

(* Goal: Algebraic solution of ZFP if n=10 and s=5 and exemplarily one coefficient, $a_{8,10}^*(5)$, is explicitly expressed as a root object (see Appendix 3 in[6]) *)

(* The proper monic Zolotarev polynomial of degree n=10 and with parameter s=5 can be generated by *)

(* $Z_{10,5}(x) = \sum_{k=0}^{10} a_{k,10}^*(5) x^k = (R_{10,\alpha,\beta}(x)) / .s \rightarrow 5 / .\alpha \rightarrow \text{Root}[F_{m(10),5}(\alpha), 4] / .\beta \rightarrow \text{Root}[G_{m(10),5}(\beta), 4]$ *)

(* The first insertion, $(R_{10,\alpha,\beta}(x)) / .s \rightarrow 5$, will yield the following polynomial of degree n=10, still depending on α and β : *)

Out[*]=

$$\begin{aligned}
 & 1 \\
 & 4128768 \\
 & (-3955365672464128 - 206438400x^9 + 4128768x^{10} - 87535\alpha^8 + \alpha^7(6031600 - 6440\beta) + \\
 & 1112172811187200\beta - 94946672592640\beta^2 - 258499116800\beta^3 + 446595548000\beta^4 - \\
 & 21430337600\beta^5 + 208135760\beta^6 + 6031600\beta^7 - 87535\beta^8 - \\
 & 2580480x^8(-1996 + \alpha^2 - 2\alpha(-10 + \beta) + 20\beta + \beta^2) + 140\alpha^6(1486684 - 331140\beta + 10477\beta^2) + \\
 & 280\alpha^5(-76536920 + 11153356\beta - 373110\beta^2 + 383\beta^3) - \\
 & 10\alpha^4(-44659554800 + 4834115360\beta + 55170696\beta^2 - 15770120\beta^3 + 296009\beta^4) + 40\alpha^3 \\
 & (-6462477920 - 20443249712\beta + 3844923440\beta^2 - 222603624\beta^3 + 3942530\beta^4 + 2681\beta^5) + \\
 & 20\alpha^2(-4747333629632 + 1577695359680\beta - 179308853552\beta^2 + \\
 & 7689846880\beta^3 - 27585348\beta^4 - 5223540\beta^5 + 73339\beta^6) - \\
 & 40\alpha(-27804320279680 + 8080729935168\beta - 788847679840\beta^2 + 20443249712\beta^3 + \\
 & 1208528840\beta^4 - 78073492\beta^5 + 1158990\beta^6 + 161\beta^7) - \\
 & 1720320x^7(49720 + \alpha^3 - 1500\beta - 60\beta^2 + \beta^3 - \alpha^2(60 + \beta) - \alpha(1500 - 160\beta + \beta^2)) - \\
 & 26880x^6(-39584336 + 15\alpha^4 + 2395680\beta + 59784\beta^2 - 3800\beta^3 + 15\beta^4 + 4\alpha^3(-950 + 21\beta) + \\
 & \alpha^2(59784 + 4760\beta - 198\beta^2) + 28\alpha(85560 - 10556\beta + 170\beta^2 + 3\beta^3)) + \\
 & 10752x^5(-984035440 + 16\alpha^5 + 99500000\beta + 480280\beta^2 - 244648\beta^3 + 4325\beta^4 + \\
 & 16\beta^5 - 5\alpha^4(-865 + 48\beta) + 4\alpha^3(-61162 + 1975\beta + 56\beta^2) + \\
 & \alpha^2(480280 + 444648\beta - 28290\beta^2 + 224\beta^3) + \\
 & \alpha(99500000 - 14946480\beta + 444648\beta^2 + 7900\beta^3 - 240\beta^4)) + \\
 & 1120x^4(78247056320 + 215\alpha^6 + \alpha^5(7860 - 1210\beta) - 11889677760\beta + \\
 & 242597712\beta^2 + 34217440\beta^3 - 1553100\beta^4 + 7860\beta^5 + 215\beta^6 - \\
 & 5\alpha^4(310620 - 24980\beta + 187\beta^2) + 4\alpha^3(8554360 - 221028\beta - 37030\beta^2 + 965\beta^3) + \\
 & \alpha^2(242597712 - 106167520\beta + 8381624\beta^2 - 148120\beta^3 - 935\beta^4) - \\
 & 2\alpha(5944838880 - 1113127088\beta + 53083760\beta^2 + 442056\beta^3 - 62450\beta^4 + 605\beta^5)) + \\
 & 320x^3(-1944334519040 + 507\alpha^7 + 414131188800\beta - 21013009440\beta^2 - 965230192\beta^3 + \\
 & 109949280\beta^4 - 2342004\beta^5 - 14210\beta^6 + 507\beta^7 - 7\alpha^6(2030 + 177\beta) - \\
 & 7\alpha^5(334572 - 43420\beta + 891\beta^2) + \alpha^4(109949280 - 10880164\beta + 81410\beta^2 + 6969\beta^3) + \\
 & \alpha^3(-965230192 - 46647360\beta + 18866968\beta^2 - 834440\beta^3 + 6969\beta^4) + \\
 & \alpha^2(-21013009440 + 6831924592\beta - 646007360\beta^2 + 18866968\beta^3 + 81410\beta^4 - 6237\beta^5) + \\
 & \alpha(414131188800 - 97102964800\beta + 6831924592\beta^2 - \\
 & 46647360\beta^3 - 10880164\beta^4 + 303940\beta^5 - 1239\beta^6)) + \\
 & 5x^2(773331957533440 + 17507\alpha^8 - 219758384921600\beta + 18935314545920\beta^2 + \\
 & 44014688000\beta^3 - 88971134560\beta^4 + 4284306880\beta^5 - 41675312\beta^6 - 1206320\beta^7 +
 \end{aligned}$$

$$\begin{aligned}
& 17507 \beta^8 + 8 \alpha^7 (-150790 + 161 \beta) - 28 \alpha^6 (1488404 - 331140 \beta + 10477 \beta^2) - \\
& 56 \alpha^5 (-76505480 + 11148516 \beta - 373110 \beta^2 + 383 \beta^3) + \\
& 2 \alpha^4 (-44485567280 + 4820126560 \beta + 55275416 \beta^2 - 15770120 \beta^3 + 296009 \beta^4) - \\
& 8 \alpha^3 (-5501836000 - 20468061296 \beta + 3840776080 \beta^2 - 222495544 \beta^3 + \\
& \quad 3942530 \beta^4 + 2681 \beta^5) - 4 \alpha^2 (-4733828636480 + 1571743581120 \beta - \\
& \quad 178839216496 \beta^2 + 7681552160 \beta^3 - 27637708 \beta^4 - 5223540 \beta^5 + 73339 \beta^6) + \\
& 8 \alpha (-27469798115200 + 8018196067520 \beta - 785871790560 \beta^2 + 20468061296 \beta^3 + \\
& \quad 1205031640 \beta^4 - 78039612 \beta^5 + 1158990 \beta^6 + 161 \beta^7) - \\
& \times (-4588218808356608 + 87535 \alpha^8 (-1 + \beta) + 5833986004439808 \beta - 2068991785196288 \beta^2 + \\
& \quad 347442809541632 \beta^3 - 32125727497632 \beta^4 + 1702670167904 \beta^5 - 50635524464 \beta^6 + \\
& \quad 804336864 \beta^7 - 8045375 \beta^8 + 87375 \beta^9 + 40 \alpha^7 (154846 - 155007 \beta + 4217 \beta^2) - \\
& \quad 140 \alpha^6 (-1454204 + 1788176 \beta - 310249 \beta^2 + 11589 \beta^3) - \\
& \quad 56 \alpha^5 (396064408 - 453567988 \beta + 72941778 \beta^2 - 3818233 \beta^3 + 58683 \beta^4) + \\
& \quad 10 \alpha^4 (48182582000 - 53365120656 \beta + 8826987800 \beta^2 - 472491152 \beta^3 + \\
& \quad \quad 5358111 \beta^4 + 115929 \beta^5) + 8 \alpha^3 (-71250619232 - 32820906128 \beta + \\
& \quad \quad 80309683104 \beta^2 - 17724980024 \beta^3 + 1465638234 \beta^4 - 51050981 \beta^5 + 609531 \beta^6) + \\
& \quad 4 \alpha^2 (-25416392105920 + 33852618515584 \beta - 11131776233264 \beta^2 + \\
& \quad \quad 1631976766416 \beta^3 - 123430771156 \beta^4 + 4879891936 \beta^5 - 91659197 \beta^6 + 590177 \beta^7) + \\
& \quad 8 \alpha (155720899510400 - 200028790253760 \beta + 66897912882592 \beta^2 - 10345756518352 \beta^3 + \\
& \quad \quad 861245913832 \beta^4 - 39822513972 \beta^5 + 974926542 \beta^6 - 10832255 \beta^7 + 37205 \beta^8) \Big)
\end{aligned}$$

(* In particular,

[\[eingesetztes Objekt\]](#)

the coefficient at the monomial x^8 still depends on α and β and reads as follows: *)

Out[*]=

$$-\frac{5}{8} (-1996 + \alpha^2 - 2 \alpha (-10 + \beta) + 20 \beta + \beta^2)$$

(* The parameter $\alpha = \text{Root}[F_{m(10),5}(\alpha), 4]$ reads as follows: *)

[\[Nullstelle\]](#)

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Root[-902 916 165 996 751 513 267 465 124 012 438 757 643 +
Nullstelle
497 030 079 043 959 165 766 487 167 410 308 521 608 z +
470 883 066 714 140 740 028 574 167 286 333 402 852 z^2 -
557 798 978 114 217 405 352 335 250 880 996 359 432 z^3 +
259 013 736 742 139 314 918 477 218 957 964 415 402 z^4 -
83 193 647 383 001 195 159 285 562 468 572 875 816 z^5 +
28 369 267 147 681 697 861 493 967 384 484 760 276 z^6 -
11 345 951 216 672 397 177 561 217 972 485 348 696 z^7 +
4 079 173 571 680 846 230 384 260 645 929 158 011 z^8 -
1 148 812 707 645 071 016 188 839 034 917 125 936 z^9 +
250 484 915 089 243 384 275 766 057 659 535 048 z^10 -
43 090 394 434 496 237 722 080 383 848 651 728 z^11 +
5 968 383 847 559 379 235 587 852 922 918 988 z^12 - 676 346 899 312 644 277 626 744 914 053 328 z^13 +
63 406 360 100 365 984 051 933 030 508 488 z^14 - 4 947 518 262 157 408 936 448 680 564 784 z^15 +
321 577 999 263 242 640 193 712 465 979 z^16 - 17 325 486 552 915 492 809 760 923 864 z^17 +
764 987 697 016 314 011 428 866 644 z^18 - 27 165 750 677 096 464 528 297 384 z^19 +
754 175 359 578 274 067 183 274 z^20 - 15 687 719 867 332 535 314 824 z^21 +
228 565 012 549 821 980 004 z^22 - 2 064 519 540 496 950 264 z^23 + 8 614 550 657 858 229 z^24, 4]

(* Its numerical approximation is  $\alpha$ =
50.009999000199949914095736354554149801844577527038115132536504820776302408330816631\
2322237947482113307485301231737058`99. *)

(* The parameter  $\beta$ =Root[Gm(10),5( $\beta$ ),4] reads as follows: *)
Nullstelle
Root[699 418 548 616 090 654 753 463 166 059 072 135 317 -
Nullstelle
84 207 831 373 473 311 471 706 906 185 414 478 888 z -
371 837 384 810 092 421 593 314 535 632 947 036 668 z^2 +
122 865 656 875 400 379 934 222 858 584 911 236 072 z^3 +
129 687 407 842 540 616 616 763 566 917 786 216 202 z^4 -
141 325 764 249 690 682 953 736 637 695 719 688 504 z^5 +
70 647 813 962 644 136 061 515 257 358 389 939 636 z^6 -
23 604 294 425 519 612 340 823 362 672 487 440 264 z^7 +
5 956 018 040 090 422 338 744 086 488 879 603 131 z^8 -
1 217 268 457 459 881 655 728 371 366 358 229 264 z^9 +
211 287 652 903 375 050 990 910 176 564 001 288 z^10 -
32 012 249 323 523 839 383 892 776 116 007 792 z^11 +
4 260 996 603 742 336 281 312 755 955 965 708 z^12 - 493 557 433 703 568 550 792 089 460 796 912 z^13 +
48 945 869 019 470 211 638 100 715 806 088 z^14 - 4 088 305 591 477 239 921 535 504 695 696 z^15 +
283 526 852 078 665 621 270 798 947 419 z^16 - 16 109 845 494 607 840 705 573 877 576 z^17 +
739 184 459 068 683 967 806 465 524 z^18 - 26 891 262 796 373 169 491 380 216 z^19 +
755 893 115 357 149 631 763 594 z^20 - 15 788 521 297 375 495 758 936 z^21 +
229 861 223 120 924 692 164 z^22 - 2 070 464 775 274 999 656 z^23 + 8 614 550 657 858 229 z^24, 4]

(* Its numerical approximation is  $\beta$ =
50.009999000199950113895866284588134304624694528294999248282575603145593382064168113\
8756828922706722601967380572112741`99. *)

```

(* After insertion,

[\[Fläche nach dem Objekt](#)

a numerical approximation of $Z_{10,5}(x) = \sum_{k=0}^{10} a_{k,10}^*(5) x^k$ reads as follows: *)

Out[*]=

```
-0.0039058594921484511669549290174387722665296024886337151766018427336596415794748193 \
67 -
1.7576758183480505261293182177834731061943591590853975499341142506605109107170872840
x +
0.1601464868352267823508830153731194355098178978651348004984165351410468691422864137 \
91 x^2 +
23.436406507741427493322485212902212331652990391924206664824085837317845467452475118 \
3056 x^3 -
1.0937093842162663560227296318739105651223673050724553518502222847734332368438768280 \
46934 x^4 -
84.372812968628159989315902008499591059633481495707839995212629446035721654641488601 \
63164243 x^5 +
2.6874437618719383747338345347968061828024297378986471781099135725672752931081758636 \
357932072 x^6 +
112.49875024993751749475164946392856444579280624460085901303992019560626935454229601 \
2165776689500 x^7 -
2.7499750049987503498950329892785762809233507282026929115815059802012292838271106300 \
1314238074701 x^8 -
50.000000000000000000000000000000000000000000000000000000000000000000000000000000 \
0000000000000000 x^9 +
1.000000000000000000000000000000000000000000000000000000000000000000000000000000 \
0000000000000000 x^10
```

(* To obtain a non-numerical explicit representation of $Z_{10,5}(x) =$

$\sum_{k=0}^{10} a_{k,10}^*(5) x^k$ would require to insert into $(R_{10,\alpha,\beta}(x)) / .$

s→5 the parameters α and β *)

(* In particular, inserting the parameters α and β into the coefficient at the

[\[eingegebenes Objekt](#)

monomial x^8 (i.e., into $-\frac{5}{8} (-1996+\alpha^2-2 \alpha (-10+\beta)+20 \beta+\beta^2)$) would yield,

after simplification with `RootReduce` or `FullSimplify`,

[\[reduziere auf Null\]](#) [\[vereinfache vollständig](#)

the following algebraic representation of $a_{8,10}^*(5)$ as a root object: *)

Root [
Nullstelle

$-1\,786\,154\,209\,802\,451\,002\,972\,759\,427\,968\,487\,151\,452\,029\,651\,728\,799\,407\,639\,295\,635\,068\,397\,082 \setminus$
 $203\,209\,400\,177\,001\,953\,125 -$
 $609\,669\,402\,283\,538\,940\,064\,962\,351\,778\,105\,590\,416\,586\,800\,600\,947\,854\,626\,322\,974\,747\,244\,589\,514 \setminus$
 $636\,993\,408\,203\,125\,000\,z +$
 $14\,065\,049\,508\,440\,152\,464\,640\,540\,540\,273\,080\,874\,274\,960\,792\,390\,609\,169\,876\,582\,299\,521\,406\,218 \setminus$
 $557\,357\,788\,085\,937\,500\,z^2 -$
 $151\,644\,396\,136\,269\,517\,420\,537\,591\,671\,668\,698\,662\,291\,198\,974\,943\,286\,974\,771\,385\,459\,752\,798\,534 \setminus$
 $393\,310\,546\,875\,000\,z^3 +$
 $1\,034\,230\,604\,547\,796\,776\,894\,311\,813\,388\,800\,222\,396\,217\,413\,745\,783\,240\,608\,643\,078\,907\,588\,289 \setminus$
 $451\,599\,121\,093\,750\,z^4 -$
 $5\,024\,914\,765\,274\,801\,563\,771\,944\,009\,102\,429\,931\,695\,283\,939\,155\,677\,138\,996\,369\,663\,362\,725\,677 \setminus$
 $490\,234\,375\,000\,z^5 +$
 $18\,525\,952\,105\,958\,034\,810\,709\,332\,606\,630\,760\,338\,002\,035\,331\,088\,431\,649\,304\,996\,067\,017\,868\,041 \setminus$
 $992\,187\,500\,z^6 -$
 $53\,864\,412\,863\,332\,279\,824\,097\,453\,356\,521\,821\,046\,949\,689\,890\,166\,470\,307\,609\,262\,109\,185\,791\,015 \setminus$
 $625\,000\,z^7 +$
 $126\,674\,139\,176\,457\,908\,238\,718\,970\,995\,577\,561\,835\,058\,163\,022\,076\,650\,012\,130\,064\,226\,837\,158\,203 \setminus$
 $125\,z^8 -$
 $245\,158\,102\,934\,475\,621\,588\,720\,737\,350\,126\,855\,222\,640\,626\,413\,210\,252\,772\,627\,352\,050\,781\,250\,000$
 $z^9 +$
 $395\,141\,149\,943\,474\,384\,086\,541\,487\,309\,901\,808\,801\,450\,907\,453\,130\,982\,006\,914\,013\,671\,875\,000\,z^{10} -$
 $534\,675\,536\,343\,861\,900\,773\,943\,400\,121\,707\,653\,199\,032\,965\,353\,920\,988\,934\,511\,718\,750\,000\,z^{11} +$
 $610\,408\,680\,346\,008\,995\,559\,084\,191\,546\,226\,619\,764\,247\,071\,897\,049\,596\,575\,195\,312\,500\,z^{12} -$
 $589\,363\,193\,243\,490\,285\,228\,380\,696\,749\,499\,045\,880\,940\,357\,279\,577\,349\,218\,750\,000\,z^{13} +$
 $481\,268\,800\,440\,643\,409\,158\,707\,973\,922\,718\,780\,320\,349\,080\,587\,281\,171\,875\,000\,z^{14} -$
 $331\,602\,009\,631\,729\,919\,433\,495\,764\,100\,455\,383\,505\,298\,260\,194\,281\,250\,000\,z^{15} +$
 $191\,838\,742\,991\,889\,260\,376\,287\,186\,647\,873\,491\,989\,258\,474\,983\,203\,125\,z^{16} -$
 $92\,446\,978\,692\,855\,823\,629\,707\,816\,684\,644\,176\,221\,635\,105\,625\,000\,z^{17} +$
 $36\,673\,710\,264\,764\,661\,364\,347\,661\,637\,419\,565\,379\,970\,187\,500\,z^{18} -$
 $11\,772\,852\,602\,784\,092\,591\,451\,221\,404\,911\,955\,573\,975\,000\,z^{19} +$
 $2\,982\,568\,473\,064\,831\,329\,440\,330\,772\,506\,936\,613\,750\,z^{20} -$
 $574\,072\,646\,933\,509\,157\,103\,499\,538\,022\,699\,000\,z^{21} + 78\,900\,603\,628\,779\,519\,972\,988\,584\,750\,300\,z^{22} -$
 $6\,898\,845\,416\,722\,982\,294\,850\,842\,760\,z^{23} + 288\,421\,901\,218\,258\,609\,913\,307\,z^{24}, 1]$

(* Its numerical value is indeed *)

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In[*]:= N[Root[
  .. [Nullstelle
    - 1 786 154 209 802 451 002 972 759 427 968 487 151 452 029 651 728 799 407 639 295 635 068 397 082 \
      203 209 400 177 001 953 125 -
    609 669 402 283 538 940 064 962 351 778 105 590 416 586 800 600 947 854 626 322 974 747 244 589 \
      514 636 993 408 203 125 000 z +
    14 065 049 508 440 152 464 640 540 540 273 080 874 274 960 792 390 609 169 876 582 299 521 406 218 \
      557 357 788 085 937 500 z2 -
    151 644 396 136 269 517 420 537 591 671 668 698 662 291 198 974 943 286 974 771 385 459 752 798 \
      534 393 310 546 875 000 z3 +
    1 034 230 604 547 796 776 894 311 813 388 800 222 396 217 413 745 783 240 608 643 078 907 588 289 \
      451 599 121 093 750 z4 -
    5 024 914 765 274 801 563 771 944 009 102 429 931 695 283 939 155 677 138 996 369 663 362 725 677 \
      490 234 375 000 z5 +
    18 525 952 105 958 034 810 709 332 606 630 760 338 002 035 331 088 431 649 304 996 067 017 868 041 \
      992 187 500 z6 -
    53 864 412 863 332 279 824 097 453 356 521 821 046 949 689 890 166 470 307 609 262 109 185 791 015 \
      625 000 z7 +
    126 674 139 176 457 908 238 718 970 995 577 561 835 058 163 022 076 650 012 130 064 226 837 158 \
      203 125 z8 -
    245 158 102 934 475 621 588 720 737 350 126 855 222 640 626 413 210 252 772 627 352 050 781 250 000
      z9 +
    395 141 149 943 474 384 086 541 487 309 901 808 801 450 907 453 130 982 006 914 013 671 875 000
      z10 -
    534 675 536 343 861 900 773 943 400 121 707 653 199 032 965 353 920 988 934 511 718 750 000 z11 +
    610 408 680 346 008 995 559 084 191 546 226 619 764 247 071 897 049 596 575 195 312 500 z12 -
    589 363 193 243 490 285 228 380 696 749 499 045 880 940 357 279 577 349 218 750 000 z13 +
    481 268 800 440 643 409 158 707 973 922 718 780 320 349 080 587 281 171 875 000 z14 -
    331 602 009 631 729 919 433 495 764 100 455 383 505 298 260 194 281 250 000 z15 +
    191 838 742 991 889 260 376 287 186 647 873 491 989 258 474 983 203 125 z16 -
    92 446 978 692 855 823 629 707 816 684 644 176 221 635 105 625 000 z17 +
    36 673 710 264 764 661 364 347 661 637 419 565 379 970 187 500 z18 -
    11 772 852 602 784 092 591 451 221 404 911 955 573 975 000 z19 +
    2 982 568 473 064 831 329 440 330 772 506 936 613 750 z20 -
    574 072 646 933 509 157 103 499 538 022 699 000 z21 +
    78 900 603 628 779 519 972 988 584 750 300 z22 -
    6 898 845 416 722 982 294 850 842 760 z23 + 288 421 901 218 258 609 913 307 z24, 1], 99]

Out[*]=
- 2.7499750049987503498950329892785762809233507282026929115815059802012292838271106300 \
  1314238074700799

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