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Series Parallel Posets with Nonfinitely Generated Clones

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Abstract. In 1986 Tardos proved that for the poset $1+2+2+2+1$, the clone of monotone operations is nonfinitely generated. We generalize his result in the class of series parallel posets. We characterize the posets with nonfinitely generated clones in this class by the property that they have a retract of the form either $1+2+2+2+1$, $2+2+1$, or $1+2+2$.

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Introduction

Throughout the paper the same boldface and slanted capital letters are used to denote a poset and its baseset, respectively.

For a poset **P** the clone of all monotone operations is called the *clone of* **P**. For $n \geqslant 3$ an n-ary operation f is a *near unanimity function*, briefly a *nuf*, if it obeys the identity

$$f(x,\dots,x,\underbrace{y}_{i},x,\dots,x)=x$$

for every $1 \leqslant i \leqslant n$. If $n=3$, then f is called a *majority function*. One of the most intriguing problems on finite bounded posets is the following, mentioned in [1], [4] and [5]. Is it true that if the clone of a finite bounded poset is finitely generated then it contains a near unanimity function? Since the property of admitting a nuf is preserved under retraction, if the answer were yes we would have the following claim. If the clone of a finite bounded poset **P** is finitely generated, then the clone of every retract of **P** is also finitely generated. By the result of Tardos in [6], this would imply that for every finite bounded poset having $1+2+2+2+1$ as a retract, the clone of monotone operations is nonfinitely generated. But to prove even the claim in the preceding sentence seems very difficult.

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A 4-element subset in a poset $\mathbf{P}$ is called an *n-subset* if it induces a 4-element fence in $\mathbf{P}$. A poset is called *series parallel* if it is finite and does not contain an n-subset. Series parallel posets are characterized in [7] as the ones that are obtained from one element posets by using linear sum and disjoint union in a finite number of steps. In this paper we give a full description of the series parallel posets with nonfinitely generated clones. In Theorem 3.8 these posets are characterized by the property that they have one of $1+2+2+2+1$, $2+2+1$, and $1+2+2$ as a retract. We also show that every bounded series parallel poset having a finitely generated clone admits a 5-ary nuf. So for bounded series parallel posets the above mentioned question has an affirmative answer.

1. Zigzags

We need some definitions and claims from [8]. We say that a poset $\mathbf{Q}$ is *contained* in a poset $\mathbf{P}$ if $\leqslant_{\mathbf{Q}} \subseteq \leqslant_{\mathbf{P}}$. If $\mathbf{Q}$ is contained in $\mathbf{P}$ we write $\mathbf{Q} \subseteq \mathbf{P}$. We say that $\mathbf{Q}$ is *properly contained* in $\mathbf{P}$ if $\mathbf{Q} \subseteq \mathbf{P}$ and $\mathbf{Q} \neq \mathbf{P}$.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A pair $(\mathbf{Q}, f)$ is called a $\mathbf{P}$*-colored poset* if f is a partially defined map from Q to P. If f can be extended to a fully defined monotone map f': $\mathbf{Q} \to \mathbf{P}$ on Q then f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$*-extendible*; otherwise f and $(\mathbf{Q}, f)$ are called $\mathbf{P}$*-nonextendible*. A $\mathbf{P}$*-zigzag* is a $\mathbf{P}$-nonextendible, $\mathbf{P}$-colored poset $(\mathbf{H}, f)$, where H is finite and for every $\mathbf{K}$, properly contained in $\mathbf{H}$, the $\mathbf{P}$-colored poset $(\mathbf{K}, f|_K)$ is $\mathbf{P}$-extendible. Roughly speaking, the $\mathbf{P}$-zigzags are the finite, minimal, nonextendible $\mathbf{P}$-colored posets. When it is clear what $\mathbf{P}$ is we omit it in the terms such as $\mathbf{P}$-zigzags, $\mathbf{P}$-extendible, etc.

For two $\mathbf{P}$-colored posets $(\mathbf{H}, f)$ and $(\mathbf{Q}, g)$ we say that $(\mathbf{H}, f)$ is *contained* in $(\mathbf{Q}, g)$ and we write $(\mathbf{H}, f) \subseteq (\mathbf{Q}, g)$ if $\mathbf{H} \subseteq \mathbf{Q}$ and $f = g|_H$. Observe that every finite nonextendible colored poset contains a zigzag. For a $\mathbf{P}$-colored poset $(\mathbf{H}, f)$ we define the set $C(\mathbf{H}, f) = \{h \in H\colon\ f(h) \text{ exists}\}$ and $N(\mathbf{H}, f) = H \setminus C(\mathbf{H}, f)$. We call the elements of $C(\mathbf{H}, f)$ *colored elements* and the elements of $N(\mathbf{H}, f)$ *noncolored elements*. If $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$ are nonempty we define the posets $\mathbf{C}(\mathbf{H}, f)$ and $\mathbf{N}(\mathbf{H}, f)$ by the restriction of $\leqslant_{\mathbf{H}}$ to $C(\mathbf{H}, f)$ and $N(\mathbf{H}, f)$, respectively. A colored poset $(\mathbf{H}, f)$ is called *monotone* if f is monotone on its domain. Observe that a zigzag $(\mathbf{H}, f)$ is monotone if and only if $|N(\mathbf{H}, f)| \geqslant 1$.

We frequently use the following claims about zigzags in the next section without explicitly mentioning them. The reader is encouraged to supply the easy proofs to them.

CLAIM 1.1. *Let $(\mathbf{H}, f)$ be a $\mathbf{P}$-zigzag. The subgraph spanned by $N(\mathbf{H}, f)$ in the covering graph of $\mathbf{H}$ is connected.*

CLAIM 1.2. *Let $(\mathbf{H}, f)$ be a monotone zigzag and let $a \in C(\mathbf{H}, f)$. For every $b \in H$ which satisfies $a \prec b$ or $b \prec a$ we have $b \in N(\mathbf{H}, f)$.*

CLAIM 1.3. *Let* $(\mathbf{H}, f)$ *be a* $\mathbf{P}$*-zigzag and let* a *and* b *be two different elements of* $C(\mathbf{H}, f)$. *Let us suppose that there exists* $c \in N(\mathbf{H}, f)$ *with* $c \prec a, b$. *Then* $f(a)$ *and* $f(b)$ *are incomparable.*

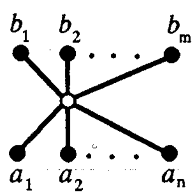
An element $a \in \mathbf{Q}$ is called *irreducible* if there is a unique $b \in Q$ with $a \prec b$ or $b \prec a$.

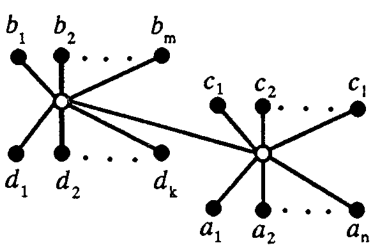
CLAIM 1.4. *Let* $(\mathbf{H}, f)$ *be a* $\mathbf{P}$*-zigzag. Then* $N(\mathbf{H}, f)$ *has no irreducible element of* $\mathbf{H}$.

CLAIM 1.5. *If* $\mathbf{P} = \mathbf{Q} + 1$, *then every maximal element of a* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *is colored.*

CLAIM 1.6. *For a* $\mathbf{P}$*-zigzag* $(\mathbf{H}, f)$ *the following hold.*

(1) *If* $|N(\mathbf{H}, f)| = 0$, *then* $(\mathbf{H}, f)$ *is a two element nonmonotone zigzag.*
(2) *If* $|N(\mathbf{H}, f)| = 1$, *then* $(\mathbf{H}, f)$ *is the first colored poset shown in the figure, where* m *and* n *are nonnegative integers such that* $m + n > 0$ *and* $n, m \neq 1$. *Moreover,* f *is an order isomorphism on its domain.*
(3) *If* $|N(\mathbf{H}, f)| = 2$, *then* $(\mathbf{H}, f)$ *is the second collored poset shown in the figure, where* $k, l \geqslant 1$ *and* m *and* n *are nonnegative integers for which* $m, n \neq 1$. *Moreover, any comparable pair in* Range(f) *not shown in the figure is of the form* $d_i < c_j$, $c_j < b_s$ *or* $a_t < d_i$ *for some* $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$, $1 \leqslant s \leqslant m$ *and* $1 \leqslant t \leqslant n$.





Tardos's remark in [6] describes via zigzags the finite posets admitting an n-ary nuf.

REMARK 1.7. Let $n \geqslant 3$. A finite poset $\mathbf{P}$ admits an n-ary near unanimity function if and only if in every $\mathbf{P}$-zigzag the number of colored elements is at most $n - 1$.

2. Series Parallel Posets with Nonfinitely Generated Clones

To get instances of series parallel posets possessing clones with no finite generating sets we begin with some claims that serve as the basis of Tardos's proof in [6].

For an algebra A let $Clo(A)$ be the set of finitary term operations on A and let $Clo(A)$ be the set of m-ary term operations on A. Recall that for any algebra A the set of n-ary relations on A admitting $Clo(A)$ concides with the set of subalgebras of the n-th power of A. A set of sets is called an *m-cover* if any m-element subset of their union is a subset of one of them. Note that for all $n \leqslant m$ any m-cover is also an n-cover.

Let A be a finite algebra. We characterize the relations admitting $Clo_m(A)$ with the help of relations admitting $Clo(A)$.

LEMMA 2.1. *An n-ary relation R admits $Clo_m(A)$ if and only if R is of the form $\bigcup_{i=1}^{s} R_i$ for some s, where each R_i, $1 \leqslant i \leqslant s$, admits $Clo(A)$ and the R_i form an m-cover.*

Proof. Let $R = \{r_1, \ldots, r_t\}$ an n-ary relation admitting $Clo_m(A)$. Then we have

$$R = \bigcup_{\{i_1,\ldots,i_m\}\subseteq\{1,\ldots,t\}} \{f(r_{i_1}, \ldots, r_{i_m})\colon\ f \in Clo_m(A)\}.$$

Each set following the cup symbol is subalgebra of A^n. So

$$R = \bigcup_{i=1}^{s} R_i,$$

where each R_i is a subalgebra of A^n and, clearly, every m-element subset of R is contained in some R_i.

Now, let R be a set of n-tuples such that $R = \bigcup_{i=1}^{s} R_i$, where each R_i admits $Clo(A)$ and every m-element subset of R is contained in some R_i. When applying any $f \in Clo_m(A)$ to R only m elements of R are involved but these elements are in R_i for some i and f preserves R_i. So f preserves R as well. Thus R admits $Clo_m(A)$. □

Lemma 2.1 has the following corollary.

COROLLARY 2.2. *$Clo(A)$ is nonfinitely generated if and only if for every m there exists an m-cover $\{R_1, \ldots, R_{s_m}\}$ of n-ary relations admitting $Clo(A)$ such that*

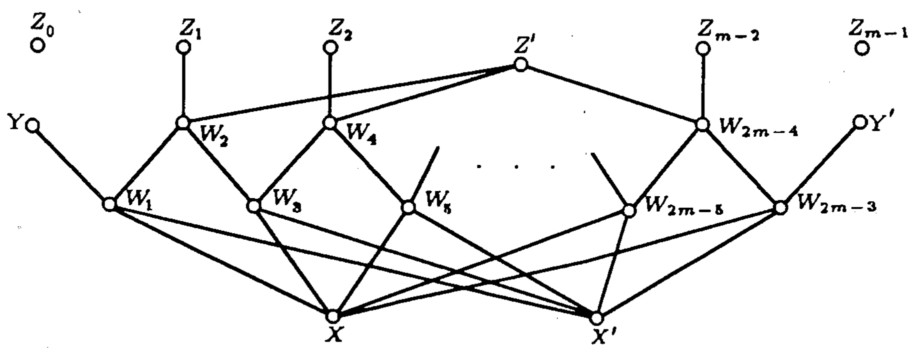
$$R = \bigcup_{i=1}^{s_m} R_i$$

does not admit $Clo(A)$, i.e., R is smaller than the join of the R_i, $1 \leqslant i \leqslant s_m$, in the subalgebra lattice of A^n.

Now we are ready to prove the main claim of the paper. The proof is based on the one contained in [6].

THEOREM 2.3. *If a series parallel poset* **P** *has at least one of $2+2+1$, $1+2+2$ and $1+2+2+2+1$ as a retract then the clone of* **P** *is nonfinitely generated.*

Proof. Let **P** be a series parallel poset and let C be its clone. First let us suppose that $1+2+2+2+1$ is a retract of **P**. We are going to show that C is nonfinitely generated. Let A be the algebra given on the universe P whose fundamental operations are all the monotone operations on **P**. We define some subalgebras of A^{m+5} with the help of the poset $\mathbf{Q}_m$, $m \geqslant 3$, in the figure below.



Let $R_0 \subseteq P^{m+5}$ be the set of those $(m+5)$-tuples of the form

$$(a, a', b, b', c_1, \ldots, c_m, c')$$

for which all the partial functions f_i, $1 \leqslant i \leqslant m$, given by

$$f_i(X) = a, \quad f_i(X') = a', \quad f_i(Y) = b, \quad f_i(Y') = b', \quad f_i(Z') = c',$$
$$f_i(Z_0) = c_i, \ldots, f_i(Z_{m-1}) = c_{(m-1)+i},$$

where the indices are considered modulo m, are **P**-extendible when restricted to both $\mathbf{Q}_m \setminus \{Y\}$ and $\mathbf{Q}_m \setminus \{Y'\}$. Let R_i contain those elements of R_0, where f_i is **P**-extendible to $\mathbf{Q}_m$. We note that in the case of $\mathbf{P} = 1+2+2+2+1$ Tardos has the same definition in [6].

It is obvious that each R_i, $0 \leqslant i \leqslant m$, is a subalgebra of A^{m+5}. We show that the R_i, $1 \leqslant i \leqslant m$, form an $[(m-1)/2]$-cover. Let $R = \bigcup_{i=1}^{m} R_i$. For an arbitrary $u \in R$ we define $S_u = \{l\colon\ u \in R_l\}$. First we prove that $|S_u| \geqslant m-2$. We assume that there is an i such that $u \notin R_i$ and $u \in R_{i+1}$ otherwise there is nothing to prove. Observe that $b' > c'$ is impossible otherwise $u \in R_0$ implies $u \in R_i$. Suppose that b' and c' are incomparable. By leaving out W_{2m-4} from the colored poset $(\mathbf{Q}_m, f_i)$, because of $u \in R_0$, the remaining colored poset is **P**-extendible via a map $f\colon\ Q_m \setminus \{W_{2m-4}\} \to P$. Now, $f(W_{2m-5})$, c', $f(W_{2m-3})$ and b' form an n-subset in **P** since $u \notin R_i$. But this contradicts the fact that **P** is series parallel. So we have $b' < c'$ and similarly $b' < c_{m-2+i}$. Let us color W_{2m-4} by b' in $(\mathbf{Q}_m, f_i)$.

The so obtained colored poset is nonextendible since $u \notin R_i$. By leaving out W_{2m-6} from this colored poset, again, because of $u \in R_0$, the remaining colored poset is **P**-extendible. By repeating the preceding argument we get $b' < c_{m-3+i}$. Similarly, by induction, we get $b' < c_{j+i}$ for all $1 \leqslant j \leqslant m-2$. By symmetry $b < c'$ and $b < c_{j+i}$ for all $1 \leqslant j \leqslant m-2$.

Let $M_j = \{p \in P\colon\ a, a' \leqslant p \leqslant c', c_{j+i+1}\}$, where $1 \leqslant j \leqslant m-2$. Let **M** be the subposet of **P** induced by the set

$$\bigcup_{j=1}^{m-2} M_j \cup \{b, b'\}.$$

Since $u \in R_{i+1}$, b and b' are connected by a path in **M**. Because $u \notin R_i$ and **M** is series parallel there exists a $d \in M$ with $b, b' < d \leqslant c_{i+m-1}, c'$.

Now, in both cases when either $u \in R_{i-1}$ or not, by the preceding two paragraphs we get $b, b' \leqslant c_i$. But then each $(\mathbf{Q}_m, f_{j+i})$, $1 \leqslant j \leqslant m-2$, is extendible by coloring W_k by b if $k < 2(m-1)-2j$, $W_{2(m-1)-2j}$ by d and W_k by b' if $k > 2(m-1)-2j$. Thus we have $|S_u| \geqslant m-2$.

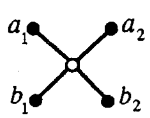
Since any $[(m-1)/2]$ elements of R avoid at most $2[(m-1)/2] \leqslant m-1$ of the R_i, where $1 \leqslant i \leqslant m$, there is at least one R_i which contains these $[(m-1)/2]$ elements. So the R_i form an $[(m-1)/2]$-cover. By Corollary 2.2, it remains to show that R is not preserved by a monotone operation on **P**. Let us select a subposet **T** of **P** that is isomorphic to $1+2+2+2+1$ such that there exists an idempotent monotone map r from **P** onto **T**. By Lemma 5 in [6] there is a monotone function g: $\mathbf{T}^{2m} \to \mathbf{T}$ that does not preserve $\bigcup_{i=1}^{m} r(R_i)$. But then $g(r(x_1), \ldots, r(x_{2m}))$ does not preserve R.

For the case when $2+2+1$ is a retract of **P**, we note that the proofs and claims in Tardos's paper [6] can simply be carried over to the poset $1+2+2$ and so can the above argument to **P** by leaving out the two minimal elements when defining $\mathbf{Q}_m$. The case when $1+2+2$ is a retract of **P** is the dual of the preceding one. □

3. Series Parallel Posets with Finitely Generated Clones

We will prove that the posets described in Theorem 2.3 are the only ones with clones not having finite generating sets among series parallel posets. First we show, if none of $2+2$, $1+2+2$, $2+2+1$ and $1+2+2+2+1$ is a retract of a series parallel poset **P** then the clone of **P** contains a 5-ary nuf.

THEOREM 3.1. *Let* **P** *be a connected series parallel poset. If* **P** *has neither of* $2+2$, $1+2+2+2+1$, $2+2+1$, *and* $1+2+2$ *as a retract then every monotone* **P**-*zigzag is one of the form*



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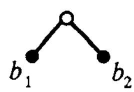
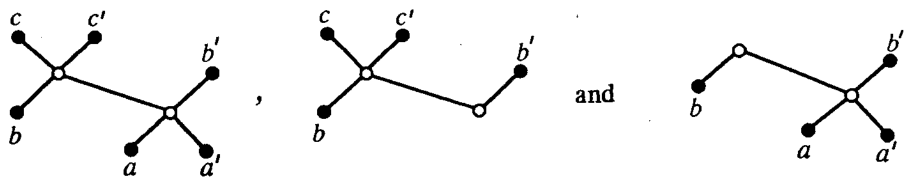
and



for some appropriate a_1, a_2, b_1 *and* b_2 *in* **P**. *In particular,* **P** *admits a majority function (when* $1+2+2+1$ *is not a retract of* **P**) *or a 5-ary nuf.*

Proof. Let **P** be a connected series parallel poset. Let us suppose that $(\mathbf{Q}, g)$ is a **P**-zigzag with $|N(\mathbf{Q}, g)| \geqslant 2$. We are going to prove that **P** has a retract of the form 2+2, 1+2+2+2+1, 2+2+1 or 1+2+2. Let h_1 and h_2 be two elements in $N(\mathbf{Q}, g)$ such that h_1 is covered by h_2. By the minimality of $(\mathbf{Q}, g)$ there is an extension g' of g to $\mathbf{Q} \setminus \{h_1, h_2\}$ such that $(\mathbf{Q}, g')$ contains a zigzag $(\mathbf{H}, f)$ with noncolored elements h_1 and h_2.

We show that $(\mathbf{H}, f)$ is one of the form

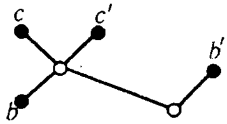


Note, if h_2 had no upper covers and h_1 had no lower covers in **H** then $(\mathbf{H}, f)$ would be **P**-extendible because **P** has no n-subsets. Hence, if $(\mathbf{H}, f)$ is not of the above form then, in $(\mathbf{H}, f)$, up to duality, either there exist three upper covers s, s' and s'' of h_2 colored by c, c' and c'', respectively, or there exist two lower covers t and t' of h_2 colored by b, b'', respectively. In the first case, let us delete s in $(\mathbf{H}, f)$. We extend the remaining colored poset coloring h_1 by u and h_2 by v. Similarly, by deleting s' in $(\mathbf{H}, f)$ the remaining colored poset is extended coloring h_1 by u' and h_2 by v'. Now, it is easy to check that v, c'', v' and c form an n-subset in **P**. In the second case, by deleting t in $(\mathbf{H}, f)$ the remaining colored poset is extended coloring h_1 by u and h_2 by v. Let $v' \in P$ an element that extends f to h_2. Now b, v', b'' and v form an n-subset in **P**. So $(\mathbf{H}, f)$ is of the desired form.

In all three cases for $(\mathbf{H}, f)$ we claim that $a, a' < b, b'$ and $b, b' < c, c'$, whichever applies. We prove this in the first case. For the others the proof is similar. If $b < b'$ then by deleting the covering edge (h_1, h_2) in $(\mathbf{H}, f)$ the remaining colored poset extends coloring h_1 by u and h_2 by v. But then u, b', b and v form an n-subset of **P**. Hence b and b' are incomparable. In this case, by the main argument in Theorem 2.3, $a, a' < b < c, c'$ holds if and only if $a, a' < b' < c, c'$ does. So if $a, a' < b, b' < c, c'$ does not hold then, up to symmetry, b is incomparable to a and b' is incomparable to c. Then b, c, a and b' form an n-subset in **P**. So we have the claim.

Finally, we show that one of $2+2$, $1+2+2+2+1$, $2+2+1$, and $1+2+2$ is a retract of **P**. Let T be the subset of P that is obtained from Range(f) by augmenting it by an element above both c and c' and an element below both a and a', provided there exist such. Let **T** be the subposet in **P** induced by T. Let **R** be a maximal subposet of **T** such that **R** is isomorphic to one of $2+2$, $1+2+2+2+1$, $2+2+1$, and $1+2+2$. We claim that **R** is an idempotent image of **P**. In other words, the colored poset $(\mathbf{P}, \mathrm{id}_R)$ is **R**-extendible, i.e., $(\mathbf{P}, \mathrm{id}_R)$ does not contain an **R**-zigzag. This claim is obvious if **R** is isomorphic to one of $1+2+2+2+1$, $2+2+1$ and

$1+2+2$ by using the form of **R**-zigzags described in Tardos [6], the facts that **P** is series parallel and that $(\mathbf{H}, f)$ is not extendible. The only more complicated case, up to duality, comes up when $(\mathbf{H}, f)$ is of the form shown here from the middle of the preceding figure



and $T = R = \{b, b', c, c'\}$. Then let S be the set of elements dominating both b, b' in **P**. So $c, c' \in S$. The subposet **S** induced by S in **P** is disconnected and c, c' belong to different components. For otherwise c and c' are connected in **S** by a path of length two which contradicts the definition of T or the fact that $(\mathbf{H}, f)$ is not extendible. Now, let $\mathbf{S}'$ be the subposet of **P** induced by $S' = P \setminus S$. Again, $\mathbf{S}'$ is disconnected and b and b' belong to different components of $\mathbf{S}'$. For otherwise b and b' are connected in $\mathbf{S}'$ by a path of length two which contradicts the definition of S or the fact that $(\mathbf{H}, f)$ is not extendible. So one can easily define an idempotent monotone map from **P** onto **R**.

Thus, if **P** has none of $2+2$, $1+2+2+2+1$, $2+2+1$, and $2+2+1$ as a retract then every **P**-zigzag has at most one noncolored element. To show that such a zigzag is of the form stated in the claim use the first half of the second paragraph of this proof. The last claim of the theorem follows by Remark 1.7. □

We have the following corollary of the previous two theorems.

COROLLARY 3.2. *Let* **P** *be a bounded series parallel poset. The following are equivalent*:

(1) **P** *admits a 5-ary near unanimity function.*
(2) **P** *has a finitely generated clone.*
(3) $1+2+2+2+1$ *is not a retract of* **P**.

Theorem 3.1 has a generalization as follows.

COROLLARY 3.3. *Every series parallel poset* **P** *that has neither of* $2+2$, $1+2+2+2+1$, $2+2+1$ *and* $1+2+2$ *as a retract admits a majority function* (*when* $1+2+2+1$ *is not a retract of* **P**) *or a 5-ary nuf.*

Proof. By Theorem 3.1 we have the claim for the connected components of **P**. Now, **P** is obtained as a retract of a product formed by a power of the two element antichain and the components of **P**. Since the property of admitting an n-ary nuf is inherited for product and retract we have the claim. □

It was shown by Demetrovics and Rónyai in [3] that the clone of every crown is finitely generated although it does not contain a nuf. Next we will show that a similar result is true for the series parallel posets having $2+2$ and not having any of $1+2+2+2+1$, $2+2+1$, and $1+2+2$ as a retract.

Let $\mathbf{P}$ and $\mathbf{P}_i$, $1 \leqslant i \leqslant n$, be posets. Let $\mathbf{Q}$ denote the product of $\mathbf{P}_i$, $1 \leqslant i \leqslant n$. For $a \in Q$ let a_i stand for the i-th component of a and let a^i stand for the $(n-1)$-tuple obtained from a by leaving out its i-th component a_i. We use the notation $a[i,p]$ for the n-tuple whose i-th component is p and whose other components coincide with the other components of a, respectively. Let $Q[i,p]$ denote the set of all the elements of the form $a[i,p]$, where $a \in Q$. For a function $f\colon Q \to P$ and $p \in P_i$ let $f_{[i,p]} = f|_{Q[i,p]}$.

LEMMA 3.4. *Let* $\mathbf{P}$ *and* $\mathbf{P}_i$, $1 \leqslant i \leqslant n$, *be connected posets. Let* $\mathbf{Q}$ *denote the product of* $\mathbf{P}_i$, $1 \leqslant i \leqslant n$. *Let* $f\colon \mathbf{Q} \to \mathbf{P}$ *be a monotone function with* $|\mathrm{Range}(f)| \geqslant 3$ *which depends on at least two of its variables. Then there exist* $a, b, c \in \mathbf{Q}$ *such that* $a_i = b_i$, $b^i = c^i$ *for some* i *and* $f(a)$, $f(b)$ *and* $f(c)$ *form a three element connected subposet of* $\mathbf{P}$.

Proof. Let $f\colon \mathbf{Q} \to \mathbf{P}$ be a monotone function with $|\mathrm{Range}(f)| \geqslant 3$ such that it depends on its i-th variables, where $i \neq j$. Since f depends on its j-th variable there is $p \in P_i$ such that $|\mathrm{Range}(f_{[i,p]})| \geqslant 2$. Since f depends on its i-th variable there is $q \in P_i$ such that $f_{[i,p]} \neq f_{[i,q]}$. By using that $|\mathrm{Range}(f)| \geqslant 3$ we choose $q \in P_i$, $e, b \in Q[i,p]$ and $d \in Q[i,q]$ in such a way that $b^i = d^i$ and f takes on different values for e, b and d. Since $\mathbf{P}_i$ is connected there is a path between p and q in $\mathbf{P}$. Let s be the closest point to p on this path such that $f(b[i,s])$ is different from both $f(e)$ and $f(b)$. Let $c = b[i,s]$. If there is a t on the path between p and s satisfying $f(b[i,t]) = f(e)$ we set $a = e$. Then for the so obtained a, b and c we get the claim. For otherwise, since the points of $Q[i,p]$ span a connected subposet of $\mathbf{Q}$ the points b and c are connected by a path in this subposet. Let a be the closest point on this path to b with the property that $f(a)$ is different from both $f(b)$ and $f(c)$. Again, by taking the so obtained a, b and c we get the claim. □

LEMMA 3.5. *Let* $\mathbf{P} = \mathbf{A}_1 + \mathbf{A}_2$, *where* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are two antichains. For every* $1 \leqslant i \leqslant n$ *let* $\mathbf{P}_i$ *be a finite connected poset whose each element is either comparable to all minimal or to all maximal elements. Let* $\mathbf{Q}$ *denote the product of* $\mathbf{P}_i$, $1 \leqslant i \leqslant n$. *Then for every monotone map* $f\colon \mathbf{Q} \to \mathbf{P}$ *depending on at least two of its variables either* $|\mathrm{Range}(f) \cap A_1| = 1$ *or* $|\mathrm{Range}(f) \cap A_2| = 1$.

Proof. Let $f\colon \mathbf{Q} \to \mathbf{P}$ be a monotone map depending on at least two of its variables. We assume that $|\mathrm{Range}(f)| \geqslant 3$, otherwise there is nothing to prove. Lemma 3.4 applies. Hence there are $a, b, c \in Q$ such that $a_i = b_i$ and $b^i = c^i$ for some i and $f(a)$, $f(b)$, $f(c)$ form a three element connected subposet of $\mathbf{P}$. Without loss of generality we assume that one of $f(a)$, $f(b)$ and $f(c)$ is minimal and the other two are maximal in $\mathbf{P}$. By contradiction, let us assume that there is a $d \in Q$ such that $f(a)$, $f(b)$, $f(c)$, and $f(d)$ form a 4-element crown in $\mathbf{P}$.

Now we replace a, b, c, and d by new elements in $\mathbf{P}^n$ such that the values of f will not change. As a result of this replacement the new a, b, c, and d will satisfy the old properties for a, b, c, and d, moreover for every $1 \leqslant j \leqslant n$ the elements a_j, b_j, c_j, and d_j will form a connected subposet in $\mathbf{P}_j$. We replace d by a minimal element of $\mathbf{Q}$ below d. If $f(a)$ is minimal we replace c by a maximal element of $\mathbf{Q}$ above c and then replace b by $c[i, b_i]$. If $f(b)$ is minimal we replace c by $c[i, p]$, where p is maximal in $\mathbf{P}_i$ above c_i and we replace a by a maximal element above a in the poset spanned by the elements of $Q[i, a_i]$ in $\mathbf{Q}$. If $f(c)$ is minimal we replace a by a maximal element of $\mathbf{Q}$ above a and then replace b by $b[i, a_i]$.

Let

$$H = \prod_{j=1}^{n} \{a_j, b_j, c_j, d_j\}.$$

Let $\mathbf{H}$ be the subposet spanned by the elements of H in $\mathbf{Q}$. Then $f|_H$: $\mathbf{H} \to \mathbf{P}$ is monotone with a 4-element crown in its range. This is impossible since each factor of $\mathbf{H}$ is a connected poset with at most three elements, so it contains an element with distance at most 1 from each extremal element, which property is preserved under taking product and monotone image. □

From Lemma 3.5 it follows immediately that the clone of the linear sum of two finite antichains, each of which having at least two elements, is contained in the Słupecki clone on the underlying set. We note that an argument similar to the one given in the last paragraph of the proof of Lemma 3.5 can be used to show for other posets, such as braids of height one defined in [2], that their clones are contained in the Słupecki clone.

Observe that a connected series parallel poset $\mathbf{P}$ which does not have any of $1+2+2+2+1$, $2+2+1$, and $1+2+2$ but does have $2+2$ as a retract is of the form $\mathbf{A}_1 + \mathbf{A}_2$, where each $\mathbf{A}_i$ is a disjoint sum of series parallel posets $\mathbf{B}_{i,1}, \ldots, \mathbf{B}_{i,l_i}$ with $l_i \geqslant 2$. Each $B_{i,j}$ has neither one of $2+2$, $1+2+2+2+1$, $2+2+1$, and $1+2+2$ as a retract and contains a largest element $b_{i,j}$ for $i = 1$ and a smallest one $b_{i,j}$ for $i = 2$. On $\mathbf{P}$ we define the monotone idempotent maps $r_{i,j}$ by $r_{i,j}(x) = x$ if $x \in B_{i,j}$ and $r_{i,j} = b_{i,j}$ otherwise.

LEMMA 3.6. *Let $\mathbf{P}$ be a connected series parallel poset that has $2+2$ but does not have any of $1+2+2+2+1$, $2+2+1$, and $1+2+2$ as a retract. We assume that $\mathbf{P}$ is given in the above form. For every $1 \leqslant i \leqslant n$ let $\mathbf{P}_i$ be a finite connected poset whose each element is either comparable to all minimal or to all maximal elements. Let $\mathbf{Q}$ denote the product of $\mathbf{P}_i$, $1 \leqslant i \leqslant n$. Let f: $\mathbf{Q} \to \mathbf{P}$ be a monotone map such that its range has a nonempty intersection with at least two of the $B_{i,j}$ for each $i = 1, 2$. Let $f_{i,j}$: $\mathbf{Q} \to \mathbf{P}$ be defined $r_{i,j} \circ f$. Then there exists a monotone map*

h: $\mathbf{P}_t \times \mathbf{P}^{l_1+l_2} \to \mathbf{P}$ *for some* $1 \leqslant t \leqslant n$ *such that*

$$f(x_1,\ldots,x_n) = h\Big(x_t, f_{1,1}(x_1,\ldots,x_n),\ldots, f_{1,l_1}(x_1,\ldots,x_n),$$
$$f_{2,1}(x_1,\ldots,x_n),\ldots, f_{2,l_2}(x_1,\ldots,x_n)\Big).$$

Proof. Let $\mathbf{P}$ be a series parallel poset of the above form and let f: $\mathbf{Q} \to \mathbf{P}$ be an arbitrary monotone map. Let r: $\mathbf{P} \to \mathbf{P}$ be an idempotent map defined by $r(p) = b_{i,j}$ for all $p \in B_{i,j}$. Then the map $r \circ f$: $\mathbf{Q} \to \mathbf{P}$ is a monotone map and its image is the linear sum of two at least two element finite antichains. So, by Lemma 3.5, $r \circ f$ is essentially unary. Hence there exists a $1 \leqslant t \leqslant n$ such that for each $q \in P_t$, Range$(f_{[t,q]})$ is a subset of some $B_{i,j}$.

We define h: $\mathbf{P}_t \times \mathbf{P}^{l_1+l_2} \to \mathbf{P}$ as follows. The value of h on

$$(q, p_{1,1},\ldots,p_{1,l_1}, p_{2,1},\ldots,p_{2,l_2})$$

is given by $r_{i,j}(p_{i,j})$ that corresponds to the (i,j) for which Range$(f_{[t,q]}) \subseteq B_{i,j}$. Now, observe that the so defined h is monotone and satisfies the identity in the claim. $\square$

Let f be an n-ary operation on P. An l-ary operation on P is called an *l-ary polymer* of f, if it is obtained from f by identifying some variables.

LEMMA 3.7. *Let C be a clone on a finite set P. Suppose that there is an l-ary $m \in C$ that is near unanimity when restricted to some $R \subseteq P$. Let $n \geqslant |P|^{l-1}$. Then every n-ary operation $f \in C$ with* Range$(f) \subseteq R$ *can be built from its $|P|^{l-1}$-ary polymers, m and the projections via composition of functions.*

Proof. Let $f \in C$ be an operation with arity $n \geqslant |P|^{l-1}$ such that Range$(f) \subseteq R$. Observe that for an arbitrary $l-1$-element subset H of P^n there is an n-ary operation which interpolates f on H and is obtained by a $|P|^{l-1}$-ary polymer of f substituted in a suitable variable of an n-ary projection. By using these operations, m and the projections, f is easily built. $\square$

THEOREM 3.8. *Let* $\mathbf{P}$ *be a series parallel poset. The clone of* $\mathbf{P}$ *is nonfinitely generated if and only if one of* $1+2+2+2+1$, $2+2+1$, *and* $1+2+2$ *is a retract of* $\mathbf{P}$.

Proof. The if part of the theorem has been shown in Theorem 2.3. Now, we will prove the only if part. Let $\mathbf{P}$ be a series parallel poset having neither of $1+2+2+2+1$, $2+2+1$, and $1+2+2$ as a retract. Suppose that $\mathbf{P}$ has k connected components. Then $\mathbf{P}^n$ has k^n connected components, say, $\mathbf{C}_1,\ldots,\mathbf{C}_{k^n}$. Let f: $\mathbf{P}^n \to \mathbf{P}$ be a monotone operation. Let r_i: $\mathbf{P}^n \to \mathbf{C}_i$ be an onto idempotent monotone map defined by componentwise idempotent maps s_l: $P \to P$, $1 \leqslant l \leqslant n$, and let $f_i = f \circ r_i$

for each $1 \leqslant i \leqslant k^n$. First we want to show that f_i is generated by the $|P|^4$-ary operations on $\mathbf{P}$. Let the image of f_i be contained in a connected component $\mathbf{C}$ of $\mathbf{P}$. If the image of f_i does not contain a four element crown that is an idempotent image of $\mathbf{C}$ then there exist a subposet $\mathbf{R}$ of $\mathbf{C}$ with Range$(f_i) \subseteq R$ and an onto idempotent map $r\colon \mathbf{P} \to \mathbf{R}$ such that $\mathbf{R}$ is connected, does not have a four element crown as a retract and so, by Theorem 3.1, it admits a 5-ary nuf m. Hence by taking $m(r(x_1), r(x_2), r(x_3), r(x_4), r(x_5))$ and applying Lemma 3.7 we get the claim. If the image of f_i contains a four element crown that is an idempotent image of $\mathbf{C}$ then Lemma 3.6 applies to the map $f|_{C_i}$. So there exist t and h such that

$$\begin{aligned} &f_i(x_1, \ldots, x_n) \\ &\quad = h\Big(s_t(x_t), f_{1,1}\big(s_1(x_1), \ldots, s_n(x_n)\big), \ldots, f_{1,l_1}\big(s_1(x_1), \ldots, s_n(x_n)\big), \\ &\qquad\qquad f_{2,1}\big(s_1(x_1), \ldots, s_n(x_n)\big), \ldots, f_{2,l_2}\big(s_1(x_1), \ldots, s_n(x_n)\big)\Big). \end{aligned}$$

The range of $f_{i',j'} \circ r_i$ is contained in $B_{i',j'}$ for all i' and j' hence the preceding argument applies for $f_{i',j'} \circ r_i$. Since h extends to the whole $\mathbf{P}^{1+l_1+l_2}$ we have the claim in this case as well.

Finally, we show that f is obtained by using only $|P|^4$-ary monotone operations on $\mathbf{P}$. Observe that f is obtained by substituting the operations $f_i(x_1, \ldots, x_n)$ in the i-th variable of the $(k^n + n)$-ary operation v_n that projects to its l-th variable, if its last n coordinates form an n-tuple in C_l. But v_n is obtained, via a $(k+1)$-ary monotone operation v_1 that projects to its j-th variable if its last coordinate is in the j-th component of $\mathbf{P}$, in the following way:

$$\begin{aligned} &v_n(y_0, \ldots, y_{k^n-1}, x_0, \ldots, x_{n-1}) \\ &\quad = v_1\Big(v_{n-1}(y_0, \ldots, y_{k^{n-1}-1}, x_0, \ldots, x_{n-2}), \ldots, \\ &\qquad\qquad v_{n-1}(y_{(k-1)k^{n-1}}, \ldots, y_{k^n-1}, x_0, \ldots, x_{n-2}), x_{n-1}\Big). \end{aligned}$$ □

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