RANDOM SPHERICAL DISC-POLYGONS AND A DUALITY

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Abstract. In this work, we consider the asymptotic behaviour of the expectation of the perimeter deviation of a uniform random spherical disc-polygon in a spherical spindle-convex disc with smooth boundary. We also introduce a notion of duality on the sphere, define a model of random circumscribed disc-polygons, and determine some asymptotic results about them.

1. Introduction

With their 1963 paper [18], Rényi and Sulanke launched the investigation of the now well-studied topic of random polytopes. For a convex body K in $\mathbb{R}^d$, the convex hull of n independently chosen points from K according to the uniform probability distribution is called a *uniform random polytope* in K. The questions of interest are mainly focused on the asymptotic behaviour of this random object, for example, the expectation of the volume of K missed by the random polytope.

More recently, similar problems started to arise in different settings. Consider the following notion of convexity in the plane: the r *spindle convex hull* of a set H is the intersection of all discs of radius r containing H. The spindle convex hull of finitely many points is called a *disc-polygon*, which is the intersection of finitely many discs. The notion of spindle convexity, including it's higher dimensional form, was studied in [3], see also [8]. With the use of this form of convexity, one can define a similar uniform model of random disc-polygons in a spindle convex body K, and investigate its asymptotic properties. Such results on the expected number of vertices of the random disc-polygon K_n, the expectation of the missed area $\mathrm{Area}(K \setminus K_n)$, and the perimeter deviation $\mathrm{Per}(K) - \mathrm{Per}(K_n)$ were given in [7], in two different settings: either choosing points from an r_0 spindle convex disc with some value $0 < r_0 < r$, or from a circular disc of radius r. In the former, the expected number of vertices has order of magnitude $n^{1/3}$, similarly to the linear convex case, but in the latter, the expectation tends to a finite constant, which has no analogue in linear convexity. In [5], Fodor generalised these results to higher dimensions, and showed a similar distinction between the two types of models. Recently, results have also been obtained for the series expansion [9] and variance [6, 11] of the expected number of vertices, as well as a central limit theorem in [10].

In a recent paper by Marynych and Molchanov [15], it was shown that the f-vector of a uniform random r ball-polytope in a ball of radius r converges in distribution, and in power moments of all orders, to the f-vector of the polar body of the zero cell of a Poisson hyperplane tessalation induced by the unit ball, or equivalently, the convex hull of a Poisson point process induced by the unit ball (see [15, Corollary 6.7]). Specifically, as the expected number of facets of the convex hull of the above point process is known, it follows that the expected number of

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facets of a uniform random ball-polytope tends to a finite number, namely $2^{-d}d!\kappa_d^2$, where κ_d denotes the volume of the unit ball of $\mathbb{R}^d$. This result is a special case of a much broader theorem, which is set in a more general notion of strong convexity with respect to an arbitrary convex body.

Another approach to random polytopes was introduced by Bárány, Hug, Reitzner and Schneider in their 2017 paper [1]. They considered the spherical counterpart of the problems discussed previously: the n random points are chosen from a half-sphere, and a random spherical polytope is obtained by taking their spherical convex hull i.e. the intersection of all half-spheres containing them. It was shown by the authors that the expected number of facets tends to a constant (dependent on the dimension d) as $n \to \infty$, and this constant coincides with the limit obtained in the corresponding dimensional Euclidean spindle convex case.

In [2], Besau et al. considered the following related problem: a uniform random spherical polytope is constructed in a so-called wedge, which in the intersection of two half-spheres whose defining hyperplanes are orthogonal. They obtained that the number of facets in this construction is of order $\log n$.

In [16], the authors introduced the notion of *spherical spindle-convexity* (see also [4]) as an extension of the previous two models. Namely, for some set X in S^2 contained in a spherical disc of radius $0 < r \leq \pi/2$, its (closed) *spherical r spindle convex hull*, denoted by $\operatorname{conv}_r(X)$, is the intersection of all spherical circular discs of radius r containing X, and X is spherically r spindle convex if it coincides with $\operatorname{conv}_r(X)$. Similarly to the planar case, for a finite point set X, $\operatorname{conv}_r(X)$ is called a spherical disc–polygon, which is the intersection of finitely many spherical discs. We note that as a special case, $r = \pi/2$ gives us half-spheres and the usual notion of spherical convexity. They determined the following result: if K is a spherically convex disc with C^2 boundary and with the property that $\kappa_g(s) > \cot r$ for every $x \in \partial K$, where κ_g denotes the geodesic curvature, and K_n is a uniform spherical disc–polygon in K, then

$$\lim_{n\to\infty} \mathbb{E}(\operatorname{SArea}(K \setminus K_n))n^{\frac{2}{3}} = \sqrt[3]{\frac{2A^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int\limits_{\partial K} \left(\kappa_g - \cot r\right)^{\frac{1}{3}} \mathrm{d}s, \tag{1}$$

where A denotes the surface area $\operatorname{SArea}(K)$, and $\Gamma(\cdot)$ is the Gamma function, and the integration is taken with respect to arc length. In the case when K is a sphericaul circular disc of radius r, the expectation tends to the constant $\pi^2/2$ without normalisation.

In this work, we first determine a similar result concerning the perimeter deviation in the above model.

Theorem 1. *Let K be a spherical convex disc with C^5 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, where $\kappa_g = \kappa_g(s)$ denotes the geodesic curvature. Then*

$$\lim_{n\to\infty} \mathbb{E}\big((\operatorname{Per}(K) - \operatorname{Per}(K_n)\big)n^{\frac{2}{3}} = \sqrt[3]{\frac{2A^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int\limits_{\partial K} \left(\kappa_g - \cot r\right)^{\frac{1}{3}} \left(\frac{3\kappa_g + \cot r}{4}\right) \mathrm{d}s,$$

where A denotes $\operatorname{SArea}(K)$.

Similarly to case of the surface area, if the points are chosen from a spherical circular disc of radius r, we obtain a different order of magnitude.

Theorem 2. *Let K be a spherical circular disc of radius r. Then*

$$\mathbb{E}(\mathrm{Per}(K) - \mathrm{Per}(K_n)) = n^{-1} \cdot \pi^3(1-\cos r)\cot r + \\ + n^{-2} \cdot 2\pi^3 \frac{(1-\cos r)^2}{\sin r}(1-\cot^2 r) + o(n^{-2}) \qquad \text{as } n \to \infty.$$

Note that this means that for $0 < r < \pi/2$, the expectation of the perimeter deviation is of order of magnitude n^{-1}. However, when $r = \pi/2$, i.e. we are in standard spherical convexity, the coefficient of n^{-1} is 0, hence the order of magnitude is actually n^{-2}.

In addition to the results on perimeter deviation, we also introduce a notion of duality, which is the spherical counterpart of the planar spindle convex duality defined in [8] (see also [3]). This provides a natural way to define a model for circumscribed disc-polygons. For a spherically spindle convex disc K, let $X_n = \{P_1, \ldots, P_n\}$ be an independently chosen set of points from K^r according to the uniform probability distribution. Then a random circumscribed disc-polygon $K_{(n)}$ can be defined as the dual of $\mathrm{conv}_r(X_n)$. This is the spherical analogue of the model defined in [11, Section 5]. In this model of random circumscribed spherical disc-polygons, we obtain the following results.

Theorem 3. *Assume that K has C^5 boundary, and $\kappa_g > \cot r$. With the notation above,*

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{SArea}(K_{(n)} \setminus K))n^{2/3} = \Gamma\left(\frac{5}{3}\right)(\sin r)^{-2/3} \cdot \\ \cdot \sqrt[3]{\frac{2((1-\cos r)\cdot 2\pi - \sin r \cdot L + \cos r \cdot A)^2}{3}} \cdot \frac{3}{4}\int_{\partial K} \Delta\kappa_g^{-1/3}\mathrm{d}s$$

and

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{Per}(K_{(n)}) - \mathrm{Per}(K))n^{2/3} = \Gamma\left(\frac{5}{3}\right)(\sin r)^{-2/3} \cdot \\ \cdot \sqrt[3]{\frac{2((1-\cos r)\cdot 2\pi - \sin r \cdot L + \cos r \cdot A)^2}{3}} \cdot \int_{\partial K} \Delta\kappa_g^{-1/3}\left(\kappa_g - \frac{\cot r}{4}\right)\mathrm{d}s,$$

where L and A denote $\mathrm{Per}(K)$ *and* $\mathrm{SArea}(K)$*, respectively.*

The paper is structured as follows. In Section 2, we state and prove some geometric lemmas necessary for the proof of Theorems 1 and 2, which are discussed in Section 3. In Section 4, we define the spherical spindle convex dual of a set, and consider the relationship between a set and its dual. Finally, in Section 5, we show the assertions of Theorem 3.

2. Geometric tools

In the section, we introduce some terminology, and prove some lemmas used in the proof of Theorems 1 and 2.

By the distance of two points, we will generally mean their spherical distance. We will also use the usual o and O notation, and write $f \sim g$ as $x \to x_0$ when $f = f(x)$ and $g = g(x)$ are asymptotically equal, that is, $f(x)/g(x) \to 1$ as $x \to x_0$.

A subset D of a spherical spindle convex disc K is called an r *spherical disc cap* if $D = \mathrm{cl}\,(K \cap B^c)$ for some spherical disc B of radius r, where $\mathrm{cl}\,(\cdot)$ and $\cdot^c$ denote the closure and complement of a set, respectively.

From now on, we fix the radius r of the spindle convexity, and will generally omit it from the notation.

Lemma 1 ([16], Lemma 2)**.** *Let $D = \mathrm{cl}\,(K \cap B^c)$ be a nonempty spherical disc-cap. Then there is a unique point x_0 of ∂K and a nonnegative real number t such that the spherical center of B is of distance $r + t$ from x_0 along the great circle determined by the normal plane at x_0.*

Lemma 2. *Let K be a spherical spindle convex disc with C^5 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, and let ϱ be an arc-length parametrisation of ∂K. We denote by $A(t, s_0)$ the area of the spherical disc-cap with vertex $\varrho(s_0)$ and height t, and by $\Delta\theta(t, s_0)$ the central angle of the circular arc that determines this disc-cap. Then*

$$\Delta\theta(t, s_0) = \vartheta_1 t^{1/2} + \vartheta_2 t^{3/2} + O(t^2) \quad \text{and} \quad A(t, s_0) = v_1 t^{3/2} + v_2 t^{5/2} + O(t^{7/2}),$$

where the coefficients are dependent on s_0, namely

$$\vartheta_1 = \frac{2}{\sin r} \cdot \sqrt{\frac{2}{\Delta\kappa_g}}, \qquad v_1 = \frac{4}{3} \cdot \sqrt{\frac{2}{\Delta\kappa_g}},$$

$$\vartheta_2 = \sqrt{\frac{2}{\Delta\kappa_g}} \frac{-\frac{1}{2}\Delta\kappa_g^4 - \frac{3}{2}\cot r \cdot \Delta\kappa_g^3 - \frac{5}{6}(1 + \cot^2 r)\Delta\kappa_g^2 - \frac{1}{6}\kappa_g''\Delta\kappa_g + \frac{5}{18}(\kappa_g')^2}{\sin r \cdot \Delta\kappa_g^3},$$

and

$$v_2 = \sqrt{\frac{2}{\Delta\kappa_g}} \frac{-\frac{1}{5}\Delta\kappa_g^4 - \frac{3}{5}\cot r \cdot \Delta\kappa_g^3 - \frac{3}{5}(1 + \cot^2 r)\Delta\kappa_g^2 - \frac{1}{15}\kappa_g''\Delta\kappa_g + \frac{1}{9}(\kappa_g')^2}{\Delta\kappa_g^3},$$

where $\Delta\kappa_g = \Delta\kappa_g(s_0) = \kappa_g(s_0) - \cot r$.

Proof. We may assume without loss of generality that ϱ is a parametrisation with positive orientation, $s_0 = 0$, $\varrho(0) = (0, 0, 1)$ and $\dot\varrho(0) = (1, 0, 0)$, hence $\varrho(0) \times \dot\varrho(0) = (0, 1, 0)$. It is straightforward to check geometrically that $\ddot\varrho = \kappa_g \cdot \varrho \times \dot\varrho - \varrho$, hence we can express the first four derivatives of ϱ in terms of the orthonormal basis $(\dot\varrho, \varrho \times \dot\varrho, \varrho)$, and by the smoothness condition on the boundary, we obtain the fourth-order series expression for $\varrho(s)$ around $\varrho(0)$:

$$\begin{aligned} \varrho(s) = \Big(&s - \frac{s^3}{6}(\kappa_g^2 + 1) - \frac{s^4}{8}\kappa_g\kappa_g' + O(s^5), \\ &\frac{s^2}{2}\kappa_g + \frac{s^3}{6}\kappa_g' + \frac{s^4}{24}(\kappa_g'' - \kappa_g - \kappa_g^3) + O(s^5), \\ &1 - \frac{s^2}{2} + \frac{s^4}{24}(\kappa_g^2 + 1) + O(s^5)\Big). \end{aligned} \tag{2}$$

Now, let $C = C(t)$ be the spherical disc that determines the spherical disc-cap with vertex $\varrho(0)$ and height t. Then clearly by Lemma 1, C has center $(0, \sin(r + t), \cos(r + t))$, and hence an othonormal basis in the plane containing ∂C is given by the vectors $(1, 0, 0)$ and $(0, -\cos(r + t), \sin(r + t))$, thus ∂C can be parametrised as

$$\Big(\sin r \cos\varphi,\ \cos r \sin(r+t) + \sin r \cos(r+t)\sin\varphi,$$
$$\cos r\cos(r+t) - \sin r \sin(r+t)\sin\varphi\Big), \qquad \varphi \in [0, 2\pi], \quad (3)$$

since the Euclidean radius of the ∂C is $\sin r$, and its center is of distance $\cos r$ from the origin. Note that the closest point of C to $\varrho(0)$ is at the parameter value $\varphi = 3\pi/2$.

Our first aim is to express the intersection points x_+ and x_- of ∂K and ∂C in terms of t.

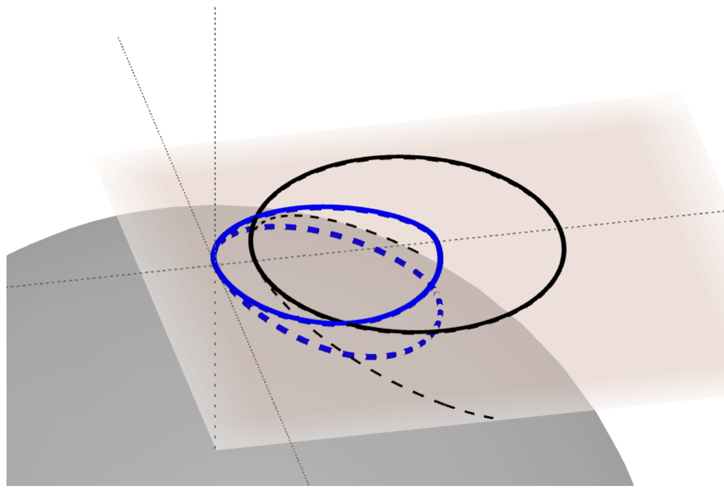

FIGURE 1. Projection of a disc-cap to the tangent plane

Consider the orthogonal projection of the construction to the tangent plane $z = 1$ to the sphere at $\varrho(0)$, or equivalently, the xy plane. It is straightforward to check from the above parametrisations that the projection of ∂K is described by the function

$$K(x) := \frac{x^2}{2}\kappa_g + \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}(\kappa_g'' + 3\kappa_g + 3\kappa_g^3) + f(x),$$

where $f(x) = O(x^5)$, and the necessary arc of the projection of ∂C by

$$C(x) := \cos r \sin(r+t) - \cos(r+t)\sqrt{\sin^2 r - x^2},$$

for arbitrary t when $r = \pi/2$, and for $t < \pi/2 - r$ when $r < \pi/2$.

Thus equating $K(x) = C(x)$ gives us an implicit relationship between t and x at the points of intersection. Using implicit differentiation (see [14]), we can derive the fourth-order Taylor expansion of t around 0, and obtain

$$t = \frac{x^2}{2}(\kappa_g - \cot r) + \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}(\kappa_g'' + 3\kappa_g^3 + 9\kappa_g - 3\cot^3 r - 9\cot^3 r) + O(x^5).$$

Note that since ∂K is of class C^5, the function $K(x)$, and by consequence the remainder term $f(x)$ is four times differentiable. If a function is $O(x^n)$ and is differentiable at 0, its derivative is $O(x^{n-1})$ around zero, which in our case implies that the first four derivatives of $f(x)$ equal 0 at $x = 0$. Since the implicit differentiation used above requires at most the fourth derivative of $K(x)$ at point $x = 0$, the term $f(x)$ does not affect the results.

By introducing the notation $\Delta\kappa_g = \kappa_g - \cot r$, the above formula can be reformulated as

$$t = \frac{x^2}{2}\Delta\kappa_g + \frac{x^3}{6}\kappa_g' + \frac{x^4}{24}(\kappa_g'' + 3\Delta\kappa_g^3 + 9\cot r\Delta\kappa_g^2 + 9(1+\cot^2 r)\Delta\kappa_g) + O(x^5).$$

We note that while this theoretically could be computed by hand, it is an extremely long calculation, which we used a computer for, see [17].

By [9] Lemma 2 (see also [13] Lemma 2 in a more general form), the inverse of the above function is of the form

$$x_+ = C_1 t^{1/2} + C_2 t + C_3 t^{3/2} + O(t^2) \quad \text{and} \quad x_- = -C_1 t^{1/2} + C_2 t - C_3 t^{3/2} + O(t^2),$$

where

$$C_1 = \sqrt{\frac{2}{\Delta\kappa_g}}, \quad C_2 = \frac{-\kappa_g'}{3\Delta\kappa_g^2}, \quad \text{and}$$

$$C_3 = \sqrt{\frac{2}{\Delta\kappa_g}}\frac{-\frac{1}{4}\Delta\kappa_g^4 - \frac{3}{4}\cot r\cdot\Delta\kappa_g^3 - \frac{3}{4}(1+\cot^2 r)\Delta\kappa_g^2 - \frac{1}{12}\kappa_g''\Delta\kappa_g + \frac{5}{36}(\kappa_g')^2}{\Delta\kappa_g^3}.$$

Now, by (3) we have $\sin r\cos\varphi = x$, hence for the point x_+ of intersection, the parameter value θ_+ can be expressed as

$$\begin{aligned}\theta_+ = \frac{3\pi}{2} + \frac{x_+}{\sin r} + \frac{x_+^3}{6\sin^3 r} + O(x_+^5) = \\ = \frac{3\pi}{2} + \frac{C_1}{\sin r}t^{1/2} + \frac{C_2}{\sin r}t + \left(\frac{C_3}{\sin r} + \frac{C_1^3}{6\sin^3 r}\right)t^{3/2} + O(t^2).\end{aligned}$$

Using a similar argument for θ_-, we obtain that

$$\Delta\theta = \theta_+ - \theta_- = \frac{2C_1}{\sin r}t^{1/2} + \frac{2}{\sin r}\left(C_3 + \frac{C_1^3}{6\sin^2 r}\right)t^{3/2} + O(t^2),$$

which yields the statement of the lemma after some algebraic manipulations.

The surface area can be computed via the integral

$$\int_{x_-}^{x_+}\int_{K(x)}^{C(x)} \frac{1}{\sqrt{1-x^2-y^2}}\mathrm{d}y\mathrm{d}x = \int_{x_-}^{x_+}\int_{K(x)}^{C(x)} 1 + \frac{x^2}{2} + \frac{3x^4}{8} + h(x) + \frac{y^2}{2} + g(x)\mathrm{d}y\mathrm{d}x,$$

where $f(x) = O(x^6)$ and $g(y) = O(y^4)$. We will also use an expansion of $C(x)$, namely

$$C(x) = t + O(t^3) + \frac{x^2}{2}\big(\cot r - t + O(t^2)\big) + \frac{x^4}{8}\big(\cot r + O(t)\big) + O(x^6).$$

Clearly, $K(x)$ and $C(x)$ are positive for all values of $x \in [x_-, x_+]$ by the construcion. For small enough t, and some $M > 0$, it also holds that

$$\begin{aligned} &x_+ < \ \ Mt^{1/2}, && C(x) < Mt, && |h(x)| < Mx^6, \\ &x_- > -Mt^{1/2}, && K(x) < Mx^2, && |g(y)| \le My^4 \end{aligned}$$

for the values of x and y that are in the examined interval for that value of t. Then

$$\left|\int_{x_-}^{x_+}\int_{K(x)}^{C(x)} h(x)\mathrm{d}y\mathrm{d}x\right| < \int_{x_-}^{x_+}\int_{0}^{C(x)} |h(x)|\mathrm{d}y\mathrm{d}x <$$

$$< \int_{x_-}^{x_+} Mt \cdot Mx^6 \mathrm{d}y\mathrm{d}x = \frac{M^2}{7} \cdot t\left(x_+^7 - x_-^7\right)\mathrm{d}x < \frac{2M^2}{7} t^{9/2}.$$

Similarly,

$$\left| \int_{x_-}^{x_+} \int_{K(x)}^{C(x)} g(y)\mathrm{d}y\mathrm{d}x \right| < \int_{x_-}^{x_+} \int_{0}^{C(x)} |g(y)|\mathrm{d}y\mathrm{d}x <$$

$$< \frac{M}{5} \int_{x_-}^{x_+} C^5(x)\mathrm{d}y\mathrm{d}x < \frac{M^6}{5} t^5 (x_+ - x_-) < \frac{2M^6}{5} \cdot t^{11/2}.$$

Hence the integrals of $f(x)$ and $g(y)$ carry no significant terms.

After integrating with respect to y, we obtain that the surface area is equal to

$$\int_{x_-}^{x_+} t + O(t^3) - \frac{x^2}{2}\left(\Delta\kappa_g + O(t^2)\right) - \frac{x^3}{6}\kappa_g' +$$
$$+ \frac{x^4}{24}\left(\kappa_g'' + 3\Delta\kappa_g^3 + 9\cot r \Delta\kappa_g^2 + 9\Delta\kappa_g(1+\cot^2 r) + O(t^2)\right)\mathrm{d}x + O(t^{7/2}).$$

Using a similar argument as before, the integral of $O(t^3)$ and $x^2 O(t^2)$ is $O(t^{7/2})$, and the integral of $x^4 O(t^2)$ is $O(t^{9/2})$, thus the terms not explicitly determined carry no significant terms. The remaining terms are all necessary, and computing the integral of the polynomial yields the desired result after a long but straightforward calculation. □

For the proof of Theorem 2, we need a similar geometric result for when K is a spherical circular disc of radius r. In that case, a disc-cap of height t is the complement of the intersection of two circular discs of radius r whose spherical centers are of distance t, or equivalently their set theoretical difference. Clearly, the surface area and central angle do not depend on the vertex of the disc-cap, hence we reduce the above notation to $A(t)$ for the surface area and $\Delta\theta(t)$ for the central angle. By Lemma 6 of [16], the corresponding central angle is

$$\Delta\theta(t) = 2\arccos\left(\frac{\sin t}{1+\cos t} \cdot \cot r\right).$$

For the area, we need an improvement on Lemma 5 of [16].

Lemma 3. *Let $A(t)$ denote the surface area of the set theoretical difference of two spherical circular discs of radius r whose distance is t. Then*

$$A(t) = 2 \cdot \left(2\arcsin\left(\frac{\sin(t/2)}{\sin r}\right) + 2\cos r \arccos\left(\frac{\tan(t/2)}{\tan r}\right) - \pi\cos r\right).$$

Moreover, the third-order expansion of $A(t)$ around $t = 0$ is

$$A(t) = 2\sin r \cdot t - \frac{1}{12}\cos r \cot r \cdot t^3 + O(t^5).$$

Proof. We determine the surface area of the intersection of two such discs, which yields the statement when substracted from $\mathrm{SArea}(K) = 2\pi(1-\cos r)$.

Let A be the center of one of the discs, B and C the two intersection points of their boundaries, and D the midpoint of the segment connecting B and C, which coincides with the midpoint of the segment connecting the two centers, see Figure 2. By the construction, $d(A,B)=r$, $d(A,D)=t/2$, and the angle ADB is $\pi/2$. Let β and α denote the angles DBA abd BAC, respectively, and let $a=d(B,D)$. We use spherical trigonometry in the triangle ABD. By the spherical sine theorem, $\sin\beta = \sin(t/2)/\sin r$. By the spherical cosine theorem used for the side AB, $\cos r = \cos a \cos(t/2)$, and for the side BD

$$\cos(\alpha/2) = \frac{\cos a - \cos r \cos(t/2)}{\sin r \sin(t/2)} = \frac{\tan(t/2)}{\tan r}.$$

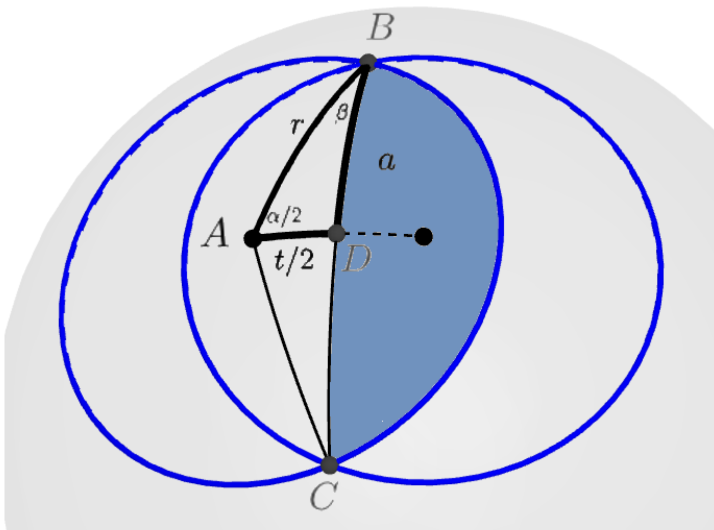

FIGURE 2. The surface area of the intersection of two spherical discs.

The surface area of half of the intersection is then obtained by subtracting the area of the triangle from the area of the circular sector:

$$\alpha \cdot (1-\cos r) - (\alpha + \beta + \pi/2 - \pi),$$

which yields the assertions after substituting in the above formulas obtained for α and β. □

3. PROOF OF THEOREMS 1 AND 2

In the proof of Theorem 1, we closely follow the proof of the corresponding planar result in [7, Theorem 1.2], whose argument is based on the technique used in [18] by Rényi and Sulanke.

Let P_1 and P_2 be two disctinct points in K. There are exactly two spherical discs or radius r that contain both points in their boundaries. Let $D^-(P_1,P_2)$ and $D^+(P_1,P_2)$ denote the spherical disc-caps determined by these discs, i.e. the closure of the subset of K that is not covered by the corresponding disc. We also introduce the notation $A^-(P_1,P_2) = A(D^-(P_1,P_2))$ and $A^+(P_1,P_2) = A(D^+(P_1,P_2))$. Furthermore, let $i(P_1,P_2)$ denote the length of the shorter r-arc connecting P_1 and P_2.

The random pair of points $P_1, P_2 \in K$ determines an edge of $\operatorname{conv}_r(K_n)$ if and only if $D^-(P_1,P_2)$ or $D^+(P_1,P_2)$ don't contain more points from K_n. Hence our goal is to determine $\mathbb{E}\left(\operatorname{Per}(K)-\operatorname{Per}(K_n)\right)=\operatorname{Per}(K)-\binom{n}{2}\cdot I_n$, where

$$I_n = \mathbb{E}\big(\mathbf{1}(P_1,P_2 \text{ is an edge of } K_n)\cdot i(P_1,P_2)\big) =$$
$$= \frac{1}{A^2}\int\limits_K\int\limits_K \left(\left(1-\frac{A^-(P_1,P_2)}{A}\right)^{n-2}+\left(1-\frac{A^+(P_1,P_2)}{A}\right)^{n-2}\right) i(P_1,P_2)\mathrm{d}P_1\mathrm{d}P_2.$$

To compute this integral, we carry out the transformation described in [16] Section 3, which can be described as follows. For every pair of distinct points $P_1, P_2 \in K$, there are exactly two disc-caps where both points are contained in the boundary of the defining spherical disc. Each of these is uniquely determined by its vertex $\varrho(s)$ and height t by Lemma 1. After fixing a circle on which the two points lie, we can define an orthonormal basis in the plane the circle's contained in, in which the points can each be described by a single parameter, say φ_1 and φ_2. In general, this transformation is twofold, but because of the properties of the specific integrand here, every cap gets counted exactly twice.

By Lemma 3 in [16], and the fact that in the new parametrisation we have $i(P_1,P_2)=\sin r|\varphi_1-\varphi_2|$, carrying out the above transformation yields

$$I_n = \frac{1}{A^2}\int_{\partial K}\int_0^{t(s)}\int_{L(s,t)}\int_{L(s,t)}\left(1-\frac{A(t,s)}{A}\right)^{n-2}\cdot\sin r\cdot|\varphi_1-\varphi_2|\cdot$$
$$\cdot\sin^2 r(\sin(r+t)\kappa_g(s)-\cos(r+t))|\sin(\varphi_1-\varphi_2)|\mathrm{d}\varphi_1\mathrm{d}\varphi_1\mathrm{d}t\mathrm{d}s.$$

After integrating with respect to φ_1 and φ_2, and we obtain

$$I_n = \frac{\sin^3 r}{A^2}\int_{\partial K}\int_0^{t(s)}\left(1-\frac{A(t,s)}{A}\right)^{n-2}(\sin(r+t)\kappa_g(s)-\cos(r+t))\cdot$$
$$\cdot 2(2-\Delta\theta\sin\Delta\theta-2\cos\Delta\theta)\ \mathrm{d}t\mathrm{d}s$$

Fix $\varepsilon>0$. There exists some $t'>0$ such that for all $0<t<t'$ the following conditions hold:

i) $\left|\sin(r+t)\kappa_g(s)-\cos(r+t)-\sin r\cdot\left(\Delta\kappa_g(s)+(1+\cot^2 r+\cot r\kappa_g(s))t\right)\right|<\varepsilon t$

by the first order series expansion of the trigonometric functions;

ii) $\left|A(t,s)-(v_1t^{3/2}+v_2t^{5/2})\right|<\varepsilon t^{5/2}$ using the expansion of $A(t,s)$ in Lemma 2;

iii) $\left|2(2-\Delta\theta\sin\Delta\theta-2\cos\Delta\theta)-\frac{1}{6}\cdot\left(\vartheta_1^4t^2+\left(4\vartheta_1^3\vartheta_2-\frac{\vartheta_1^6}{15}\right)t^3\right)\right|<\frac{\varepsilon}{6}t^3$

using the expansion of $\Delta\theta$ in Lemma 2, as well as the sixth-order Taylor polynomial $x^4/6-x^6/90+o(x^6)$ of $2(2-x\sin x-2\cos x)$, and

iv) $\left|(1+\cot^2 r+\cot r\kappa_g(s)\pm\varepsilon)\left(4\vartheta_1^3\vartheta_2-\frac{\vartheta_1^6}{15}\pm\varepsilon\right)\right|t^4<\varepsilon t^3$

by the boundedness of the expression in the absolute value on the left hand side.

Using essentially the same analytical tools as in [7] (see (5.17) and the preceding formula), we obtain that

$$\lim_{n\to\infty}\mathbb{E}(\mathrm{Per}(K)-\mathrm{Per}(K_n))n^{2/3}=\lim_{n\to\infty}\left(\mathrm{Per}(K)-\binom{n}{2}\int_{\partial K}\hat{I}_n(s)\mathrm{d}s\right)n^{2/3},$$

where

$$\hat{I}_n(s):=\frac{\sin^3 r}{A^2}\int_0^{t'}\left(1-\frac{A(t,s)}{A}\right)^n\big(\sin(r+t)\kappa_g-\cos(r+t)\big)\cdot$$
$$\cdot\,2(2-\Delta\theta\sin\Delta\theta-2\cos\Delta\theta)\ \mathrm{d}t.$$

By the choice of t',

$$\hat{I}_n(s)\leq\frac{\sin^4 r}{6A^2}\int_0^{t'}\left(1-\frac{A(t,s)}{A}\right)^n\big(\Delta\kappa_g(s)+(1+\cot^2 r+\cot r\kappa_g(s)+\varepsilon)t\big)\cdot$$
$$\cdot\left(\vartheta_1^4t^2+\left(4\vartheta_1^3\vartheta_2-\frac{\vartheta_1^6}{15}+\varepsilon\right)t^3\right)\ \mathrm{d}t$$
$$\leq\frac{\sin^4 r}{6A^2}\int_0^{t'}\left(1-\frac{v_1}{A}t^{3/2}-\frac{v_2-\varepsilon}{A}t^{5/2}\right)^n\cdot\big(D_1t^2+D_1D_2^{\varepsilon}t^3\big)\,\mathrm{d}t,$$

where $D_1:=\Delta\kappa_g\vartheta_1^4$ and

$$D_1D_2^{\varepsilon}=\vartheta_1^4(1+\cot^2 r+\cot r\Delta\kappa_g+\varepsilon)+\Delta\kappa_g\cdot\left(4\vartheta_1^3\vartheta_2-\frac{\vartheta_1^6}{15}+\varepsilon\right)+\varepsilon.$$

By the same analytical computation as in the proof of Theorem 1.2 of [7], using the tools (5.13)-(5.16) therein, we obtain that

$$\hat{I}_n(s)\leq\frac{\sin^4 rD_1}{9n^2v_1^2}\left[1+n^{-2/3}A^{2/3}\left(\frac{D_2^{\varepsilon}}{v_1^{2/3}}\Gamma\left(\frac{8}{3}\right)-\frac{v_2-\varepsilon}{v_1^{5/3}}\Gamma\left(\frac{11}{3}\right)+2\varepsilon\right)\right].$$

We can obtain a similar lower bound, and as ε was arbitrary, $\hat{I}_n(s)$ is equal to the formula on the right hand side with $\varepsilon=0$.

If we write in the formulas for D_1 and v_1, we see that $\sin^4 rD_1/(18v_1^2)=1$, and since $\int_{\partial K}1\mathrm{d}s=\mathrm{Per}(K)$, in the limit the perimeter cancels out, hence

$$\lim_{n\to\infty}\mathbb{E}(\mathrm{Per}(K)-\mathrm{Per}(K_n))n^{2/3}=\lim_{n\to\infty}\left(\mathrm{Per}(K)-\binom{n}{2}\int_{\partial K}\hat{I}_n(s)\mathrm{d}s\right)n^{2/3}=$$
$$=\int_{\partial K}\frac{\sin^4 rD_1A^{2/3}}{18v_1^2}\left(\frac{v_2}{v_1^{5/3}}\Gamma\left(\frac{11}{3}\right)-\frac{D_2^0}{v_1^{2/3}}\Gamma\left(\frac{8}{3}\right)\right)\mathrm{d}s.$$

After expanding the formulas, we obtain that the limit of the expectation is finally equal to

$$\sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\frac{1}{4}\int_{\partial K}\Delta\kappa_g^{1/3}(3\Delta\kappa_g+4\cot r)-\frac{5(\kappa_g')^2-3\kappa_g''\Delta\kappa_g}{3\Delta\kappa_g^{8/3}}\mathrm{d}s=$$
$$=\sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\frac{1}{4}\int_{\partial K}\Delta\kappa_g^{1/3}(3\kappa_g+\cot r)-\frac{5(\kappa_g')^2-3\kappa_g''\Delta\kappa_g}{3\Delta\kappa_g^{8/3}}\mathrm{d}s.$$

By an elementary calculation, the last term in the integral is exactly the derivative of the expression $\kappa_g'/\Delta\kappa_g^{5/3}$, hence its integral is zero. The remaining expression is exactly the statement of the theorem.

In the case when K is the circular disc of radius r, we follow similar steps. The integral transformation used is geometrically similar to the previous one, and its Jacobian is given by Lemma 1 of [16], hence we obtain

$$\lim_{n\to\infty} \mathbb{E}(\operatorname{Per}(K_n)) = \lim_{n\to\infty} \binom{n}{2} \frac{2\sin^3 r}{A^2} \times \\ \times \int_0^{t'} \left(1 - \frac{A(t)}{A}\right)^n \sin t \cdot (2 - \Delta\theta \sin\Delta\theta - 2\cos\Delta\theta)\mathrm{d}t,$$

where $A := \operatorname{SArea}(K) = 2\pi(1-\cos r)$, and the upper bound of the integral t' is some appropriately small fixed number. Using the appropriate series expansions, we obtain that the integral is of the form

$$\int_0^{t'} \left(1 - \frac{2\sin r}{A} \cdot t + \frac{\cot r \cos r}{12A} \cdot t^3 + O(t^5)\right)^n \times \\ \times \left(t - \frac{t^3}{6} + O(t^5)\right)\left(4 - \pi\cot r \cdot t + O(t^3)\right)\mathrm{d}t.$$

Using similar analytical tools as above, we obtain the statement of the theorem.

4. Duality

We define the spherical r-convex dual of the set H as

$$H^r = \{y \in S^2 \mid H \subseteq B(y,r)\} = \bigcap_{x\in H} B(x,r).$$

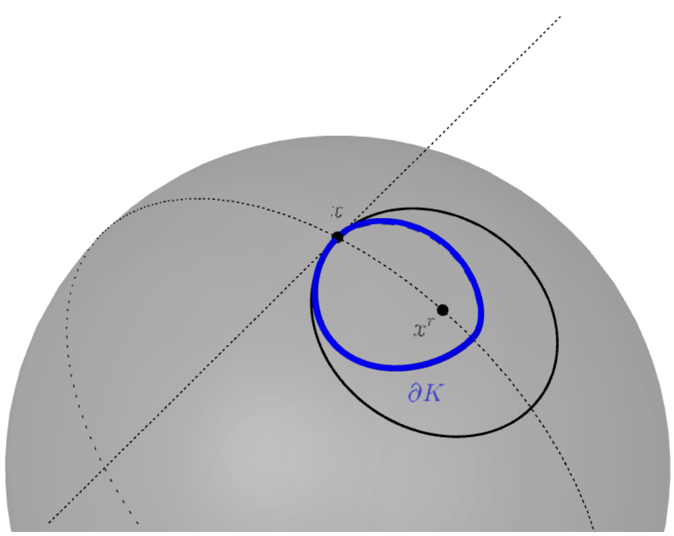

FIGURE 3. The spherical spindle convex dual

It is clear that if $H = K$ is an r spindle convex disc, $K = (K^r)^r$, justifying the name 'duality'. If K has C^1 boundary, then for any $x = \varrho(s) \in \partial K$, the smoothness guarantees that there is a unique point, denoted by $x^r := x^r(s)$, such that the spherical disc with center x^r and radius r *supports* K at x, i.e. $B(x^r, r)$ contains K, and their supporting great circles at x coincide (see Figure 3. By the definition of the dual set, $x^r \in K^r$, and more specifically it is a boundary point. It is also straightforward to check that $(x^r)^r = x$.

We note that in the special case of $r = \pi/2$, it is easy to see that $K^{\pi/2} = -K^\circ$, where $K^\circ = \{y \in S^2 \mid \langle y, x\rangle \leq 0 \ \forall x \in K\}$ is the spherical polar body.

Lemma 4. *Let $K \subseteq S^2$ be a spherically convex disc. Then for the r-spindle convex dual K^r of K we have*

$$(i)\ \operatorname{Per}(K^r) = \sin r \cdot 2\pi - \cos r \cdot \operatorname{Per}(K) - \sin r \cdot \operatorname{SArea}(K) \ \textit{and}$$
$$(ii)\ \operatorname{SArea}(K^r) = (1 - \cos r) \cdot 2\pi - \sin r \cdot \operatorname{Per}(K) + \cos r \cdot \operatorname{SArea}(K).$$

Furthermore, if K has C^2 boundary with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, the equality

$$(iii)\ \sin^2 r \cdot (\kappa_g^r(x^r) - \cot r)(\kappa_g(x) - \cot r) = 1$$

also holds.

Proof. First, we assume K is of class C^2 and $\kappa_g > \cot r$, and let $\varrho(s)$ be an arc-length parametrisation of ∂K with positive orientation. Then the center x^r of the support r-disc at $x = \varrho(s) \in \partial K$ can be explicitly determined using the parametrisation of ∂K, namely

$$x^r = \cos r \cdot \varrho(s) + \sin r \cdot \varrho(s) \times \dot{\varrho}(s).$$

Hence a parametrisation of ∂K^r is given by

$$\zeta(s) = \cos r \cdot \varrho(s) + \sin r \cdot \varrho(s) \times \dot{\varrho}(s), \qquad s \in [0, \operatorname{Per}(K)]. \tag{4}$$

Differentiating and using the relationship $\ddot{\varrho} = \kappa_g \cdot \varrho \times \dot{\varrho} - \varrho$, we obtain that

$$\dot{\zeta}(s) = -\dot{\varrho}(s) \cdot \sin r \cdot (\kappa_g(s) - \cot r).$$

Hence

$$\operatorname{Per}(K^r) = \int_0^{\operatorname{Per}(K)} |\dot{\zeta}(s)| \mathrm{d}s = \int_0^{\operatorname{Per}(K)} \sin r(\kappa_g(s) - \cot r)\mathrm{d}s =$$
$$= \sin r \int_0^{\operatorname{Per}(K)} \kappa_g(s)\mathrm{d}s - \cos r \cdot \operatorname{Per}(K),$$

and as the Gaussian curvature of S^2 is constant 1, by the Gauss-Bonnet theorem we have

$$\int_0^{\operatorname{Per}(K)} \kappa_g(s)\mathrm{d}s = 2\pi - \int_K 1 \mathrm{d}A = 2\pi - \operatorname{SArea}(K),$$

which yields the first part of the assertion. Since $(K^r)^r = K$, using the corresponding formula for $K := K^r$ and some algebraic manipulations, we obtain the second expression.

For the curvature, again a simple computation yields that

$$|\dot{\zeta}(s) \times \ddot{\zeta}(s)| = \sin^2 r(\kappa_g(s) - \cot r)^2 \kappa(s),$$

hence the curvature $\kappa^r(s)$ of ∂K^r at $x^r(s)$ is

$$\kappa^r(s) = \frac{|\dot{\zeta} \times \ddot{\zeta}|}{|\zeta|^3} = \frac{\kappa(s)}{\sin(r) \cdot (\kappa_g(s) - \cot r)}$$

and since $(x^r)^r = x$, we can express $\kappa(s)$ in terms of $\kappa^r(s)$ in a similar way, which yields the desired equality.

As the duality transformation is continuous, for general K we can obtain the results for surface area and perimeter by using standard approximating tools. □

Corollary 1. *For the spherical polar body $K^\circ = -K^{\pi/2}$ we have*

$$\mathrm{Per}(K^\circ) + \mathrm{SArea}(K) = 2\pi.$$

We note that if we consider the perimeter and surface area as special intrinsic volumes, the case for the polar body follows from combining Theorem 4.3.1 and Corollary 4.3.3 of [12] in dimension 2.

Lemma 5. *Let K be a spherical convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$. Then*

$$\int_{\partial K^r} F(x^r(s))\mathrm{d}s = \int_{\partial K} F(x(s)) \cdot \sin r \cdot (\kappa_g(s) - \cot r)\mathrm{d}s$$

for any integrable function F.

Proof. Let ϱ be an arc-length parametrisation of ∂K of positive orientation again. Using the parametrisation of ∂K^r described in (4), as well as the second-order series expansion of $\varrho(\Delta s)$ and $\varrho(\Delta s) \times \dot{\varrho}(\Delta s)$ around $\varrho(0)$ and $\varrho(0) \times \dot{\varrho}(0)$, respectively, we obtain that

$$\langle \zeta(0), \zeta(\Delta s)\rangle = 1 - \frac{\left(\Delta s \cdot \sin r(\kappa_g(s) - \cot r)\right)^2}{2} + o(\Delta s^2).$$

On the other hand,

$$\langle \zeta(0), \zeta(\Delta s)\rangle = \cos d = 1 - \frac{d^2}{2} + o(d^2),$$

where d denotes the spherical distance of $\zeta(0)$ and $\zeta(\Delta s)$, or equivalently, the angle between them as vectors. As it is well-known that d is asymptotically equal to the length of the arc of ∂K^r between $\zeta(0)$ and $\zeta(\Delta s)$, it is enough to show that d is asymptotically equal to $\Delta s \cdot \sin r(\kappa_g(0) - \cot r)$ as $\Delta s \to 0$. From the above formulas, we have that

$$\frac{(\Delta s \cdot \sin r(\kappa_g(0) - \cot r))^2}{d^2} = 1 - \frac{o(d^2)}{d^2} + \frac{o(\Delta s^2)}{d^2}. \tag{5}$$

It is clear geometrically that $\Delta s \to 0$ as $d \to 0$ and $d \to 0$ as $\Delta s \to 0$. Assume that for some sequence of the quotient $\Delta s_n \sin r(\kappa_g(0) - \cot r)/d_n \to c \in [0, \infty]$ as $n \to \infty$. Then the limit of the formulas in (5) exist, and we obtain that

$$c^2 = \lim_{n\to\infty} \frac{(\Delta s_n \cdot \sin r(\kappa_g(0) - \cot r))^2}{d_n^2} = \lim_{n\to\infty} 1 - \frac{o(d_n^2)}{d_n^2} + \frac{o(\Delta s_n^2)}{\Delta s_n^2} \cdot \frac{\Delta s_n^2}{d_n^2}.$$

If $c < \infty$, the rightmost limit is 1, hence $c^2 = 1$, but since all values are positive, we have $c = 1$. If $c = \infty$, we have that $d_n/\Delta s_n \sin r(\kappa_g(0) - \cot r) \to 0$, and as the formulas are symmetric in d_n and $\Delta s_n \sin r(\kappa_g(0) - \cot r)$, this leads to a contradiction by the previous point. □

5. A circumscribed model

In this section, we prove the asymptotic formulae stated in Theorem 3, regarding the expectation of the area- and perimeter deviation of a spherical spindle convex disc K and a circumscribed spherical disc-polygon $K_{(n)}$. Recall that $K_{(n)}$ is obtained by taking the spherical spindle convex dual of the spherical spindle convex hull of n uniformly distributed, independent points chosen from K^r.

By Lemma 4, we have

$$\begin{aligned}\operatorname{Per}(K_{(n)})-\operatorname{Per}(K)&=\cos r\cdot(\operatorname{Per}(K^r)-\operatorname{Per}(K^r_{(n)}))+\sin r\cdot\operatorname{SArea}(K^r\setminus K^r_{(n)}),\\ \operatorname{SArea}(K_{(n)}\setminus K)&=\sin r\cdot(\operatorname{Per}(K^r)-\operatorname{Per}(K^r_{(n)}))-\cos r\cdot\operatorname{SArea}(K^r\setminus K^r_{(n)}),\end{aligned}\tag{6}$$

and by definition, $K^r_{(n)}$ is a uniform disc-polygon in K^r.

To be concise, we use the notation $\Delta\kappa_g(s)=\kappa_g(s)-\cot r$ as previously, and $\Delta\kappa^r_g(s)=\kappa^r_g(s)-\cot r$ for the dual disc, as well as $A(\cdot)$ for $\operatorname{SArea}(\cdot)$. From (1), and using the integral transformation in Lemma 5,we obtain

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}(\operatorname{SArea}(K^r\setminus K^r_{(n)}))\cdot n^{2/3}&=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K^r}\left(\Delta\kappa^r_g(s)\right)^{1/3}\mathrm{d}s=\\ &=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\frac{1}{\sin^2 r\Delta\kappa_g}\right)^{1/3}\sin r\Delta\kappa_g\mathrm{d}s=\\ &=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)(\sin r)^{1/3}\int_{\partial K}\Delta\kappa_g^{2/3}\mathrm{d}s.\end{aligned}$$

Similarly, from Theorem 1 we have

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}(\operatorname{Per}(K^r)-\operatorname{Per}(K^r_{(n)}))\cdot n^{2/3}=\\ &=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K^r}(\Delta\kappa^r_g)^{1/3}\cdot\left(\frac{3}{4}\Delta\kappa^r_g+\cot r\right)\mathrm{d}s=\\ &=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\frac{1}{\sin^2 r\Delta\kappa_g}\right)^{1/3}\left(\frac{3}{4\sin^2 r\Delta\kappa_g}+\cot r\right)\sin r\Delta\kappa_g\mathrm{d}s=\\ &\qquad=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)(\sin r)^{1/3}\int_{\partial K}\Delta\kappa_g^{2/3}\left(\frac{3}{4\sin^2 r\Delta\kappa_g}+\cot r\right)\mathrm{d}s.\end{aligned}$$

Combining these results with (6), we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}(\operatorname{SArea}(K_n\setminus K))n^{\frac{2}{3}}&=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)\cdot\\ &\cdot(\sin r)^{1/3}\int\limits_{\partial K}\Delta\kappa_g^{2/3}\left(\sin r\left(\frac{3}{4\sin^2 r\Delta\kappa_g}+\cot r\right)-\cos r\right)\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}(\operatorname{Per}(K_n)-\operatorname{Per}(K))n^{2/3}&=\sqrt[3]{\frac{2A(K^r)^2}{3}}\Gamma\left(\frac{5}{3}\right)\cdot\\ &\cdot(\sin r)^{1/3}\int\limits_{\partial K}\Delta\kappa_g^{2/3}\left(\cos r\left(\frac{3}{4\sin^2 r\Delta\kappa_g}+\cot r\right)+\sin r\right)\mathrm{d}s,\end{aligned}$$

which yield the desired formulas after simplification of the integrand, and substituting $A(K^r)$ with $(1-\cos r)2\pi - \sin r\mathrm{Per}(K) + \cos A(K)$ from Lemma 4.

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