Results (assuming parameter interval pmi=(T0,1) and parameter t = T)

0.777 …

In[ ]:= T0 = Root[8 T ^ 3 – 20 T ^ 2 + 12 T – 1, 2]; N[T0, 4]

Out[ ]= 0.7775

In[1]:= $OO = \text{Sqrt}\left[\left(4 - 11\,T^{1} + 8\,T^{2}\right) / T\right];$

In[2]:=
$$\gamma\gamma = \frac{5 - 24\,T^{1} + 23\,T^{2} + 4\,T^{3} - 6\,T^{4} + 2\,T^{5}}{98\,(-1 + T)^{3}\,T^{2}} + \frac{\left(1 - 4\,T^{1} + 5\,T^{2} - 6\,T^{3}\right)\Omega}{98\,(-1 + T)^{3}\,T^{2}};$$

$$\beta\beta = \frac{3 - 31\,T^{1} + 135\,T^{2} - 328\,T^{3} + 484\,T^{4} - 440\,T^{5} + 240\,T^{6} - 72\,T^{7} + 8\,T^{8}}{8\,(-1 + T)^{3}\,T^{2}\,\left(-1 + 5\,T^{1} - 6\,T^{2} + T^{3}\right)} + \frac{\left(-1 + 9\,T^{1} - 31\,T^{2} + 52\,T^{3} - 44\,T^{4} + 16\,T^{5}\right)\Omega}{8\,(-1 + T)^{3}\,T^{2}\,\left(-1 + 5\,T^{1} - 6\,T^{2} + T^{3}\right)};$$

In[ ]:= Limit[γγ /. Ω → OO, T → 1, Direction → 1]

Out[ ]= ∞

In[ ]:= RootReduce [(γγ /. Ω → OO /. T → T0)] – 1

Out[ ]= 0

In[ ]:= Limit[ββ /. Ω → OO, T → 1, Direction → 1]

Out[ ]= ∞

In[ ]:= RootReduce [(ββ /. Ω → OO /. T → T0)] – RootReduce [(1 + 2 Tan[π / 14] ^ 2) ^ 2]

Out[ ]= 0

Checks, now symbolically

In[4]:=
$$p1 = \left(4 - 11\,T^{1} + 8\,T^{2}\right) - T\,\Omega^{2};$$
$$p2 = \left(5 - 24\,T^{1} + 23\,T^{2} + 4\,T^{3} - 6\,T^{4} + 2\,T^{5}\right) + \Omega\left(\left(1 - 4\,T^{1} + 5\,T^{2} - 6\,T^{3}\right)\right) - C\left(98\,(-1 + T)^{3}\,T^{2}\right);$$
$$p3 = \left(3 - 31\,T^{1} + 135\,T^{2} - 328\,T^{3} + 484\,T^{4} - 440\,T^{5} + 240\,T^{6} - 72\,T^{7} + 8\,T^{8}\right) + \Omega\left(-1 + 9\,T^{1} - 31\,T^{2} + 52\,T^{3} - 44\,T^{4} + 16\,T^{5}\right) - B\left(8\,(-1 + T)^{3}\,T^{2}\,\left(-1 + 5\,T^{1} - 6\,T^{2} + T^{3}\right)\right);$$

```
In[7]:= qv1 = Collect[
          Factor[GroebnerBasis[{p1, p2, p3, T (T - 1) (-1 + 5 T - 6 T^2 + T^3) u + 1}, {B, C}, {u, Ω, T}]][[1]], B]

Out[7]= -357 040 905 841 - 11 029 369 540 700 C + 4 073 432 372 590 C^2 +
         1 162 003 193 912 820 C^3 + 17 764 404 166 656 B^9 C^3 - 10 733 709 417 518 479 C^4 +
         10 554 199 561 654 728 C^5 + 354 369 783 878 840 772 C^6 - 2 239 411 199 778 656 120 C^7 +
         3 248 884 525 414 763 361 C^8 + 24 107 034 903 466 286 996 C^9 - 134 210 591 912 583 385 682 C^10 +
         265 867 831 303 643 187 332 C^11 - 191 581 231 380 566 414 401 C^12 +
         B^8 (-13 323 303 124 992 C^2 - 552 176 896 180 224 C^3 - 1 275 459 546 382 336 C^4) +
         B^7 (793 053 757 440 C + 455 688 689 025 024 C^2 +
            5 145 879 104 192 512 C^3 - 21 486 355 249 364 992 C^4 + 38 857 313 330 855 936 C^5) +
         B^6 (-5 664 669 696 - 26 939 280 850 944 C - 4 669 578 255 663 104 C^2 + 18 004 192 781 926 400 C^3 -
            44 614 165 731 000 320 C^4 + 192 088 231 276 462 080 C^5 - 653 170 564 475 785 216 C^6) +
         B^5 (190 238 490 624 + 268 898 566 078 464 C + 11 754 456 016 027 648 C^2 -
            86 294 372 223 295 488 C^3 + 169 551 610 084 347 904 C^4 + 518 182 937 619 515 392 C^5 -
            3 073 830 541 163 798 528 C^6 + 6 601 921 055 234 656 256 C^7) +
         B^4 (-1 844 896 657 408 - 686 362 058 375 168 C - 13 198 792 834 390 016 C^2 +
            163 202 730 146 498 560 C^3 - 771 087 147 529 891 584 C^4 + 2 660 888 570 937 476 096 C^5 -
            11 161 564 391 713 071 616 C^6 + 33 724 246 215 141 981 184 C^7 - 41 100 361 620 926 062 336 C^8) +
         B^3 (4 249 005 537 280 + 788 900 381 425 408 C + 5 476 918 586 671 616 C^2 -
            124 419 611 327 361 280 C^3 + 590 136 331 739 164 672 C^4 + 265 678 882 008 517 376 C^5 -
            15 445 824 135 567 533 568 C^6 + 75 356 247 912 628 334 848 C^7 -
            171 275 611 706 165 076 992 C^8 + 155 910 831 518 338 029 056 C^9) +
         B^2 (-4 217 822 981 568 - 446 724 304 132 288 C + 427 535 891 642 384 C^2 +
            50 291 367 190 572 928 C^3 - 288 850 768 037 151 936 C^4 + 564 309 726 625 564 416 C^5 -
            3 853 140 188 541 522 592 C^6 + 44 347 064 302 715 470 464 C^7 - 209 423 825 638 732 343 296 C^8 +
            435 112 113 361 671 972 800 C^9 - 345 979 266 666 445 636 336 C^10) +
         B (1 966 405 795 984 + 116 786 878 825 288 C - 613 907 085 842 400 C^2 - 9 942 391 930 995 384 C^3 -
            12 195 569 705 462 304 C^4 + 670 185 359 408 210 576 C^5 - 2 242 573 767 808 638 208 C^6 +
            5 905 304 572 649 991 376 C^7 - 56 379 533 779 605 818 096 C^8 + 269 157 226 771 422 294 312 C^9 -
            543 864 087 084 523 398 816 C^10 + 406 621 389 052 630 757 096 C^11)
```

```
In[9]:= qv1b =
  Collect[Factor[GroebnerBasis[{p1, p2, p3, T (T - 1) (-1 + 5 T - 6 T^2 + T^3) u + 1, β^2 - B, γ^2 - C},
      {β, γ}, {u, Ω, T, B, C}]][[1, 1]], β]
```

Out[9]= $-597529 - 9229150\ \gamma^2 + 4214784\ \beta^9\ \gamma^3 + 74683105\ \gamma^4 - 181179460\ \gamma^6 - 1516142663\ \gamma^8 + 9604158466\ \gamma^{10} - 13841287201\ \gamma^{12} + \beta^8 (2107392\ \gamma^2 - 100803584\ \gamma^4) + \beta^7 (-1053696\ \gamma - 115906560\ \gamma^3 + 1054135040\ \gamma^5) + \beta^6 (75264 - 13948928\ \gamma^2 + 1265269376\ \gamma^4 - 6317280704\ \gamma^6) + \beta^5 (16307200\ \gamma + 246520960\ \gamma^3 - 6979879872\ \gamma^5 + 23876158656\ \gamma^7) + \beta^4 (-1263808 - 93556288\ \gamma^2 - 1461459888\ \gamma^4 + 22765552096\ \gamma^6 - 58939325424\ \gamma^8) + \beta^3 (-14749840\ \gamma + 298272800\ \gamma^3 + 4232674880\ \gamma^5 - 45301453344\ \gamma^7 + 94911683664\ \gamma^9) + \beta^2 (1645448 + 59195724\ \gamma^2 - 506092384\ \gamma^4 - 6417988248\ \gamma^6 + 53820182136\ \gamma^8 - 96041584660\ \gamma^{10}) + \beta (3958836\ \gamma - 108109484\ \gamma^3 + 458091592\ \gamma^5 + 4921492968\ \gamma^7 - 35026930876\ \gamma^9 + 55365148804\ \gamma^{11})$

```
In[10]:= p7[x_] = x^7 + (-7 s) x^6 + a5 x^5 + a4 x^4 + a3 x^3 + a2 x^2 + a1 x + a0
```

Out[10]= $a0 + a1\ x + a2\ x^2 + a3\ x^3 + a4\ x^4 + a5\ x^5 - 7\ s\ x^6 + x^7$

```
In[ ]:= Solve[{p7[1] + L == 0, p7[-1] + L == 0}, {a0, a1}]
```

Out[ ]= {{a0 → -a2 - a4 - L + 7 s, a1 → -1 - a3 - a5}}

```
In[11]:= p7a[x_] = p7[x] /. {a0 → -a2 - a4 - L + 7 s, a1 → -1 - a3 - a5}
```

Out[11]= $-a2 - a4 - L + 7\ s + (-1 - a3 - a5)\ x + a2\ x^2 + a3\ x^3 + a4\ x^4 + a5\ x^5 - 7\ s\ x^6 + x^7$

Coefficient Comparison for obtaining polynomials (all should vanish!)

```
In[12]:= L1 = Drop[
    CoefficientList[Numerator[Together[((1 - x^2) (x - α) (x - β) / (49 (x - γ)^2) D[p7a[x], x]^2) -
        (L^2 - p7a[x]^2)]], x] /. α → (2 s - β + 2 γ), -1];
```

Tailoring the ideal by known values for ,α,β

```
In[13]:= L2 = {p7a[2 s - β + 2 γ] + L, p7a[β] - L};

In[14]:= LL = Join[L1, L2];

In[ ]:= Union[Cases[LL, _Symbol, ∞]]
```

Out[ ]= {a2, a3, a4, a5, L, s, β, γ}

In[15]:=
```
q2 = Collect[
  Factor[GroebnerBasis[Join[L1, L2, {(γ^2 - 1) (49 γ^2 - 1) (49 γ^2 - 9) (49 γ^2 - 25) u + 1}],
    {β, γ}, {u, a2, a3, a4, a5, s, L}]][[1]], β]
```

Out[15]=
$$-597529 - 9229150\,\gamma^2 + 4214784\,\beta^9\gamma^3 + 74683105\,\gamma^4 - 181179460\,\gamma^6 - 1516142663\,\gamma^8 + 9604158466\,\gamma^{10} - 13841287201\,\gamma^{12} + \beta^8\left(2107392\,\gamma^2 - 100803584\,\gamma^4\right) + \beta^7\left(-1053696\,\gamma - 115906560\,\gamma^3 + 1054135040\,\gamma^5\right) + \beta^6\left(75264 - 13948928\,\gamma^2 + 1265269376\,\gamma^4 - 6317280704\,\gamma^6\right) + \beta^5\left(16307200\,\gamma + 246520960\,\gamma^3 - 6979879872\,\gamma^5 + 23876158656\,\gamma^7\right) + \beta^4\left(-1263808 - 93556288\,\gamma^2 - 1461459888\,\gamma^4 + 22765552096\,\gamma^6 - 58939325424\,\gamma^8\right) + \beta^3\left(-14749840\,\gamma + 298272800\,\gamma^3 + 4232674880\,\gamma^5 - 45301453344\,\gamma^7 + 94911683664\,\gamma^9\right) + \beta^2\left(1645448 + 59195724\,\gamma^2 - 506092384\,\gamma^4 - 6417988248\,\gamma^6 + 53820182136\,\gamma^8 - 96041584660\,\gamma^{10}\right) + \beta\left(3958836\,\gamma - 108109484\,\gamma^3 + 458091592\,\gamma^5 + 4921492968\,\gamma^7 - 35026930876\,\gamma^9 + 55365148804\,\gamma^{11}\right)$$

In[16]:=
```
qv1b - q2
```

Out[16]= 0

The Weierstrass normal form for the p(B,CC)=0 genus 1 elliptic planar curve (check it via Maple algcurves)

$wf = -1518 + b0^2 - 225\,c0 + 3\,c0^3 = 0$

In[ ]:=
```
b0 /. Solve[-1518 + b0^2 - 225 c0 + 3 c0^3 == 0, b0][[-1]]
```

Out[ ]= $\sqrt{3}\,\sqrt{506 + 75\,c0 - c0^3}$

a symbolic check for p(βs,γs)=p(B,CC)=0

In[17]:=
```
Together[qv1 /. {B → ββ, C → γγ} /. {Ω → OO}]
```

Out[17]= 0

a symbolic check for p(β,γ)=0

Lemma $\beta\gamma$=lexpr ∧ p(β,γ)=p1(B,CC)+p2(B,CC)$\beta\gamma$

In[18]:=
```
lexpr = 1 / (28 T^2 (T - 1)^3) ((1 - 2 T - 7 T^2 + 18 T^3 - 12 T^4 + 4 T^5) + Ω ((-1) (-1 + 6 T - 9 T^2 + 6 T^3)))
```

Out[18]=
$$\frac{1 - 2\,T - 7\,T^2 + 18\,T^3 - 12\,T^4 + 4\,T^5 + \left(1 - 6\,T + 9\,T^2 - 6\,T^3\right)\Omega}{28\,(-1 + T)^3\,T^2}$$

In[19]:=
```
Together[ββ γγ - lexpr^2 /. Ω → OO]
```

Out[19]= 0

In[20]:=
```
q2b = Collect[Expand[q2 /. {β → Sqrt[B], γ -> Sqrt[C]} /.
     {(B)^(3/2) → (B) S2, (B)^(5/2) → (B)^2 S2, (B)^(7/2) → (B)^3 S2,
       (B)^(9/2) → (B)^4 S2,
       (B)^(11/2) → (B)^5 S2,
       (C)^(3/2) → (C) S1,
       (C)^(5/2) → (C)^2 S1,
       (C)^(7/2) → (C)^3 S1,
       (C)^(9/2) → (C)^4 S1,
       (C)^(11/2) → (C)^5 S1
     } /. {Sqrt[B] → S2, Sqrt[C] → S1}], {S2, S1}]
```

Out[20]=
$$-597\,529 + 1\,645\,448\, B - 1\,263\,808\, B^2 + 75\,264\, B^3 - 9\,229\,150\, C + 59\,195\,724\, B\, C - 93\,556\,288\, B^2\, C - 13\,948\,928\, B^3\, C + 2\,107\,392\, B^4\, C + 74\,683\,105\, C^2 - 506\,092\,384\, B\, C^2 - 1\,461\,459\,888\, B^2\, C^2 + 1\,265\,269\,376\, B^3\, C^2 - 100\,803\,584\, B^4\, C^2 - 181\,179\,460\, C^3 - 6\,417\,988\,248\, B\, C^3 + 22\,765\,552\,096\, B^2\, C^3 - 6\,317\,280\,704\, B^3\, C^3 - 1\,516\,142\,663\, C^4 + 53\,820\,182\,136\, B\, C^4 - 58\,939\,325\,424\, B^2\, C^4 + 9\,604\,158\,466\, C^5 - 96\,041\,584\,660\, B\, C^5 - 13\,841\,287\,201\, C^6 + (3\,958\,836 - 14\,749\,840\, B + 16\,307\,200\, B^2 - 1\,053\,696\, B^3 - 108\,109\,484\, C + 298\,272\,800\, B\, C + 246\,520\,960\, B^2\, C - 115\,906\,560\, B^3\, C + 4\,214\,784\, B^4\, C + 458\,091\,592\, C^2 + 4\,232\,674\,880\, B\, C^2 - 6\,979\,879\,872\, B^2\, C^2 + 1\,054\,135\,040\, B^3\, C^2 + 4\,921\,492\,968\, C^3 - 45\,301\,453\,344\, B\, C^3 + 23\,876\,158\,656\, B^2\, C^3 - 35\,026\,930\,876\, C^4 + 94\,911\,683\,664\, B\, C^4 + 55\,365\,148\,804\, C^5)\, S1\, S2$$

In[21]:=
```
Together[RootReduce[q2b /. {S1 S2 → lexpr} /. {B → ββ, C → γγ} /. Ω → OO]]
```

Out[21]= 0