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GENERATION OF FINITE PARTITION LATTICES

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INTRODUCTION

Let $\underline{n} = \{1,\ldots,n\}$, and denote by P_n the lattice of all partitions of $\underline{n}$. For any subset H of P_n there exists a least sublattice L of P_n containing H such that L is the congruence lattice of some algebra on $\underline{n}$ (as the intersection of congruence lattices on $\underline{n}$ is again a congruence lattice). So we may say that H generates L in the sense of congruence lattices. To contrast with generation in the usual sense of lattices, in this case we will briefly say that H cl-generates L, or H is a cl--generating set of L. Obviously, if H generates a congruence lattice $L \subseteq P_n$, then H cl-generates L as well.

In this paper we construct cl-generating sets and generating sets of minimum cardinality for the partition lattices P_n which are clearly congruence lattices. STRIETZ [2] has shown that the minimum number of generators of $P_n (n \geqq 4)$ equals 4, and, as a union of ordered chains, a 4-element generating set is of the form 1+1+1+1 or 1+1+2. In [2] he gave a generating set of type 1+1+1+1 for all $n \geqq 4$, and one of type 1+1+2 for all $n \geqq 10$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

These results obviously imply that the minimum number r of cl-generators of P_n is at most 4. We prove here that $r = 3$ provided $n \geq 3$, $n \neq 4$. This answers a question of LÄNGER and PÖSCHEL [1]. Using the 3-element cl-generating set of P_n given in the proof of Theorem 1. we construct new 4-element generating sets of the type 1+1+1+1 for $n \geq 4$, and ones of the type 1+1+2 for $n \geq 7$. So the existence of a generating set of the form 1+1+2 remains open for $n = 5,6$.

Let $U \in P_n$ and let f be an operation on $\underline{n}$. We will say that f preserves U, or U admits f, if U is a partition corresponding to a congruence relation of the algebra $\langle \underline{n}; f \rangle$. A subset A of P_n admits f if every member of A has this property.

We assume throughout that $n \geq 3$. We will denote the least (greatest) partition of $\underline{n}$ by E (T). In the description of the partitions we will not indicate the 1-element blocks.

1. CL-GENERATION

Clearly, if A admits only constants and projections, then the least congruence lattice containing A equals P_n, i.e. A cl-generates P_n. The following well-known lemma (cf. [1]) shows that in order to check this property of A it suffices to deal with unary operations. For brevity, constants and projections will be called trivial operations.

LEMMA 1. *Let $A \subseteq P_n$. If A admits a nontrivial operation then it admits a nontrivial unary operation as well.*

We now establish some straightforward properties of the cl-generating sets of P_n.

LEMMA 2. *Suppose* $A \subseteq P_n \setminus \{E,T\}$ *admits only trivial operations.*

(a) *Then A contains at least three pairwise incomparable partitions.*

(b) *If A is the union of three chains, then the join (meet) of the greatest (least) partitions of any two of these chains equals T (E).*

PROOF. (a) Let U_1 be a maximal, and U_2 a minimal partition in A. There exists a nontrivial unary operation f on $\underline{n}$ which takes on two values from a non-singleton block of U_2 and is constant on every block of U_1. It is easy to see that every partition $U \in A$ comparable with U_1 or U_2 admits f. Since, by assumption, A admits no nontrivial operation, it follows that U_1 and U_2 can be chosen incomparable, furthermore A contains a partition incomparable with both U_1 and U_2.

(b) Let U_1 denote the join of the greatest elements of two of the chains and U_2 the least element of the third chain. Were $U_1 \neq T$, the above construction would yield a nontrivial operation admitted by every partition in A. Similar argument, with the role of U_1 and U_2 interchanged, applies for the meet of the least partitions of the chains.

From this lemma it follows that at least three partitions are necessary for cl-generating P_n, furthermore, if there exists a 3-element generating set, then it is contained in a sublattice of P_n of the form

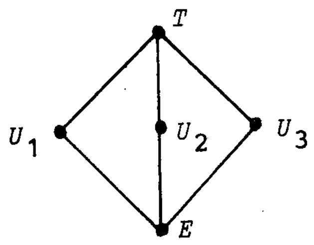

Figure 1

THEOREM 1. *For* $n \geq 3$, $n \neq 4$, *the partition lattice* P_n, *has a 3-element* cl-*generating set.*

PROOF. In view of Lemma 1 it suffices to construct three partitions U_1, U_2 and U_3 such that every unary operation admitted by them is a constant or a projection.

First we consider the case $n = 2k+1$ $(k \geq 1)$, and define

$$U_1 = \langle(1,3,\ldots,2k+1)(2,4,\ldots,2k)\rangle$$
$$U_2 = \langle(1,2)(3,4)\ldots(2k-1,2k)\rangle$$
$$U_3 = \langle(2,3)(4,5)\ldots(2k,2k+1)\rangle.$$

It will be convenient to follow the proof keeping in mind the graphs corresponding to the partitions.
The vertices of this graph are the elements of the base set, and two vertices are connected with an edge (solid, wavy, and dotted ones for U_1, U_2, and U_3, respectively) iff they are distinct and belong to the same block of the corresponding partition. (In fact the solid edges are not

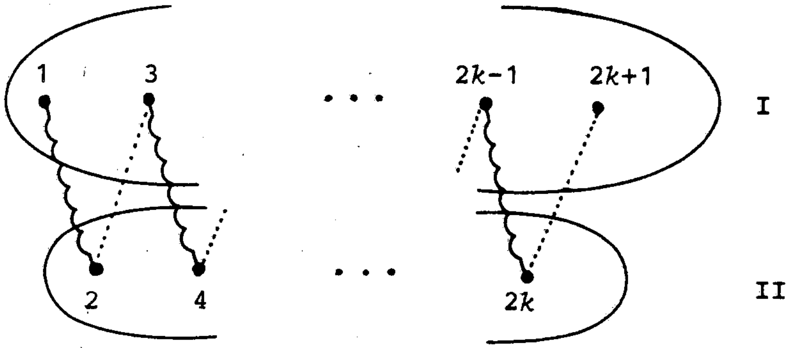

Figure 2

drawn in Figure 2, only the two blocks are indicated.)

Let f be a unary operation preserving U_1, U_2, U_3. Observe first that if the sides of a triangle belong to distinct partitions, then f assumes either three distinct values or a single value on the three vertices. Indeed, let $a,b,c \in \underline{n}$ be such that $a\ U_i\ b\ U_j\ c\ U_k\ a$ ($\{i,j,k\} = \{1,2,3\}$). If, say, $f(a) = f(b)$, then $f(a) = f(b)$ $(U_j \wedge U_k)$ $f(c)$, therefore, since $U_j \wedge U_k = E$, we get $f(a) = f(b) = f(c)$.

Since f preserves U_1, on block I it takes on values only from block I, or only from block II; f behaves similarly on block II. First assume that the values of f are from the same block of U_1. Since $1\ U_2\ 2$, it follows that $f(1)\ U_2\ f(2)$. However, by assumption, $f(1)\ U_1\ f(2)$, so $U_1 \wedge U_2 = E$ implies $f(1) = f(2)$.

Applying now the previous note consecutively for the triangles 1,2,3; 2,3,4 and so on, we get that f is constant.

Assume now that f maps block I into block II and vice versa. Observe that the value $f(2k+1)$, which is in block II, uniquely determines $f(2k)$, namely $f(2k) = f(2k+1)+1$. Indeed, $f(2k+1)\ U_3\ f(2k)$, since $2k+1\ U_3\ 2k$; on the other hand, $f(2k+1)$ and $f(2k)$ are distinct, as they belong to different blocks of U_1. Therefore $f(2k)$ is the unique vertex in block I, connected with $f(2k+1)$ by a dotted edge, i.e. it equals $f(2k+1)+1$. A similar argument with U_2 in place of U_3 shows that $f(2k-1) = f(2k)+1$, and so on. Hence we get that the equality $f(h+1)+1 = f(h)$ must hold for $h = 2k+1, 2k\ldots$ until we reach an element h^* in block II with $f(h^*) = 2k+1$. Such an element does exist, since $f(2k+1) \geqq 2$. Then $h^*\ U_2\ h^*-1$, so $2k+1 = f(h^*)\ U_2\ f(h^*-1)$. However, no wavy edge starts from $2k+1$, hence $f(h^*) = f(h^*-1)$, contradicting the assumption that they belong to different blocks of U_1.

Suppose finally that f maps blocks I and II into themselves. In the same way as before, we have $f(h+1) = f(h)+1$ for $h = 1,2,\ldots$ until we reach an element h^* such that $f(h^*) = 2k+1$. Clearly, if $f(1) = 1$, then f is the identity. Otherwise $h^* < 2k+1$, implying by $h^*\ U_2\ h^*+1$ that $2k+1 = f(h^*)\ U_2\ f(h^*+1)$. This is again a contradiction, since no wavy edge starts from $2k+1$.

We consider now the case $n = 2k+2$ $(k \geqq 2)$, and define

$$U_1 = \langle(1,3,\ldots,2k+1)(2,4,\ldots,2k)\rangle,$$
$$U_2 = \langle(1,2,2k+2)(3,4)\ldots(2k-1,2k)\rangle,$$
$$U_3 = \langle(2,3)(4,5)\ldots(2k,2k+1,2k+2)\rangle$$

(see Figure 3).

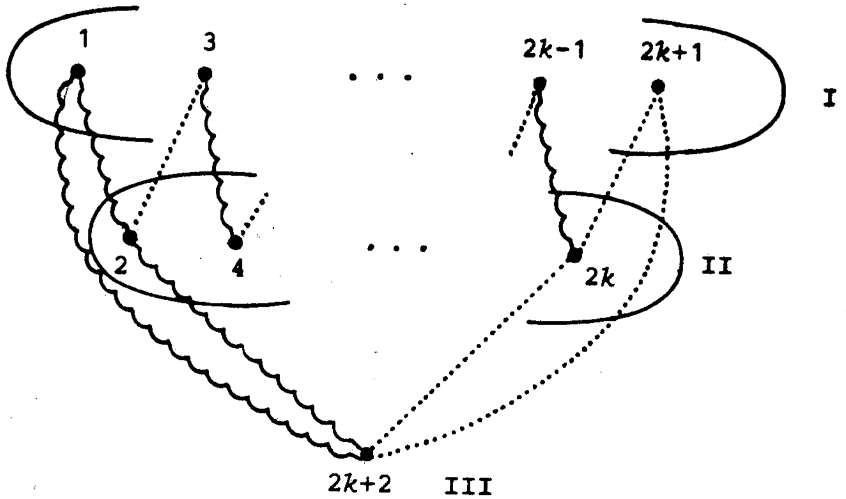

Figure 3

Let f be a unary operation preserving U_1, U_2 and U_3.

Since f preserves U_1, it maps each block of U_1 into some block of U_1. If f maps block I or II into block III, i.e. f is constant with value $2k+2$ on one of the nonsingleton blocks of U_1, then the above remark on the triangles shows that f is the constant with value $2k+2$.

It remains to discuss the operations f which map the union of blocks I and II into itself. Then, forgetting block III for a moment, we can see as in the odd case that f', the restriction of f to $\{1,\dots,2k+1\}$, is a constant or the identity. If f' is a constant, then looking at the triangle $1, 2k+1, 2k+2$ we get that $f(1) = f(2k+1) = f(2k+2)$, hence f is a constant, too. If f' is the identity, then $1\ U_2\ 2\ U_2\ 2k+2$ and $2k\ U_3\ 2k+1\ U_3\ 2k+2$

imply $1 \; U_2 \; 2 \; U_2 \; f(2k+2)$ and $2k \; U_3 \; 2k+1 \; U_3 \; f(2k+2)$, whence $f(2k+2) = 2k+2$, i.e. f is the identity, too.

This completes the proof of Theorem 1.

Note that for $n = 4$, P_n has a 4-element cl-generating set (cf. [2]), however, it has no 3-element cl-generating set. As we have seen after Lemma 2, if there existed a 3-element cl-generating set $\{U_1,U_2,U_3\}$, then its members would form, together with E and T, a sublattice, as shown in Figure 1. It is easy to check that we have one of the following two possibilities ($\{a,b,c,d\} = \{1,2,3,4\}$):

$$\begin{array}{lcl} U_1 = <(a,b)(c,d)>, & & U_1 = <(a,b)(c,d)>, \\ U_2 = <(a,c)(b,d)>, & \text{or} & U_2 = <(a,c)(b,d)>, \\ U_3 = <(a,d)(b,c)>, & & U_3 = <(b,c)>. \end{array}$$

In both cases, the nontrivial unary operation f defined by $f(a) = d$, $f(b) = c$, $f(c) = b$, $f(d) = a$ preserves U_1, U_2 and U_3.

2. GENERATION IN THE SENSE OF LATTICES

Since a generating set of P_n is a generating set in the sense of congruence lattices as well, Lemma 2 and the remark after it imply that for $n \geq 4$, a generating set of P_n has at least four elements; moreover, a 4--element generating set is in the usual notation, either of type 1+1+1+1 (four pairwise incomparable elements) or of type 2+1+1 (only two of the four elements are comparable).

STRIETZ [2] has found in P_n a generating set of type 1+1+1+1 for all $n \geq 4$, and that of type 2+1+1 for all $n \geq 10$. Using the partitions occurring in the proof of Theorem 1 we construct in P_n a new generating set of type 1+1+1+1 for $n \geq 4$ and a generating set of type 2+1+1 for $n \geq 7$.

THEOREM 2. *P_n has a generating set of type* 1+1+1+1 *for all $n \geq 4$, and one of type* 2+1+1 *for all $n \geq 7$.*

PROOF. First let $n = 2k+1$ $(k \geq 2)$, and consider the partitions U_1, U_2, U_3 defined in the proof of Theorem 1, and let

$$U_4 = \langle(1,2,2k,2k+1)\rangle$$

(see Figure 4).

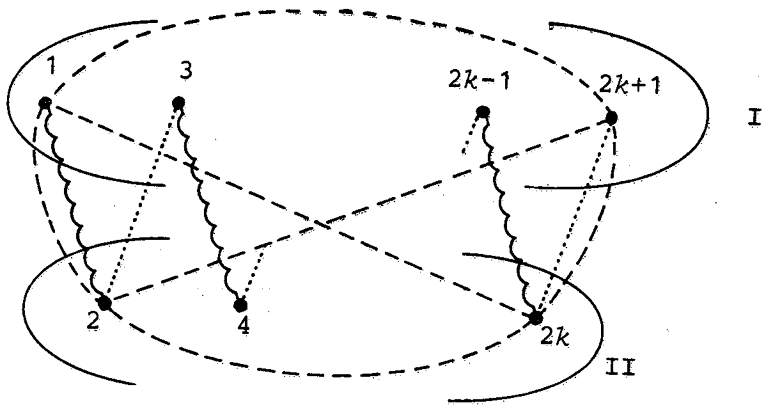

Figure 4

Set

$$M_1 = U_2 \wedge U_4 = \langle(1,2)\rangle$$

and

$$M_2 = U_3 \wedge U_4 = \langle(2k,2k+1)\rangle.$$

We show that U_1, U_2, U_3, M_1 and M_2 generate P_n. For the time being let L denote the sublattice of P_n generated by $\{U_1, U_2, U_3, M_1, M_2\}$. To show $L = P_n$, clearly, it suffices to prove that all partitions having one 2--element block and $n-2$ 1-element blocks belong to L. Define

$$K_l = \langle(1,2)\dots(2l-1,2l)\rangle,\ 1 \le l \le k,$$
$$K_l^* = \langle(2,3)\dots(2l,2l+1)\rangle,\ 1 \le l \le k.$$

It can be easily seen that $K_1 = M_1$,

$$K_l^* = (((K_l \vee U_3) \wedge U_1) \vee K_l) \wedge U_3,\ 1 \le l \le k,$$
$$K_{l+1} = (((K_l^* \vee U_2) \wedge U_1) \vee K_l^*) \wedge U_2,\ 1 \le l \le k.$$

Hence it follows by induction on l that $K_l, K_l^* \in L (1 \le l \le$ $\le k)$. We get symmetrically that the partitions

$$Z_j = \langle(2k,2k-1)\dots(2(k-j),2(k-j)-1)\rangle,$$
$$0 \le j \le k-1,$$
$$Z_j^* = \langle(2k+1,2k)\dots(2(k-j)+1,2(k-j))\rangle,$$
$$0 \le j \le k-1,$$

also belong to L. So $\langle(i,i+1)\rangle \in L$ for $i = 1,\dots,2k$, because $\langle(2l-1,2l)\rangle = K_l \cap Z_{k-l}$ and $\langle(2l,2l+1)\rangle = K_l^* \cap Z_{k-l}^*$

if $1 \leq l \leq k$. Hence

$$<(i,i+2)> = U_1 \wedge (<(i,i+1)> \vee <(i+1,i+2)>) \in L$$

for $i = 1,\ldots,2k-1$. Furthermore, if i,j ($1 \leq i < j \leq$ $\leq 2k+1$) are of the same parity and $i+2 < j$, then

$$<(i,j)> = (<(i,i+2)> \vee \ldots \vee <(j-2,j)>) \wedge$$
$$\wedge (<(i,i+1)> \vee <(j-1,j)> \vee <(i+1,i+3> \vee \ldots$$
$$\ldots \vee <(j-3,j-1)>) \in L;$$

if i is odd, j is even, and $|i-j| > 1$, then, according to whether $i < j$ or $i > j$, we have

$$<(i,j)> = (<(i,i+1)> \vee <(i+1,j)>) \wedge (<(i,j-1)> \vee$$
$$\vee <(j-1,j)>) \in L,$$

or

$$<(i,j)> = (<(j,j+1)> \vee <(j+1,i)>) \wedge (<(j,i-1)> \vee$$
$$\vee <(i-1,i)>) \in L,$$

respectively. Thus $L = P_n$, what was to be proved.

If $n = 2k+1 \geq 7$, then the partitions U_1, U_2, U_3 and $U_4' = <(1,2k+1)(2,2k)>$ ($\leq U_1$) also generate P_n, since $M_1 = (U_4' \vee U_3) \wedge U_2$ and $M_2 = (U_4' \vee U_2) \wedge U_3$. Obviously, they form a generating set of type 2+1+1.

Now let $n = 2k+2$, $k \geq 1$. It is easy to see that P_4 has a generating set of type 1+1+1+1; take, for example, $<(1,2)>, <(2,3)>, <(3,4)>, <(2,4)>$. From now on assume $k \geq 2$, and let

$$U_1 = \langle(1,3,\ldots,2k+1)(2,4,\ldots,2k+2)\rangle,$$
$$U_2 = \langle(1,2)(3,4)\ldots(2k+1,2k+2)\rangle,$$
$$U_3 = \langle(2,3)(4,5)\ldots(2k,2k+1)\rangle,$$
$$U_4 = \langle(1,2,2k,2k+1)\rangle,$$

(see Figure 5).

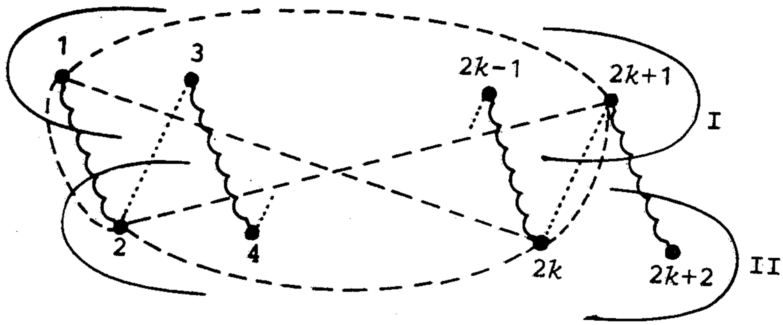

Figure 5

Set

$$M_1 = U_2 \wedge U_4 = \langle(1,2)\rangle \ , \quad M_2 = U_3 \wedge U_4 = \langle(2k,2k+1)\rangle.$$

Again we want to show that U_1, U_2, U_3, M_1 and M_2 generate the lattice P_n. Let L denote the sublattice of P_n generated by $\{U_1, U_2, U_3, M_1, M_2\}$. In the same way as in the odd case we get that

$$\langle(i,j)\rangle \in L \text{ for } 1 \leq i < j \leq 2k+1.$$

Hence

$$<(1,2,2k+1,2k+2)> = ((<(1,2k+1)>\vee U_2)\wedge U_1)\vee M_1 \in L,$$
$$<(2k-1,2k)(2k+1,2k+2)> = (((M_2\vee U_2)\wedge U_1)\vee M_2)\wedge U_2 \in L,$$

implying that their meet

$$<(2k+1,2k+2)> \in L.$$

Thus, repeating the argument used in the odd case for the partitions $U_1, U_3, U_2, <(2k+1,2k+2)>$ and $<(2,3)> (\subseteq L)$, which are situated symmetrically to U_1, U_2, U_3, M_1 and M_2, we get that

$$<(i,j)> \in L \text{ for } 2 \leq i < j \leq 2k+2.$$

Finally,

$$<(1,2k+2)> = (<(1,2k+1)>\vee<(2k+1,2k+2)>)\wedge$$
$$\wedge(<(1,2)>\vee<(2,2k+2)>)\in L,$$

completing the proof of $L = P_n$.

If $n = 2k+2 \geq 7$, then the partitions U_1, U_2, U_3 and $U_4' = <(1,2k+1)(2,2k)> (\leq U_1)$ also generate P_n, since $M_1 = (U_4'\vee U_3)\wedge U_2$ and $M_2 = (U_4'\vee U_2)\wedge U_3$. This yields a generating set of type 2+1+1.

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[2] H. STRIETZ: Über Erzeugendenmengen endlicher Partitionenverbände, *Studia Sci. Math.*, 12(1977), 1-17.