RANDOM SPHERICAL DISC-POLYGONS IN A SPHERICAL SPINDLE CONVEX DISC

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Abstract. In this paper, we consider the asymptotic behaviour of the expectation of the number of vertices of a uniform random spherical disc-polygon. This provides a connection between the corresponding results in spherical convexity, and in euclidean spindle-convexity, where the expectation tends to the same constant. We also extend the result to a more general case, where the random points generating the uniform random disc-polygon are chosen from spherical convex disc with smooth boundary.

1. Introduction

Following the prominent work of Rényi and Sulanke [13], the 2017 paper [3] of Bárány, Hug, Reitzner and Schneider examines the following stochastic model: let K denote a halfsphere of S^{d-1}, and let $X_n = \{x_1, \dots, x_n\}$ be a uniform, independent sample of n points from K. Let K_n denote the spherical convex hull of X_n, that is, the intersection of all halfspheres containing X_n. The main questions of this area are concerned with the properties of the obtained spherical polytope K_n asymptotically, as $n \to \infty$. Such properties include the number of vertices and facets of K_n, or the missed surface area $\mathrm{SArea}(K \setminus K_n)$. One of the results proven by the authors in [3] is that the number of facets tends to a finite number that is only dependent on the dimension of the sphere.

The spherical model defined above has a close relative in the Eucledian plane, in spindle convexity. If a subset $X \subseteq \mathbb{R}^2$ is contained in some disc of radius r, we define its *closed r spindle convex hull* or *r-convex hull* as the intersection of all discs of radius r containing X. For a finite subset X, we call its spindle convex hull a *disc–polygon*, which is the intersection of finitely many discs. Spindle convexity in the Euclidean space has been extensively studied, we refer the interested reader to [5] and to the recent book [11], and for more on the history of spindle convexity see [9].

In 2014, Fodor, Kevei and Vígh [9] proved that by taking n i.i.d. points from a disc of radius r, and considering their r-convex hull, the number of the convex hull's vertices (or equivalently, edges) tends to a finite number as $n \to \infty$. In 2020, Fodor [8] generalised these results to higher dimensions, showing that for random ball-polytope obtained in a similar way, the number of its facets tends to a constant as $n \to \infty$, and it is only dependent on the dimension.

A point of interest of these results is that the limiting constants obtained on S^{d-1} in [3], and the limiting constants in the spindle convex model of $\mathbb{R}^{d-1}$ [8,9] coincide. The goal of this paper is to find a connection between the two models that gives a better understanding of this phenomenon. We show the analogue of these results

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in a spherical spindle convex model on S^2, and point out the connection between the results above. We also obtain some general results that make the picture more complete.

We mention that our results fit in the broad topic of random polytopes that is very well-studied. For a survey on (linearly) convex random polytopes see for example [2]. The spindle convex results and how they relate to linear convexity were first investigated in [9], see also the recent paper [10].

2. New notions and results

By the distance of two points, we will generally mean their spherical distance. For $0 < r \leq \pi/2$, the spherical disc of radius r, or spherical cap in Euclidean terminology, is the set points of S^2 whose distance from its (spherical) center is not greater than r. (Note that we only consider spherically convex sets.) We define the distance of two discs of the same radius as the spherical distance of their (spherical) centers , which is a direct analogue of the Hausdorff distance. As a special case, $r = \pi/2$ gives us halfspheres.

Now, let $X \subseteq S^2$ be contained in some spherical disc of radius r. We denote by $\operatorname{conv}_r(X)$ the *closed spherical r spindle convex hull* of X, that is, the intersection of all discs of radius r containing X. We remark that the spindle convex hull can be defined as in the Euclidean space (see [5]), however for the sake of compactness we introduce only the closed version. We say that a closed, spherical convex set $K \subset S^2$ is a spherically r spindle convex disc if $K = \operatorname{conv}_r(K)$. Furthermore, if X is finite, we say that $\operatorname{conv}_r(X)$ is a *spherical disc–polygon*, which can be written as the intersection of finitely many discs. Because of this, the notion of edges and vertices can be defined naturally. A subset D of a spherical spindle convex disc K is called a r *spherical disc cap* if $D = \operatorname{cl}(K \cap B^c)$ for some spherical disc B of radius r, where $\operatorname{cl}(\cdot)$ and $\cdot^c$ denote the closure and complement, respectively.

From now on, we fix the radius r of the spindle convexity, and will generally omit it from the notation.

We note that the concept of spindle convexity on the sphere seems to be new, although very natural. In this paper we restricted ourselves to keep the introduction as short as possible, but spherical spindle convexity is surely an interesting topic in itself.

The stochastic model considered in the paper is the following: let K be a spherical spindle-convex disc, and take n i.i.d. points $X_n = \{P_1, \ldots, P_n\}$ from K. Let K_n denote the spherical r spindle convex hull of the sample. Furthermore, let $f_0(\cdot)$ denote the number of vertices of a spherical disc–polygon.

Our main results are the following.

Theorem 1. *Let K be a spherical circular disc of radius r. Then*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n)) = \frac{\pi^2}{2}$$

and

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{SArea}(K \setminus K_n)) \cdot n = \pi^3(1 - \cos r).$$

Theorem 2. *Let K be a spherical convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, where κ_g denotes the geodesic curvature.*

Then

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n)) \cdot n^{-1/3} = \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa_g(s) - \cot r\right)^{1/3}\mathrm{d}s$$

and

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{SArea}(K \setminus K_n)) \cdot n^{2/3} = \sqrt[3]{\frac{2A^2}{3}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}\left(\kappa_g(s) - \cot r\right)^{1/3}\mathrm{d}s,$$

where A denotes $\mathrm{SArea}(K)$.

Note that the results for number of vertices and the missed area are linked by the well-known Efron identity (see (4) and [2,12]).

From Theorem 1 with $r = \pi/2$, we obtain the corresponding result of [3] in dimension two. Also, as $r \to 0+$, one can see that formally we obtain the planar spindle convex results, hence our model gives a continuous connection between the models mentioned in the introduction. However we note that taking the limit $r \to 0+$ is only "intutively clear", we don't have a rigorous statement there.

3. Two integral transformations

In the proofs of Theorems 1 and 2, we need to compute integrals over pairs of points from K, where K is either a circular disc or a spherical convex disc with smooth boundary. In this chapter, we prove two integralgeometric lemmas that resemble to the Blaschke-Petkantschin formulas (see [12, Section 7.2.]).

First, consider a pair of points, $P_1, P_2 \in K$, where K is the circular disc with spherical center $(0, 0, 1)$ and radius r. Let $K_{t,\varphi}$ be the spherical disc of radius r whose center is $C = C(t, \varphi) = (\sin t \cos\varphi, \sin t \sin\varphi, \cos t)$, hence the distance between the center of $K_{t,\varphi}$ and the center of K is exactly t. Now, it is easy to see that the triad (C, w_1, w_2) forms an orthonormal basis of $\mathbb{R}^3$ with vectors $w_1 = (\sin\varphi, \cos\varphi, 0)$ and $w_2 = (\cos t \cos\varphi, \cos t \sin\varphi, -\sin t)$. Thus we can express the points as

$$P_i = \cos r \cdot C(t, \varphi) + \sin r \cos\varphi_i \cdot w_1 + \sin r \sin\varphi_i \cdot w_2,$$

which describes the pair of points as follows: if $K_{t,\varphi}$ is a disc containing both points in its boundary, then its boundary is the intersection of the unit sphere with the plane containing $\cos r C(t, \varphi)$ and having normal vector $C(t, \varphi)$. In this plane, w_1 and w_2 form an orthonormal basis, in which the coordinates of the corresponding point are determined by φ_1 and φ_2.

Note that for every pair of points in K, there are exactly two circular discs of radius r containing both points in their boundary, hence two pairs of (t, φ) for which $K_{t,\varphi}$ is as described. Hence every pair of points is considered exactly twice, and we have a twofold reparametrisation.

Lemma 1. *With the construction above, an integral over a pair of points (P_1, P_2) in K can be rewritten as*

$$\int_K\int_K F(P_1, P_2)\mathrm{d}P_1\mathrm{d}P_2 =$$
$$= \frac{\sin^2 r}{2}\int_0^{2\pi}\int_0^{2r}\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta_2} F \cdot \sin t \cdot |\sin(\varphi_1 - \varphi_2)|\mathrm{d}\varphi_1\mathrm{d}\varphi_1\mathrm{d}t\mathrm{d}\varphi,$$

where $[\theta_1, \theta_2]$ is the interval (depending on t and φ) parametrising $\partial K_{t,\varphi} \cap K$.

Proof. Let $P_1 = (x_1, y_1, \sqrt{1-x_1^2-y_1^2})$ and $P_2 = (x_2, y_2, \sqrt{1-x_2^2-y_2^2})$. Transforming the integral to the variables x_1, y_1, x_2, y_2 yields

$$\iint\limits_{\substack{(x_1,y_1)\\ \in B^2\cdot\sin r}} \iint\limits_{\substack{(x_1,y_1)\in\\ B^2\cdot\sin r}} F \cdot \frac{1}{\sqrt{1-x_1^2-y_1^2}} \cdot \frac{1}{\sqrt{1-x_2^2-y_2^2}} \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2. \tag{1}$$

Now, consider the reparametrisation $(x_1, y_1, x_2, y_2) = \Phi(\varphi, t, \varphi_1, \varphi_2)$ given by

$$\left(x_i, y_i, \sqrt{1-x_i^2-y_i^2}\right) = \cos r \cdot C + \sin r \cos\varphi_i \cdot w_1 + \sin r \sin\varphi_i \cdot w_2,$$

as described above.

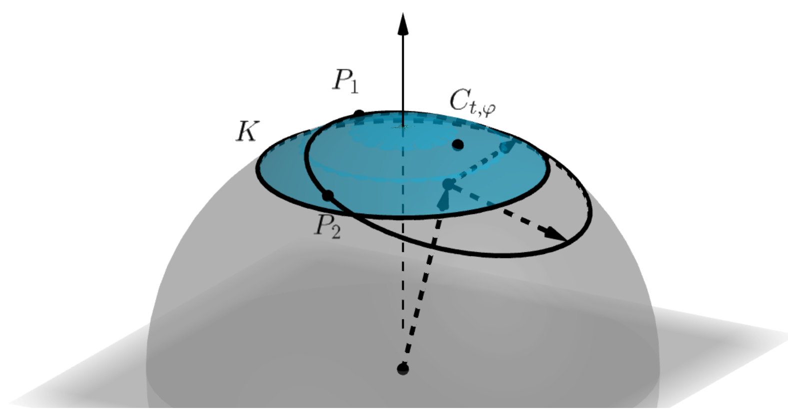

Figure 1. The reparametrisation.

The Jacobian of the transformation is

$$|J\Phi| = \Big| \sin^2 r \cdot \sin t \cdot \sin(\varphi_1 - \varphi_2) \times \\ \times (\cos r \cos t - \sin r \sin t \sin\varphi_1) \cdot (\cos r \cos t - \sin r \sin t \sin\varphi_2) \Big|,$$

whose computation can be read in Appendix A. The last two factors of the product are equal to the last coordinates of the corresponding P_i, that is, $\sqrt{1-x_i^2-y_i^2}$, in this parametrisation; these are the reciprocals of the surface volume elements, hence cancel out after carrying out the transformation. In addition, because the construction gives a twofold reparametrisation, the integral over the pair of points is exactly half of the one obtained here. □

Now, let K be an r-spherical spindle convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$. Note that $\cot r$ is the geodesic curvature of a circle of radius r on the sphere.

To define the integral transform in the general case, we need an important result regarding caps of spindle convex discs. This is the direct analogue of Lemma 4.1 in [9] and Lemma 3.1 in [8]. It can be proven using the same argument as in [8], hence we omit the proof.

Lemma 2. *Let $D = \mathrm{cl}\,(K \cap B^c)$ be a nonempty spherical disc-cap. Then there is a unique point x_0 of ∂K and a nonnegative real number t such that the spherical center of B is of distance $r + t$ from x_0 along the great circle determined by the normal plane at x_0.*

We can now turn to the description of the general transform. Let $\varrho(s) = \big(x(s), y(s), z(s)\big)$ be an arc-length parametrisation of ∂K with positive orientation, and let $C = C(t,s) \in S^2$ be the point of distance $r+t$ from $\varrho(s)$ along the great circle determined by the normal plane at $\varrho(s)$, i.e.

$$C = C(t,s) = \cos(r+t)\cdot\varrho(s) + \sin(r+t)\cdot\varrho(s)\times\dot{\varrho}(s),$$

and let $K_{t,s}$ denote the spherical disc with center $C(t,s)$ and radius r.

An orthonormal basis in the orthogonal complement of C is given by

$$\begin{aligned} &w_1(s) = \dot{\varrho}(s) \qquad \text{and} \\ &w_2(s) = -\sin(r+t)\cdot\varrho(s) + \cos(r+t)\cdot\varrho(s)\times\dot{\varrho}(s) \end{aligned}$$

Any pair of points (P_1, P_2) in K determine exactly two disc-caps, and by Lemma 2, each has a unique vertex $\varrho(s)$ and height t. Hence the points can be reparametrised as

$$P_i = \cos r + \sin r\cdot\big[\cos(\varphi_i)\cdot w_1 + \sin(\varphi_i)\cdot w_2\big],$$

which, similarly to Lemma 1, produces a twofold reparametrisation.

Lemma 3. *Let K be a spherical convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$. With the construction above, an integral over a pair of points (P_1, P_2) in K can be rewritten as*

$$\int_K\int_K F(P_1,P_2)\mathrm{d}P_1\mathrm{d}P_2 =$$

$$= \frac{\sin^2 r}{2}\int\limits_{\partial K}\int\limits_0^{t(s)}\int\limits_{\theta_1}^{\theta_2}\int\limits_{\theta_1}^{\theta_2} F\cdot\left|\sin(\varphi_1-\varphi_2)\cdot\big(\kappa_g\sin(r+t)-\cos(r+t)\big)\right|\mathrm{d}\varphi_1\mathrm{d}\varphi_1\mathrm{d}t\mathrm{d}s,$$

where $t(s)$ denotes the greatest value of t for which $K \cap K_{t,s}$ is nonempty, and $[\theta_1, \theta_2]$ is the interval (depending on t and s) parametrising $\partial K_{t,s} \cap K$.

Proof. First, let P_1 and P_2 be transformed to (x,y) coordinates, as in (1). We show in Appendix B that the Jacobian of the transformation from (x_1, y_1, x_2, y_2) to $(s, t, \varphi_1, \varphi_2)$ is given by

$$\left|\sin^2 r\sin(\varphi_1-\varphi_2)\cdot\big(\kappa_g\sin(r+t)-\cos(r+t)\big)Z_1Z_2\right|,$$

where Z_i denotes the last coordinate of the point P_i, hence they cancel out with the surface volume elements obtained from the (x,y) transform. Lastly, we get the division by two due to the twofold nature of the transformation. $\square$

We close this section by citing a technical lemma necessary in the proofs of the main theorems.

Lemma 4 (E. Artin [1], and Böröczky, Fodor, Reitzner, Vígh [6])**.** *For any $\beta \geq 0$, $\omega > 0$ and $\alpha > 0$, it holds that*

$$\int_0^{g(n)} t^\beta\cdot(1-\omega t^\alpha)^n\mathrm{d}t \sim \frac{1}{\alpha\cdot\omega^{(\beta+1)/\alpha}}\cdot\Gamma\left(\frac{\beta+1}{\alpha}\right)\cdot n^{-(\beta+1)/\alpha}$$

if

$$\left(\frac{(\alpha+\beta+1)\cdot\ln n}{\alpha\omega n}\right)^{1/\alpha} < g(n) < \omega^{-1/\alpha}$$

for sufficiently large n, where by $f_1(n) \sim f_2(n)$ we mean that $f_1(n)/f_2(n) \to 1$ as $n \to \infty$.

4. Geometric tools

In this section, we state and prove some geometric lemmas used in the proof of Theorem 1. We note that these results follow from [7] Theorem 5.1, but we include elementary proofs for the sake of completeness.

Lemma 5. *Let $A(t)$ denote the surface area of the intersection of two discs of radius r whose distance is t. Then*

$$\lim_{t\to 0}\frac{A(t)}{t} = 2\sin r.$$

Proof. Let the two discs be denoted by D_1 and D_2. We may assume without loss of generality that the planes of ∂D_1 and ∂D_2 are perpendicular to the xy plane, and their intersections with the xy plane are the lines

$$e : x = \cos r \qquad \text{and} \qquad f : x = \frac{\cos r - x\cos t}{\sin t},$$

respectively. See the orthogonal projection of the construction on Figure 2. Then the projection of $D_1 \cap D_2$ – the shaded area in Figure 2 – is precisely the region determined by the lines e and f, and the boundary of the unit circular disc.

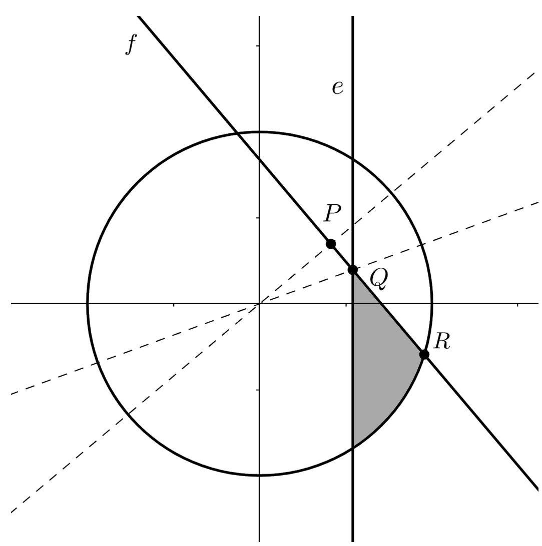

Figure 2. The intersection of two spherical discs.

We compute the surface area via an integral, where the domain of integration is the region described above, and hence the surface area is

$$A(t) = 2\int_{\cos r}^{\cos(r-t)} \int_{-\sqrt{1-x^2}}^{\frac{\cos(r)-x\cdot\cos(t)}{\sin(t)}} \frac{1}{\sqrt{1-x^2-y^2}}\,\mathrm{d}y\mathrm{d}x.$$

The integral with respect to y can be directly computed, and we get

$$A(t) = 2\int_{\cos r}^{\cos(r-t)} \arcsin\left(\frac{\cos r - x\cos t}{\sin t\cdot\sqrt{1-x^2}}\right) + \frac{\pi}{2}\,\mathrm{d}x.$$

The integrand is bounded, and the length of the interval tends to 0 as $t \to 0$, hence $A(t) \to 0$. Thus by l'Hôpital's rule, we have that

$$\lim_{t\to 0}\frac{A(t)}{t} = 2\cdot\lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\cos r}^{\cos(r-t)} \arcsin\left(\frac{\cos r - x\cos t}{\sin t\cdot\sqrt{1-x^2}}\right) + \frac{\pi}{2}\,\mathrm{d}x.$$

By the Leibniz integral rule, in the limit we have the expression

$$\left[\arcsin\left(\frac{\cos r - \cos(r-t)\cos t}{\sin t\sin(r-t)}\right) + \frac{\pi}{2}\right]\cdot\sin(r-t) +$$
$$+\int_{\cos r}^{\cos(r-t)} \frac{\mathrm{d}}{\mathrm{d}t}\left[\arcsin\left(\frac{\cos(r) - x\cos(t)}{\sin(t)\cdot\sqrt{1-x^2}}\right) + \frac{\pi}{2}\right]\mathrm{d}x.$$

The first factor of the first term can be simplified with trigonometric identities: the expression in the arcsin is equal to -1, hence the first term is equal to 0. In the second term, after taking the derivative, we get

$$\frac{1}{\sin t}\cdot\int_{\cos r}^{\cos(r-t)} \frac{x - \cos r\cos t}{\sqrt{(\sin t\sin r)^2 - (x-\cos r\cos t)^2}}\,\mathrm{d}x. \tag{2}$$

This integral can be computed directly, and we obtain

$$\frac{1}{\sin t}\cdot\sqrt{\sin^2 t\sin^2 r - \cos^2 r\cdot(1-\cos t)^2} = \sqrt{\sin^2 r - \cos^2 r\cdot\left(\frac{1-\cos t}{\sin t}\right)^2},$$

which in the limit $t \to 0$ gives us $\sin r$, since the trigonometric factor containing t tends to 0 by an elementary calculation. □

Lemma 6. *Let $\theta(t)$ denote the central angle corresponding to an arc of the intersection of two discs of radius r whose distance is t. Then we have*

(i) $\theta(t) = 2\arccos\left(\frac{\sin t}{1+\cos t}\cdot\cot r\right)$ *and*

(ii) $\theta(t) - \sin\theta(t) \nearrow \pi$ *as* $t \to 0$.

Proof. Part (ii) is a simple analytical consequence of (i), as $\theta(t)$ is monotonically decreasing and the function $x - \sin x$ is monotonically increasing. Part (i) we show as follows: we consider again the construction in Figure 2, described in Lemma 5. The intersection Q of the lines e and f is on the line

$$y = x\cdot\tan\frac{t}{2} = x\cdot\frac{\sin t}{1+\cos t}.$$

Furthermore, the Euclidean center of the disc D_2 is in the xy plane, and it is the intersection of f and is line $y = x\cdot\tan t$: this is denoted by P in Figure 2. The projection of the arc we're interested in is the line segment $\overline{QR}$. The plane spanned by ∂D_2 is shown in Figure 3.

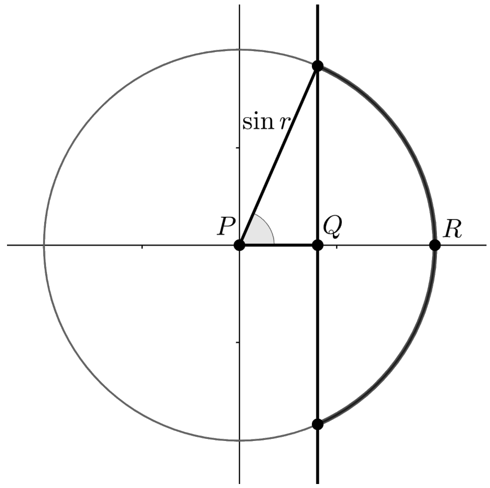

FIGURE 3. The examined arc in the plane of the intersection circle.

From the construction we have that

$$P = \big(\cos r \cos t,\ \cos r \sin t\big) \qquad \text{and} \qquad Q = \left(\cos r,\ \cos r \cdot \frac{\sin t}{1+\cos t}\right),$$

hence their Euclidean distance is

$$d(P,Q) = \frac{\sin t}{1+\cos t} \cdot \cos r,$$

and for the corresponding central angle we have

$$\cos\left(\frac{\theta(t)}{2}\right) = \frac{d(P,Q)}{\sin r} = \frac{\sin t}{1+\cos t} \cdot \frac{\cos r}{\sin r},$$

which yields the assertion. □

5. PROOF OF THEOREM 1.

Let $K \subseteq S^2$ be the disc of radius r whose center is the north pole, i.e. the point $(0,0,1)$ in $\mathbb{R}^3$, and introduce the notation $A = A(K)$. Let P_1 and P_2 be two disctinct points in K. Then there are exactly two spherical discs or radius r that contain both points in their boundaries. Let $D^-(P_1,P_2)$ and $D^+(P_1,P_2)$ denote the spherical disc-caps determined by these discs, i.e. the closure of the subset of K that is not covered by the corresponding disc. We also introduce the notation $A^-(P_1,P_2) = A(D^-(P_1,P_2))$ and $A^+(P_1,P_2) = A(D^+(P_1,P_2))$.

We now turn to the proof of Theorem 1. Our argument is based on the technique used by Rényi and Sulanke in [13]. The random pair of points $P_1, P_2 \in K$ determines an edge of $\operatorname{conv}_r(X_n)$ if and only if $D^-(P_1,P_2)$ or $D^+(P_1,P_2)$ don't contain more points from X_n. Hence

$$\mathbb{E}\left(f_0(K_n)\right) = \binom{n}{2} \cdot I, \qquad \text{where}$$

$$I = \frac{1}{A^2} \int\limits_K \int\limits_K \left(1 - \frac{A^-(P_1,P_2)}{A}\right)^{n-2} + \left(1 - \frac{A^+(P_1,P_2)}{A}\right)^{n-2} \mathrm{d}P_1 \mathrm{d}P_2.$$

Using the transformation described in Section 3, we obtain

$$I = \frac{\sin^2 r}{A^2} \int_0^{2\pi} \int_0^{2r} \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{\theta_2} \left(1 - \frac{A(t)}{A}\right)^{n-2} \cdot \sin t \cdot |\sin(\varphi_1 - \varphi_2)| \mathrm{d}\varphi_1 \mathrm{d}\varphi_1 \mathrm{d}t \mathrm{d}\varphi.$$

Note that while the transform in Lemma 1 is twofold, here every cap gets counted exactly once due to the nature of the specific integrand. Also note that the area of the disc-cap depends only on t of this new parametrisation, and is equal to the quantity defined in Lemma 5.

After integration with respect to φ_1 and φ_2, we get an integrand that depends on $\theta_2 - \theta_1$. This is the quantity defined in Lemma 6, which is now only dependent on t, but not φ. Hence after integration w.r.t. φ_1, φ_2 and φ, we have

$$I = \frac{4\pi \sin^2 r}{A^2} \int_0^{2r} \left(1 - \frac{A(t)}{A}\right)^{n-2} \cdot \sin t \cdot \big(\theta(t) - \sin\theta(t)\big) \mathrm{d}t\ .$$

Fix $\varepsilon > 0$. Then there exists a constant $0 < \gamma_1 < r$ such that for every $0 \leq t \leq \gamma_1$ we have

$$\begin{aligned} &\text{(i)} \quad 2\sin r \cdot t \cdot (1+\varepsilon)^{-1} \leq A(t) \leq 2\sin r \cdot t \cdot (1+\varepsilon), \\ &\text{(ii)} \quad \pi \cdot (1+\varepsilon)^{-1} \leq \theta(t) - \sin\theta(t) \leq \pi \cdot (1+\varepsilon), \text{ and} \\ &\text{(iii)} \quad t \cdot (1+\varepsilon)^{-1} \leq \sin t \leq t \cdot (1+\varepsilon). \end{aligned}$$

The existence of such a constant comes from Lemmas 5 and 6, and the limit $\sin x / x \to 1$ $(x \to 0)$, respectively. As $A(t)$ is a monotonically increasing, we have that for $t \geq \gamma_1$, $A(t) \geq A(\gamma_1) = \delta \cdot A$ for some $\delta > 0$, hence

$$\lim_{n\to\infty} \binom{n}{2} \int_{\gamma_1}^{2r} \left(1 - \frac{A(t)}{A}\right)^{n-2} \sin t \big(\theta(t) - \sin\theta(t)\big) \mathrm{d}t \leq$$
$$\leq \lim_{n\to\infty} \binom{n}{2} \int_{\gamma_1}^{2r} (1-\delta)^{n-2} \cdot \pi \mathrm{d}t = 0.$$

Thus it is enough to consider the integral on the interval $[0, \gamma_1]$.

By the choice of γ_1, we have

$$\begin{aligned} (1+\varepsilon)^{-2} \cdot \frac{4\pi^2 \sin^2 r}{A^2} \int_0^{\gamma_1} \left(1 - \frac{2\sin r \cdot (1+\varepsilon)}{A} \cdot t\right)^{n-2} t \ \mathrm{d}t \leq \\ \leq \int_0^{\gamma_1} \left(1 - \frac{A(t)}{A}\right)^{n-2} \sin t \big(\theta(t) - \sin\theta(t)\big) \mathrm{d}t \leq \\ \leq (1+\varepsilon)^2 \cdot \frac{4\pi^2 \sin^2 r}{A^2} \cdot \int_0^{\gamma_1} \left(1 - \frac{2\sin r \cdot (1+\varepsilon)^{-1}}{A} t\right)^{n-2} t \ \mathrm{d}t. \end{aligned} \tag{3}$$

We are going to apply Lemma 4 to the integrals in (3). With the notations of the lemma, we have $\alpha = \beta = 1$, in the upper bound $\omega^+ = 2\sin r \cdot (1+\varepsilon)^{-1}/A$, in the lower $\omega^- = 2\sin r \cdot (1+\varepsilon)/A$. Let also $g^+(n) = \gamma_2^+ = 1/(2\omega^+)$ and $g^-(n) = \gamma_2^- = 1/(2\omega^-)$. Then the conditions of the lemma are satisfied, and hence

$$\int_0^{\gamma_2^+} \left(1 - \frac{2\sin r \cdot (1+\varepsilon)^{-1}}{A} \cdot t\right)^{n-2} t \ \mathrm{d}t \sim \frac{A^2 \cdot (1+\varepsilon)^2}{4\sin^2 r} \cdot n^{-2}, \quad \text{és}$$
$$\int_0^{\gamma_2^-} \left(1 - \frac{2\sin r \cdot (1+\varepsilon)}{A} \cdot t\right)^{n-2} t \ \mathrm{d}t \sim \frac{A^2 \cdot (1+\varepsilon)^{-2}}{4\sin^2 r} \cdot n^{-2}.$$

All in all, if $\gamma = \min\{\gamma_1, \gamma_2^+, \gamma_2^-\}$,

$$(1+\varepsilon)^{-4} \leq \lim_{n\to\infty} \frac{\displaystyle\int_0^{\gamma} \left(1 - \frac{A(t)}{A}\right)^{n-2} \cdot \sin t \cdot (\theta(t) - \sin\theta(t))\mathrm{d}t}{n^{-2}\cdot \pi^2} \leq (1+\varepsilon)^4,$$

which yields the assertion on the number of vertices.

For the proof of the claim on the area difference, we use the spherical spindle convex counterpart of the Efron identity, which states

$$\mathbb{E}(f_0(K_n)) = \frac{n\mathbb{E}(A(K\setminus K_n))}{A}. \tag{4}$$

An elementary calculation shows that $A = 2\pi(1-\cos r)$, from which the second assertion of the theorem follows.

6. Sketch of proof of Theorem 2

In the proof of Theorem 2, we use similar asymptotic results as the ones shown in Section 4.

Lemma 7. *Let K be a spherical spindle convex disc with C^2 boundary and with the property that $\kappa_g(x) > \cot r$ for every $x \in \partial K$, and let ϱ be an arc-length parametrisation of ∂K. We denote by $A(t,s)$ the area of the spherical disc-cap with vertex $\varrho(s)$ and height t, and by $l(t,s)$ the arc length of the circular arc that determines the disc-cap. Then*

$$\lim_{t\to 0} \frac{A(t,s)}{t^{3/2}} = \frac{4}{3}\sqrt{\frac{2}{\kappa_g(s) - \cot r}} \qquad \textit{and} \qquad \lim_{t\to 0}\frac{l(t,s)}{t^{1/2}} = 2\sqrt{\frac{2}{\kappa_g(s)-\cot r}}.$$

Proof. Assume without loss of generality that $\varrho(s) = (0,0,1)$, $\dot\varrho(s) = (1,0,0)$ and $\ddot\varrho(s)$ is in the yz-plane, and fix $\varepsilon > 0$. We consider the projection of the disc cap to the tangent plane of the sphere at $(0,0,1)$, i.e. the plane $z = 1$. Let $A_p(t,s)$ and $l_p(t,s)$ denote the area and arc length of the projection. As the endpoints of the arc determining the disc-cap tend to $\varrho(s)$ as $t \to 0$, there exists some $t_0 > 0$ such that

$$\begin{aligned} A_p(t,s) \leq A(t,s) \leq A_p(t,s)\cdot(1+\varepsilon) \\ l_p(t,s) \leq l(t,s) \leq l_p(t,s)\cdot(1+\varepsilon) \end{aligned} \tag{5}$$

holds for every $0 < t < t_0$. Now, we determine the behaviour of $A_p(t,s)$ and $l_p(t,s)$ as $t \to 0$ with an argument similar to the corresponding proof in Lemma 4.2 of [9]. For simplicity in notation, we consider the projection as part of $\mathbb{R}^2$, and omit the last coordinate. By the assumptions on the position of ∂K, the projection of the fixed point $\varrho(s)$ is the origin, and the projection has tangent line $x = 0$, normal line $y = 0$, and curvature $\kappa_g(s)$ at $(0,0)$. The projection of the circular arc determining the disc-cap is the arc of an ellipse with semi-major and semi-minor axes parallel to the x- and y-axis, and of length $\sin r$ and $\sin r\cos(r+t)$, respectively. Furthermore, note that the ellipse has a co-vertex at $(0, \sin t)$, which tends to the origin as $t \to 0$. Hence for small enough t, there is a sufficiently small neighbourhood of the origin where they can both be represented as graphs of twice differentiable functions f and g_t, respectively, and Taylor's theorem yields

$$f(x) = \frac{\kappa_g(s)}{2}x^2 + o(x^2) \quad \text{and} \quad g_t(x) = \sin t + \frac{\cos(r+t)}{2\sin r}x^2 + o(x^2)$$

as $x \to 0$. By the condition $\kappa_g(s) > \cot r$, we also have $\kappa_g(s) > \cos(r+t)/\sin r$, and the graphs of the function f and g_t intersect. The points of intersection, which correspond to the endpoints of the circular arc defining the disc-cap, have x-coordinates

$$x_{+/-} = \pm \sqrt{\frac{2\sin t}{\kappa_g(s) - \cos(r+t)/\sin r}} + o\left(t^{1/2}\right).$$

Hence the area of the projection is

$$A_p(t,s) = \int_{x_-}^{x_+} g_t(x) - f(x)\mathrm{d}x = \frac{4}{3}\sqrt{\frac{2}{\kappa_g(s) - \cos(r+t)/\sin r}}\,(\sin t)^{3/2} + o\left(t^{3/2}\right),$$

and

$$\lim_{t\to 0} \frac{A_p(t,s)}{t^{3/2}} = \frac{4}{3}\cdot\sqrt{\frac{2}{\kappa_g(s) - \cot r}},$$

which yields the assumption together with (5). As the ratio of the length of an arc and its corresponding chord tends to 1 as the length of the arc tends to 0, the claim on the arc length follows. □

Outline of proof of Theorem 2. We follow similar steps as in the proof of Theorem 1. With the same notation for D^-, D^+, A^-, A^+ and the integral I in Section 5, we will now compute the limit

$$\lim_{n\to\infty} n^{-1/3}\mathbb{E}(f_0(K_n)) = \lim_{n\to\infty} n^{-1/3}\binom{n}{2} I.$$

Using the general integral transform described in Section 3, we have

$$I = \frac{\sin^2 r}{A^2}\int_{\partial K}\int_0^{t(s)}\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta_2}\left(1 - \frac{A(t,s)}{A}\right)^{n-2} \times \\ \times |\kappa_g(s)\sin(r+t) - \cos(r+t)| \cdot |\sin(\varphi_1 - \varphi_2)|\mathrm{d}\varphi_1\mathrm{d}\varphi_1\mathrm{d}t\mathrm{d}s.$$

We can again integrate with respect to φ_1 and φ_2, and obtain an expression that is a function of $\theta_2 - \theta_1$. This quantity is equal to $l(t,s)/\sin r$, where $l(t,s)$ is the length of the circular arc as described in Lemma 7. By the compactness of the domain of s, for every $\delta > 0$ there exists some constant γ independent of s such that whenever $t \geq \gamma$, we have $A(t,s) \geq \delta$. Hence for $t \geq \gamma$, the integrand is exponentially small in n, hence is negligible in limit. Also note that the expression $\kappa_g(s)\sin(r+t) - \cos(r+t) = \sin(r+t)\cdot(\kappa_g(s) - \cot(r+t))$ is positive. Thus we need to determine the limit

$$\lim_{n\to\infty}\int_{\partial K}\theta_n(s)\mathrm{d}s,$$

where

$$\theta_n(s) = n^{-1/3}\binom{n}{2}\frac{\sin^2 r}{A^2}\int_0^{\gamma}\left(1 - \frac{A(t,s)}{A}\right)^{n-2}\sin(r+t)\times \\ \times\cdot(\kappa_g(s) - \cot(r+t))\left(\frac{l(t,s)}{\sin r} - \sin\left(\frac{l(t,s)}{\sin r}\right)\right)\mathrm{d}t.$$

The remainder of the argument closely follows the proof of Theorem 1.1 of [9]. There exists some function $h(n)$ such that on the interval $[h(n), \gamma]$, the corresponding limit is 0, and on the interval $[0, h(n)]$ the function $\theta_n(s)$ has a universal upper bound. Lebesgue's dominated convergence theorem thus implies that

$$\lim_{n\to\infty} \mathbb{E}(f_0(K_n))n^{-1/3} = \int_{\partial K} \lim_{n\to\infty} \theta_n(s)\mathrm{d}s.$$

Lastly, the limit of $\theta_n(s)$ can be determined by using the geometric limits in Lemma 7, and the asymptotic result in Lemma 4. We omit the details.

7. Concluding remarks

As mentioned in the Introduction, the limit of the expected number of facets in the case of spherical polytopes on S^d and (euclidean) ball-polytopes in $\mathbb{R}^d$ coincide for any $d > 2$ as well. A potential generalisation of the results of this paper would be defining spherical ball-polytopes on S^d for $d > 2$, and investigating the analogue of the uniform model. This would most likely entail finding integralgeometric transformations similar to the ones used here. We repeat here that taking the limit $r \to 0+$ in Theorem 2 only formally gives back the main results of [9], but we do not have a rigorous proof of this.

We note that random (and best) approximation by (generalized) polygons and polytopes on the sphere is a new area, one can formulate many interesting questions based on the known results from Euclidean space, for more on the topic we refer to the very recent article [4].

8. Appendix A

We compute the Jacobian of the transform described in (3) via the exterior product of differential forms. For $i = 1, 2$ we have

$$\begin{aligned}\mathrm{d}x_i = {} & \cos\varphi \cdot (\cos r\cos t - \sin r\sin t\sin\varphi_i)\ \mathrm{d}t + \\ & + \left[\cos\varphi\cdot(\sin r\cos\varphi_i) - \sin\varphi\cdot(\cos r\sin t + \sin r\cos t\sin\varphi_i)\right]\ \mathrm{d}\varphi + \\ & + (-\sin\varphi\sin\varphi_i + \cos\varphi\cos t\cos\varphi_i)\cdot\sin r\ \mathrm{d}\varphi_i,\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}y_i = {} & \sin\varphi \cdot (\cos r\cos t - \sin r\sin t\sin\varphi_i)\ \mathrm{d}t + \\ & + \left[\sin\varphi\cdot(\sin r\cos\varphi_1) + \cos\varphi\cdot(\cos r\sin t + \sin r\cos t\sin\varphi_i)\right]\ \mathrm{d}\varphi + \\ & + (\cos\varphi\sin\varphi_i + \sin\varphi\cos t\cos\varphi_i)\cdot\sin r\ \mathrm{d}\varphi_i.\end{aligned}$$

For the computation of $\mathrm{d}x_i\mathrm{d}y_i$, we group the terms by φ-factor: terms containing $\sin\varphi\cos\varphi$ cancel out, while terms containing $\sin^2\varphi$ and $\cos^2\varphi$ have the same coefficients, hence can be collected and simplified via $\sin^2\varphi + \cos^2\varphi = 1$. Moreover, let the coefficient of $\mathrm{d}t\mathrm{d}\varphi$ be denoted by $U = U(t, \varphi, \varphi_1, \varphi_2)$ – the exact formula is not necessary, which will be explained later on. By this method, we have

$$\begin{aligned}\mathrm{d}x_i\mathrm{d}y_i = {} & \sin r\sin\varphi_i\cdot(\cos r\cos t - \sin r\sin t\sin\varphi_i)\ \mathrm{d}t\mathrm{d}\varphi_i + \\ & + \left[\sin r\cos\varphi_i\sin\varphi_i - \cos t\cos\varphi_i(\cos r\sin t + \sin r\cos t\sin\varphi_i)\right]\ \mathrm{d}\varphi\mathrm{d}\varphi_i + \\ & + U\mathrm{d}t\mathrm{d}\varphi,\end{aligned}$$

and further simplifying the second factor yields

$$\mathrm{d}x_i\mathrm{d}y_i = (\cos r\cos t - \sin r\sin t\sin\varphi_i)\cdot\big[\sin r\sin\varphi_i\cdot \mathrm{d}t\mathrm{d}\varphi_i + \cos\varphi_i\sin t\,\mathrm{d}\varphi\mathrm{d}\varphi_i\big] + U\mathrm{d}t\mathrm{d}\varphi.$$

As either $\mathrm{d}t$ or $\mathrm{d}\varphi$ appears in each term of the sum, if we take the wedge product of the expression $U\mathrm{d}t\mathrm{d}\varphi$ in $\mathrm{d}x_1\mathrm{d}y_1$ with any of the terms in $\mathrm{d}x_2\mathrm{d}y_2$, we get the wedge product of either $\mathrm{d}\varphi$, or $\mathrm{d}t$, with itself, hence zero. Thus for the quadruple product we have

$$\begin{aligned}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2 = (\cos r\cos t - \sin r\sin t\sin\varphi_1)(\cos r\cos t - \sin r\sin t\sin\varphi_2)\times\\ \times\sin r\sin t\cdot(\sin\varphi_1\cos\varphi_2 - \cos\varphi_2\sin\varphi_1)\ \mathrm{d}t\mathrm{d}\varphi_1\mathrm{d}\varphi\mathrm{d}\varphi_2.\end{aligned}$$

9. Appendix B

Using the notation and construction described in Section 3, for any point $P_i = (x_i, y_i, z_i)$ in K we have

$$\begin{aligned}P_i &= \cos r\cdot C(s,t) + \sin r\cdot\big[\cos(\varphi_i)\cdot w_1 + \sin(\varphi_i)\cdot w_2\big] =\\ &= \varrho\cdot\big[\cos r\cos(r+t) - \sin r\sin(r+t)\sin(\varphi_i)\big] + \dot\varrho\cdot\sin r\cos(\varphi_i) +\\ &\qquad + \varrho\times\dot\varrho\cdot\big[\cos r\sin(r+t) + \sin r\cos(r+t)\sin(\varphi_i)\big]\end{aligned}$$

Let $C = C(s,t)$, and to further abbreviate the derivation of the formula, we introduce the notation $A_i = A_i(t,\varphi_i)$ and $B_i = B_i(t,\varphi_i)$ for the coefficient of ϱ and $\varrho\times\dot\varrho$, respectively, in the expression above. Note that

$$\frac{\partial}{\partial t}A_i = -B_i \quad \text{and} \quad \frac{\partial}{\partial t}B_i = A_i.$$

Using the notation $\varrho = (x,y,z)$ and $\varrho\times\dot\varrho = (n_x, n_y, n_z)$, and making use of the simplifications noted above, we have

$$\begin{aligned}\mathrm{d}x_i = \big[x'\cdot(A_i - \kappa_g B_i) + x''\cdot\sin r\cos\varphi_i\big]\mathrm{d}s + \cdot\big[-x\cdot B_i + n_x\cdot A_i\big]\mathrm{d}t +\\ + \sin r\cdot\big[-x\cdot\sin(r+t)\cos\varphi_i - x'\cdot\sin\varphi_i + n_x\cdot\cos(r+t)\cos\varphi_i\big]\mathrm{d}\varphi_i\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}y_i = \big[y'\cdot(A_i - \kappa_g B_i) + y''\cdot\sin r\cos\varphi_i\big]\mathrm{d}s + \big[-y\cdot B_i + n_y\cdot A_i\big]\mathrm{d}t +\\ + \sin r\cdot\big[-y\cdot\sin(r+t)\cos\varphi_i - y'\cdot\sin\varphi_i + n_y\cdot\cos(r+t)\cos\varphi_i\big]\mathrm{d}\varphi_i\end{aligned}$$

While computing the exterior product $\mathrm{d}x_i\mathrm{d}y_i$, we first consider the terms as functions of s. The expressions we obtain are the last coordinates of a given cross product: for example, $xy' - xy'$ is the last coordinate of $\varrho\times\dot\varrho$, and $x'n_y - n_x y'$ of $\dot\varrho\times(\varrho\times\dot\varrho) = \varrho$. Our aim is to express everything in terms of only coordinates of ϱ, $\dot\varrho$ and $\varrho\times\dot\varrho$; to achieve this, we use the relationship $\ddot\varrho = \kappa_g\cdot\varrho\times\dot\varrho + \kappa_n\cdot\varrho$, where $\kappa_g = \kappa_g(s)$ and $\kappa_n = \kappa_n(s)$ denote the geodesic and normal curvatures, respectively, of ∂K at $\varrho(s)$. This can be shown by a straightforward differential geometric consideration. Also note that by Meusnier's theorem, the normal curvature at any point is equal to the normal curvature of a great circle of the sphere, hence $\kappa_n \equiv 1$.

Using this, and by denoting the coefficient of $\mathrm{d}s\mathrm{d}t$ by $U = U(t,s,\varphi_1,\varphi_2)$ –which similarly to Appendix A, doesn't need to be expressed explicitly–, we obtain

$$\mathrm{d}x_i\mathrm{d}y_i = \mathrm{d}s\mathrm{d}\varphi_i\cdot\sin r\cos\varphi_i\cdot\Big[z\cdot\big((A_i - \kappa_g B_i)\cos(r+t) + \kappa_g\sin r\sin\varphi_i\big) +$$

$$+ z' \cdot \big(-\kappa_g \sin r \sin(r+t) \cos\varphi_i + \sin r \cos(r+t) \cos\varphi_i \big) +$$
$$+ n_z \cdot \big((A_i - \kappa_g B_i) \sin(r+t) + \sin r \sin\varphi_i \big) \Big] +$$
$$+ \mathrm{d}t \mathrm{d}\varphi_i \cdot \sin r \cdot \Big[z \cdot A_i \sin\varphi_i + n_z \cdot B_i \sin\varphi_i +$$
$$+ z' \cdot \big(B_i \cos(r+t) \cos\varphi_i - A_i \sin(r+t) \cos\varphi_i \big) \Big] + U \mathrm{d}t \mathrm{d}s.$$

By expanding A_i and B_i, and collecting like terms, we get

$$\mathrm{d}x_i \mathrm{d}y_i = \Big[\mathrm{d}s \mathrm{d}\varphi_i \cdot \cos\varphi_i \cdot \big(\cos(r+t) - \kappa_g \sin(r+t) \big) + \mathrm{d}t \mathrm{d}\varphi_i \cdot \sin\varphi_i \Big] \times$$
$$\times \sin r \cdot \big(z \cdot A_i + z' \cdot \sin r \cos\varphi_i + n_z \cdot B_i \big) + U \mathrm{d}t \mathrm{d}s.$$

Hence the Jacobian is

$$\sin^2 r \sin(\varphi_1 - \varphi_2) \cdot \big(\cos(r+t) - \kappa_g \sin(r+t) \big) Z_1 Z_2,$$

where

$$Z_i = z A_i + z' \sin r \cos\varphi_i + n_z B_i,$$

which is exactly the last coordinate of the point P_i.

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