $\mathbf{Q}^m$. So $\mathbf{P}$ is a retract of a finite power of $\mathbf{Q}$. Obviously, $\mathbf{Q}$ is a retract of a finite power of $\mathbf{P}$. Thus $\mathbf{P} \equiv \mathbf{Q}$. □

We say that a reset $\mathbf{P}$ is a *reset for an algebra A* if the underlying sets of $\mathbf{P}$ and A are the same and the set of morphisms from finite powers of $\mathbf{P}$ to $\mathbf{P}$ coincides with the set of finitary term operations of A.

Theorem 2.3. *Let A and B be algebras. Then there exists a reset for A and it can be chosen to be finite when A is finite. Moreover, the following are equivalent.*

(1) $A \equiv_c B$.
(2) *If $\mathbf{P}$ is a reset for A then there exists a reset $\mathbf{Q}$ for B such that $\mathbf{P}$ and $\mathbf{Q}$ are of the same type and $\mathbf{P} \equiv \mathbf{Q}$.*
(3) *There exist two resets $\mathbf{P}$ and $\mathbf{Q}$ of the same type such that $\mathbf{P}$ is a reset for A, $\mathbf{Q}$ is a reset for B and $\mathbf{P} \equiv \mathbf{Q}$.*

Proof. In order to prove the first statement of the theorem, define a reset for algebra A by equipping the base set of A with the relations which are subalgebras generated by the m projections in algebra A^{A^m} where $0 \le m < \omega$.

The proof of the second statement is as follows. First we show (1) $\Rightarrow$ (2). Suppose that $A \equiv_c B$. Then by Theorem 2.1, there exist a positive integer n and an idempotent, invertible, unary term r for $A^{[n]}$ such that B is weakly isomorphic to $r(A^{[n]})$. Let $\mathbf{P}$ be an arbitrary reset for A. Observe that the set of finitary term operations of $A^{[n]}$ equals the set of morphism from finite powers of $\mathbf{P}^n$ to $\mathbf{P}^n$. By using this fact there exists a reset $\mathbf{Q}$ for B such that $\mathbf{Q}$ is isomorphic to $r(\mathbf{P}^n)$ where $r : \mathbf{P}^n \to \mathbf{P}^n$ is an idempotent, invertible morphism. By Theorem 2.2, this means that $\mathbf{P} \equiv \mathbf{Q}$. By using the first statement of the theorem, (2) $\Rightarrow$ (3) immediately. Finally we prove (3) $\Rightarrow$ (1). Let $\mathbf{P}$ and $\mathbf{Q}$ be resets of the same type such that $\mathbf{P}$ is a reset for A and $\mathbf{Q}$ is a reset for B and $\mathbf{P} \equiv \mathbf{Q}$. Then, by Theorem 2.2, there exist a positive integer n and an idempotent, invertible morphism $r : \mathbf{P}^n \to \mathbf{P}^n$ such that $\mathbf{Q}$ is isomorphic to $r(\mathbf{P}^n)$. Hence, by the above observation B is weakly isomorphic to $r(A^{[n]})$ where r is an idempotent, invertible, unary term for $A^{[n]}$. So by Theorem 2.1, we have that $A \equiv_c B$. □

For congruence primal algebras we have the following.

Theorem 2.4. *Let A be a congruence primal algebra and let B be any algebra. Let P and Q denote the underlying sets of A and B, respectively. Then*

$A \equiv_c B$ if and only if B is congruence primal, Con(A) is isomorphic to Con(B) and $(P, \text{Con}(A)) \equiv (Q, \text{Con}(B))$.

Proof. Let A be a congruence primal algebra and let B be an algebra such that $A \equiv_c B$. Because of Theorem 2.1 it suffices to show that B is congruence primal, Con(A) is isomorphic to Con(B) and $(P, \text{Con}(A)) \equiv (Q, \text{Con}(B))$ when $B = r(A^{[n]})$ for some positive integer n and idempotent, invertible term r for $A^{[n]}$. Let $\mathbf{P} = (P, \text{Con}(A))$. By properties of matrix power and idempotent, invertible image, it follows that $\mathbf{P}^n = (P^n, \text{Con}(A^{[n]}))$, $r(\mathbf{P}^n) = (r(P^n), \text{Con}(r(A^{[n]})))$ and Con(A), Con$(A^{[n]})$, and Con$(r(A^{[n]}))$ are isomorphic lattices. Since A is congruence primal we get that $\mathbf{P}^n$ is a reset for $A^{[n]}$ and $r(\mathbf{P}^n)$ is a reset for $B = r(A^{[n]})$. Hence B is congruence primal and Con(A) is isomorphic to Con(B). Moreover by Theorem 2.2, $\mathbf{P} \equiv r(\mathbf{P}^n)$, i.e., $(P, \text{Con}(A)) \equiv (Q, \text{Con}(B))$. The other part of the claim immediately follows from Theorem 2.3. □

In the case of finite algebras we have the following corollary of Corollary 1.15 and Theorem 2.3.

Theorem 2.5. *For finite algebras A and B the following are equivalent.*

(1) *$A \equiv_c B$.*

(2) *If $\mathbf{P}$ is a finite reset for A then there exists a finite reset $\mathbf{Q}$ for B such that $\mathbf{P}$ and $\mathbf{Q}$ are of the same type and have the same minimal resets up to isomorphism.*

(3) *There exist a reset $\mathbf{P}$ for A and a reset $\mathbf{Q}$ for B such that $\mathbf{P}$ and $\mathbf{Q}$ are of the same type and have representations by the same resets.*

Next we show some examples how to apply the results in Sec. 1 and Theorem 2.5 in the study of algebras. A sublattice of the lattice of equivalences on a set P is called *arithmetical*, if it is distributive and permutable. We shall examine the resets of the form $\mathbf{P} = (P, (r^s_{\mathbf{P}})_{s \in S})$ where P is a finite set and the $r^s_{\mathbf{P}}$ are equivalences on P which generate (via lattice operations) an arithmetical $0,1$-sublattice of the lattice of equivalences on P. It is allowed that different relational symbols have the same interpretation. The $0,1$-sublattice associated with $\mathbf{P}$ is denoted by $L_{\mathbf{P}}$. Let V denote the class of resets of the above form. A class of finite resets of the same type is called a *finite relation variety* if it is closed under retract and finite product. It is easy to check that V is a finite relation variety.

When $\mathbf{P} \in V$ every $\mathbf{P}$-colored reset $(\mathbf{H}, f)$ can be imagined as a directed graph on the set H where each edge is labeled by some r^s (meaning that the given edge belongs to r^s in $\mathbf{H}$) and accordingly to f some points of H are colored by elements from P. All the r^s that do not occur as a label are empty in $\mathbf{H}$. For example, Fig. 1 indicates a $\mathbf{P}$-colored reset $(\mathbf{H}, f)$. The set H has $m+1$ elements. Each of them is symbolized by a vertex. For each i with $1 \le i \le m$, the relation θ_i in $\mathbf{H}$ contains a single pair, say (h, h'), for which we draw an edge connecting the vertices h and h' and label it by θ_i. In this special case $\mathbf{H}$ is visualized as a path (not necessarily a directed one). According to f the two endpoints of this path are the only colored elements of $(\mathbf{H}, f)$, one of them is colored by a and the other by b, as shown in Fig. 1.

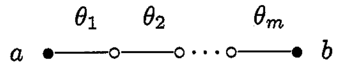

Fig. 1. A **P**-colored reset of the form $o(a, b, \theta_1, \ldots, \theta_m)$.

Theorem 2.6. *Let* $\mathbf{P}$ *be a reset in* V. *Then every* $\mathbf{P}$*-obstruction* $(\mathbf{H}, f)$ *is of the form* $o(a, b, \theta_1, \ldots, \theta_m)$ *in the figure where* $\theta_1, \ldots, \theta_m \in \{r^s : s \in S\}$ *and* $a, b \in P$ *such that* $(a, b) \notin \theta_1 \vee \cdots \vee \theta_m$ *in* $\mathbf{P}$. (*The direction of edges not shown in Fig. 1 is arbitrary in* $\mathbf{H}$.)

Proof. Since all relations of $\mathbf{P}$ are reflexive, the $\mathbf{P}$-obstructions have at least two colored elements. Due to Pixley [12], $\mathbf{P}$ admits a majority function. Hence Theorem 1.17 gives that every $\mathbf{P}$-obstruction has exactly two colored elements. Observe that in every $\mathbf{P}$-obstruction each colored element has degree one; otherwise we could split the colored elements to obtain a new obstruction with more than two colored elements which is impossible.

Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-obstruction. We prove the theorem via an induction on the number $|H|$. If $|H| = 2$ it is clear that the $\mathbf{P}$-obstruction $(\mathbf{H}, f)$ is of the required form. Now, suppose that $|H| \geq 3$. Let h_1 and h_2 be the two colored elements of $(\mathbf{H}, f)$. Let $f(h_1) = a$ and $f(h_2) = b$. The unique element connected to h_1 in $\mathbf{H}$ is denoted by h. During the proof keep in mind that we work in an arithmetical equivalence lattice and so $\theta_1 \vee \cdots \vee \theta_m = \theta_1 \circ \cdots \circ \theta_m$ for every $\theta_1, \ldots, \theta_m$ in $L_{\mathbf{P}}$. We create a system $(*)$ of equivalence equations in $L_{\mathbf{P}}$ as follows. We take the equation $(x, a) \in \theta$ where θ is the label of the unique edge connecting h_1 and h in $\mathbf{H}$ and for each path connecting h and h_2 we take the equation $(x, b) \in \theta_1 \vee \cdots \vee \theta_m$ where the edges in the path are labelled by $\theta_1, \ldots, \theta_m$ in $\mathbf{H}$.

Suppose that $(*)$ is solvable and has solution $u \in P$. Now, delete h_1 in $(\mathbf{H}, f)$ and color h by u. The so obtained colored poset is nonextendible since $(\mathbf{H}, f)$ is. So it contains an obstruction. By the induction hypothesis this obstruction is a path connecting h and h_2 in $\mathbf{H}$ such that the color of h is u, the color of h_2 is b and $(u, b) \notin \theta_1 \vee \cdots \vee \theta_m$ in $L_{\mathbf{P}}$ where $\theta_1, \ldots, \theta_m$ are the labels of the edges of the path. This contradicts the choice of u. So $(*)$ is not solvable. Hence by Baker and Pixley [1] the system $(*)$ contains a two-element subsystem that is not solvable. This implies that that there exists a path connecting h_1 and h_2 such that $(a, b) \notin \theta \vee \theta_1 \vee \cdots \vee \theta_m$ in $L_{\mathbf{P}}$ where $\theta, \theta_1, \ldots, \theta_m$ are the labels of the edges of the path, i.e. we have an obstruction of the required form contained in $(\mathbf{H}, f)$. Because the only obstruction contained by $(\mathbf{H}, f)$ is $(\mathbf{H}, f)$ itself we get the claim. $\square$

With $\mathbf{P}$-obstructions on hand we show that every $\mathbf{P} \in V$ has a finite representation by two element resets in V. Note that the two element resets are always irreducible. For $a, b \in P$ let $\theta_{a,b}$ denote the equivalence generated by (a, b) in $L_{\mathbf{P}}$.

Proposition 2.7. *Let* $\mathbf{P} \in V$. *Let* c *and* d *be two different elements in* P. *Let* $\theta_{a,b} \leq \theta_{c,d}$ *in* $L_{\mathbf{P}}$ *such that* $\theta_{a,b}$ *is join irreducible and* a *is different from* b. *Define the partial map* φ *by* $\varphi(c) = a$ *and* $\varphi(d) = b$. *Then* φ *extends to a morphism* $\varphi_{a,b} : \mathbf{P} \to \mathbf{P}|_{\{a,b\}}$ *where* $\mathbf{P}|_{\{a,b\}} = (\{a, b\}, (r^s_{\mathbf{P}}|_{\{a,b\}})_{s \in S})$.

Proof. Let $\mathbf{Q} = \mathbf{P}|_{\{a,b\}}$. Note that $\mathbf{Q} \in V$. If φ is $\mathbf{Q}$-nonextendible then there is a $\mathbf{Q}$-obstruction $(\mathbf{H}, f)$ contained in the $\mathbf{Q}$-colored reset $(\mathbf{P}, \varphi)$. So $(\mathbf{H}, f)$ is of the form $o(a, b, \theta_1, \ldots, \theta_m)$ where $(a, b) \notin \theta_1 \vee \cdots \vee \theta_m$ in ‘. Since $\mathbf{Q}$ is two-element, this gives that (a, b) is not in any of $\theta_1, \ldots, \theta_m$ in ‘. As $(\mathbf{H}, f) \subseteq (\mathbf{P}, \varphi)$ we have that $(c, d) \in \theta_1 \vee \cdots \vee \theta_m$ in $L_{\mathbf{P}}$. So we get that $\theta_{a,b} \leq \theta_{c,d} \leq \theta_1 \vee \cdots \vee \theta_m$ in $L_{\mathbf{P}}$. Since $\theta_{a,b}$ is join irreducible it is join prime in the distributive lattice $L_{\mathbf{P}}$. Hence $\theta_{a,b} \leq \theta_i$ for some i in $L_{\mathbf{P}}$. So $(a, b) \in \theta_i$ in ‘, which gives a contradiction. □

Corollary 2.8. *Let $\mathbf{P} \in V$. For every $a, b \in P$ with $\theta_{a,b}$ is join irreducible in $L_{\mathbf{P}}$, the reset $\mathbf{P}|_{\{a,b\}}$ is an idempotent image of $\mathbf{P}$.*

Proposition 2.9. *Let $\mathbf{P}$ be a reset in V. For every $\mathbf{P}$-obstruction $(\mathbf{H}, f)$ of the form $o(c, d, \theta_1, \ldots, \theta_m)$ there exist a and b such that $\theta_{a,b} \leq \theta_{c,d}$ and $\theta_{a,b}$ is join irreducible in $L_{\mathbf{P}}$, $a \neq b$ and $\varphi_{a,b}$ given in Proposition 2.7 separates $(\mathbf{H}, f)$.*

Proof. Since $L_{\mathbf{P}}$ is algebraic, $\theta_{c,d} = \bigvee_{(a,b) \in C} \theta_{a,b}$ where C is the set of ordered pairs (a, b) with the property that $\theta_{a,b} \leq \theta_{c,d}$, $\theta_{a,b}$ is join irreducible in $L_{\mathbf{P}}$ and $a \neq b$. If all the $(\mathbf{H}, \varphi_{a,b} f)$, $(a, b) \in C$, were extendible then for every $(a, b) \in C$ there would exists an i such that $(a, b) \in \theta_i$ in $\mathbf{P}|_{\{a,b\}}$ and so $(a, b) \in \theta_i$ in $L_{\mathbf{P}}$. Hence $\theta_{c,d} = \bigvee_{(a,b) \in C} \theta_{a,b} \leq \theta_1 \vee \cdots \vee \theta_m$ in $L_{\mathbf{P}}$. But this contradicts the fact that $(\mathbf{H}, f)$ is a $\mathbf{P}$-obstruction. □

By Theorem 1.9, we have the following corollary of Proposition 2.7 and 2.9.

Corollary 2.10. *Every $\mathbf{P}$ with $\mathbf{P} \in V$ has a finite representation by two element resets in V. This representation is unique for the whole $\equiv$-class of $\mathbf{P}$ in V in the strong sense that for all $\mathbf{Q} \equiv \mathbf{P}$ all finite representations of $\mathbf{Q}$ by irreducibles have the same members.*

Proof. The first statement is clear by Theorem 1.9. The uniqueness part follows from Theorem 1.14 and the fact that the two-element retracts of every reset in V are minimal resets. The proof of this fact is as follows. Let $\mathbf{Q}$ be an irreducible idempotent image of $\mathbf{P} \in V$. We know that $\mathbf{Q}$ is two-element, let $Q = \{a, b\}$. Let $\theta_1, \ldots, \theta_m$ be the list of those relation symbols in $\mathbf{Q}$ that interpret as 0 in $\mathbf{Q}$. Consider a $\mathbf{Q}$-obstruction of the form $o(a, b, \theta_1, \ldots, \theta_m)$. There exists a minimal reset $\mathbf{R}$ and a morphism $g : \mathbf{Q} \to \mathbf{R}$ that separates this obstruction. Of course, $\mathbf{R}$ is also two-element. By Theorem 2.6 every $\mathbf{Q}$-obstruction is of the form $o(a, b, \gamma_1, \ldots, \gamma_l)$ where $\{\gamma_1, \ldots, \gamma_l\} \subseteq \{\theta_1, \ldots, \theta_m\}$. Since $\gamma_1 \vee \cdots \vee \gamma_l \leq \theta_1 \vee \cdots \vee \theta_m$ in $L_{\mathbf{P}}$ and $(g(a), g(b)) \notin \theta_1 \vee \cdots \vee \theta_m$ all $\mathbf{Q}$-obstructions are separated by g. Moreover g is one-to-one. So by Corollary 1.6, $\mathbf{Q}$ is a retract of $\mathbf{R}$. Both being two-element, we have that $\mathbf{Q}$ is isomorphic to $\mathbf{R}$. □

An algebra is *arithmetical* if its congruence lattice is arithmetical. The next theorem is due to Bergman and Berman, see [2]. We give a proof of it by using the description of irreducible resets in a particular V of the above form.

Theorem 2.11. *Let A be a finite, arithmetical, congruence primal algebra and let B be any algebra. Then $A \equiv_c B$ if and only if B is finite, arithmetical, congruence primal and $Con(A) \cong Con(B)$.*

Proof. It is well known, see [8], that categorical equivalence between algebras preserves finiteness and the Maltsev property for arithmeticity. So by Theorem 2.4 we have one direction of the claim. To prove the other direction which is the essential part we use Theorem 2.5.

Let A and B be finite, arithmetical, congruence primal algebras such that Con(A) is isomorphic to Con(B). Let $\mathbf{P}$ the be the finite reset $(P, r_1, \ldots, r_n)$ where P is the base set of the algebra A and $r_1, \ldots, r_n$ are the elements of Con(A). Clearly, $\mathbf{P}$ is a reset for A. By Proposition 2.7 and 2.9, the irreducibles in a representation of $\mathbf{P}$ are given by taking a join irreducible r_s and a pair (a, b) that generates r_s in Con(A) and restricting the reset $\mathbf{P}$ to $\{a, b\}$. So every irreducible retract $\mathbf{T}$ of $\mathbf{P}$ is determined by a principal filter given by a join irreducible element r_s in Con(A) such that $r_t = 1$ in $\mathbf{T}$ if and only if $r_t \geq r_s$ in Con(A).

Let ι :Con$(A) \to$Con(B) be a lattice isomorphism. We form the relational set $\mathbf{Q} = (Q, \iota(r_1), \ldots, \iota(r_n))$ of the same type as $\mathbf{P}$ on the base set Q of B. Since ι is an order isomorphism and the order of Con(B) determines the irreducible retracts of $\mathbf{Q}$ in the same way as the order of Con$A$ does the ones of $\mathbf{P}$ the irreducible retracts of $\mathbf{Q}$ coincide with the irreducible retracts of $\mathbf{P}$. So by Theorem 2.5 we get that $A \equiv_c B$. □

Let A be an algebra on a finite set P. Let A^* be the extension of A by all constant operations on P. Let $\mathbf{P} = (P, r_1, \ldots, r_n, \ldots)$ be a finite reset for A^* such that the elements of Con(A) occur among $r_1, \ldots, r_n, \ldots$. In [6], for algebra A, Hobby and McKenzie define the notion of an $\langle\alpha, \beta\rangle$-minimal set where $\langle\alpha, \beta\rangle$ is a quotient in Con(A) and the notion of a tame quotient. It is interesting to remark that every $\langle\alpha, \beta\rangle$-minimal set U where the quotient $\langle\alpha, \beta\rangle$ is tame is a base set of some minimal reset of $\mathbf{P}$ which is associated with a minimal $\mathbf{P}$-obstruction of the form $o(a, b, r_i)$ for some $a, b \in U$ and $r_i = \alpha$ in $\mathbf{P}$ where $(a, b) \in \beta \setminus \alpha$. The proof of this fact is based on the properties of $\langle\alpha, \beta\rangle$-minimal sets listed in Theorem 2.9 of [6] and is left to the reader. In the arithmetical, congruence primal case we get that the set of $\langle\alpha, \beta\rangle$-minimal sets where $\langle\alpha, \beta\rangle$ is tame coincides with the base sets of minimal resets of $\mathbf{P}$. This is not the case in general as the following example shows.

Example 2.12. Let A be a congruence primal algebra defined on the four element set $\{1, 2, 3, 4\}$ such that A has two nontrivial congruences θ_1 with the only two element block $\{2, 3\}$ and θ_2 with two two-element blocks $\{1, 2\}$ and $\{3, 4\}$. Let $\mathbf{P}$ be the corresponding reset $(\{1, 2, 3, 4\}, 0, \theta_1, \theta_2, 1)$. Then the $\langle\alpha, \beta\rangle$-minimal sets are $\{1, 2\}, \{2, 3\}, \{3, 4\}$ and the base sets of the minimal resets of $\mathbf{P}$ are $\{1, 2, 3\}$ and $\{2, 3, 4\}$ besides the $\langle\alpha, \beta\rangle$-minimal sets.

We note that in the case of preprimal algebras Theorem 2.5 comes down to a result of Denecke and Lüders in [4]. They proved that every reset $\mathbf{P}$ given on a finite set P by a relation of Rosenberg's type [13] different from a bounded partial order has a finite representation by a distinguished reset of the same type as $\mathbf{P}$. One can easily show that the distinguished reset is irreducible in each case and conclude that the equivalence class of $\mathbf{P}$ contains an irreducible element which is in turn a unique minimal reset of every element in the class. On the contrary, if $\mathbf{P}$ is a finite bounded poset it might easily happen that the equivalence class of $\mathbf{P}$ has no irreducible element. For example, take $\mathbf{P}$ as the product of two finite bounded irreducible posets such that none of them is a retract of a power of the other. The interested reader can find examples of such irreducible posets in [5] or [11]. Variants of Theorem 2.3 for order primal algebras were proven in [3] and [8].

References

1. K. A. Baker and A. F. Pixley, *Polynomial interpolation and the Chinese remainder theorem for algebraic systems*, Math. Z. **143** (1975), 165–174.
2. C. Bergman and J. Berman, *Morita equivalence of almost-primal clones*, J. Pure and Applied Algebra **108** (1996), 175–201.
3. B. A. Davey, R. W. Quackenbush and D. Schweigert, *Monotone clones and the varieties they determine*, Order **7** (1991), 145–167.
4. K. Denecke and O. Lüders, *Category equivalences of clones*, Algebra Universalis **34** (1995), 608–618.
5. D. Duffus and I. Rival, *A structure theory for ordered sets*, Discrete Math. **35** (1981), 53–118.
6. D. Hobby and R. McKenzie, *The structure of finite algebras*, Contemporary Mathematics, AMS, Providence-Rhode Island **76** (1988).
7. M. Kabil and M. Pouzet, *Injective envelope of graphs and transition systems*, preprint (1993).
8. R. N. McKenzie, *An algebraic version of categorical equivalence for varieties and more general algebraic categories*, preprint (1995).
9. P. Nevermann, *κ-embeddings of ordered sets*, Order **5** (1988), 173–186.
10. P. Nevermann and I. Rival, *Holes in ordered sets*, Graphs and Comb. **1** (1985), 339–350.
11. P. Nevermann and R. Wille, *The strong selection property and ordered sets of finite length*, Algebra Universalis **18** (1984), 18–28.
12. A. F. Pixley, *Completeness in arithmetical algebras*, Algebra Universalis **2** (1972), 179–196.
13. I. G. Rosenberg, *Über die funktionale Vollständingkeit in den mehrwertigen Logiken*, Roz. Ceskoslovenské Akad. Véd **80** (1970), 3–90.
14. G. Tardos, *A maximal clone of monotone operations which is not finitely generated*, Order **3** (1986), 211–218.
15. L. Zadori, *Order varieties generated by finite posets*, Order **8** (1992), 341–348.