(* Example 2.6 related to Proposition 2.5 in [6] *)

(* The reduced relation curve (w.r.t. α and β) for n = n_0 = 7 is H7 = $H^7_{m(7)}(\alpha,\beta) = H^7_{12}(\alpha,\beta)$ with *)

In[∘]:=
```
H7 = 4096 - 2048 α^2 - 8448 α^4 + 7424 α^6 - 1040 α^8 + 184 α^10 + α^12 + 12 288 α β -
    17 408 α^3 β + 1536 α^5 β + 4992 α^7 β - 144 α^9 β + 12 α^11 β - 10 240 β^2 +
    25 088 α^2 β^2 - 8448 α^4 β^2 - 4288 α^6 β^2 + 1976 α^8 β^2 - 118 α^10 β^2 -
    13 312 α β^3 + 33 792 α^3 β^3 - 10 624 α^5 β^3 - 4032 α^7 β^3 + 364 α^9 β^3 +
    14 080 β^4 - 29 952 α^2 β^4 + 23 712 α^4 β^4 - 3472 α^6 β^4 - 441 α^8 β^4 + 5632 α β^5 -
    19 328 α^3 β^5 + 11 680 α^5 β^5 - 168 α^7 β^5 - 9984 β^6 + 10 560 α^2 β^6 - 7632 α^4 β^6 +
    1260 α^6 β^6 - 5760 α β^7 + 3648 α^3 β^7 - 1800 α^5 β^7 + 1776 β^8 - 3624 α^2 β^8 +
    1311 α^4 β^8 + 1136 α β^9 - 484 α^3 β^9 + 280 β^10 + 42 α^2 β^10 + 28 α β^11 - 7 β^12;
```

(* Formula (35) in [6] then reads, with ν_7 = 1+2 $\left(\mathrm{Tan}\left[\frac{\pi}{14}\right]\right)^2$ *)

In[∘]:=
$$\mathrm{Reduce}\left[H7 == 0 \land 1 < \alpha < \beta \land 1 + 2\left(\mathrm{Tan}\left[\frac{\pi}{14}\right]\right)^2 < \beta,\ \{\beta\}\right]$$

reduziere

$$\alpha > 1\ \&\&\ \Big(\beta == \mathrm{Root}\big[-4096 + 2048\,\alpha^2 + 8448\,\alpha^4 - 7424\,\alpha^6 + 1040\,\alpha^8 - 184\,\alpha^{10} - \alpha^{12} + (-12\,288\,\alpha + 17\,408\,\alpha^3 - 1536\,\alpha^5 - 4992\,\alpha^7 + 144\,\alpha^9 - 12\,\alpha^{11})\,\#1 + (10\,240 - 25\,088\,\alpha^2 + 8448\,\alpha^4 + 4288\,\alpha^6 - 1976\,\alpha^8 + 118\,\alpha^{10})\,\#1^2 + (13\,312\,\alpha - 33\,792\,\alpha^3 + 10\,624\,\alpha^5 + 4032\,\alpha^7 - 364\,\alpha^9)\,\#1^3 + (-14\,080 + 29\,952\,\alpha^2 - 23\,712\,\alpha^4 + 3472\,\alpha^6 + 441\,\alpha^8)\,\#1^4 + (-5632\,\alpha + 19\,328\,\alpha^3 - 11\,680\,\alpha^5 + 168\,\alpha^7)\,\#1^5 + (9984 - 10\,560\,\alpha^2 + 7632\,\alpha^4 - 1260\,\alpha^6)\,\#1^6 + (5760\,\alpha - 3648\,\alpha^3 + 1800\,\alpha^5)\,\#1^7 + (-1776 + 3624\,\alpha^2 - 1311\,\alpha^4)\,\#1^8 + (-1136\,\alpha + 484\,\alpha^3)\,\#1^9 + (-280 - 42\,\alpha^2)\,\#1^{10} - 28\,\alpha\,\#1^{11} + 7\,\#1^{12}\ \&,\ 4\big]\ ||$$

Nullstelle

$$\beta == \mathrm{Root}\big[-4096 + 2048\,\alpha^2 + 8448\,\alpha^4 - 7424\,\alpha^6 + 1040\,\alpha^8 - 184\,\alpha^{10} - \alpha^{12} + (-12\,288\,\alpha + 17\,408\,\alpha^3 - 1536\,\alpha^5 - 4992\,\alpha^7 + 144\,\alpha^9 - 12\,\alpha^{11})\,\#1 + (10\,240 - 25\,088\,\alpha^2 + 8448\,\alpha^4 + 4288\,\alpha^6 - 1976\,\alpha^8 + 118\,\alpha^{10})\,\#1^2 + (13\,312\,\alpha - 33\,792\,\alpha^3 + 10\,624\,\alpha^5 + 4032\,\alpha^7 - 364\,\alpha^9)\,\#1^3 + (-14\,080 + 29\,952\,\alpha^2 - 23\,712\,\alpha^4 + 3472\,\alpha^6 + 441\,\alpha^8)\,\#1^4 + (-5632\,\alpha + 19\,328\,\alpha^3 - 11\,680\,\alpha^5 + 168\,\alpha^7)\,\#1^5 + (9984 - 10\,560\,\alpha^2 + 7632\,\alpha^4 - 1260\,\alpha^6)\,\#1^6 + (5760\,\alpha - 3648\,\alpha^3 + 1800\,\alpha^5)\,\#1^7 + (-1776 + 3624\,\alpha^2 - 1311\,\alpha^4)\,\#1^8 + (-1136\,\alpha + 484\,\alpha^3)\,\#1^9 + (-280 - 42\,\alpha^2)\,\#1^{10} - 28\,\alpha\,\#1^{11} + 7\,\#1^{12}\ \&,\ 5\big]\ ||$$

Nullstelle

$$\beta == \mathrm{Root}\big[-4096 + 2048\,\alpha^2 + 8448\,\alpha^4 - 7424\,\alpha^6 + 1040\,\alpha^8 - 184\,\alpha^{10} - \alpha^{12} + (-12\,288\,\alpha + 17\,408\,\alpha^3 - 1536\,\alpha^5 - 4992\,\alpha^7 + 144\,\alpha^9 - 12\,\alpha^{11})\,\#1 + (10\,240 - 25\,088\,\alpha^2 + 8448\,\alpha^4 + 4288\,\alpha^6 - 1976\,\alpha^8 + 118\,\alpha^{10})\,\#1^2 + (13\,312\,\alpha - 33\,792\,\alpha^3 + 10\,624\,\alpha^5 + 4032\,\alpha^7 - 364\,\alpha^9)\,\#1^3 + (-14\,080 + 29\,952\,\alpha^2 - 23\,712\,\alpha^4 + 3472\,\alpha^6 + 441\,\alpha^8)\,\#1^4 + (-5632\,\alpha + 19\,328\,\alpha^3 - 11\,680\,\alpha^5 + 168\,\alpha^7)\,\#1^5 + (9984 - 10\,560\,\alpha^2 + 7632\,\alpha^4 - 1260\,\alpha^6)\,\#1^6 + (5760\,\alpha - 3648\,\alpha^3 + 1800\,\alpha^5)\,\#1^7 + (-1776 + 3624\,\alpha^2 - 1311\,\alpha^4)\,\#1^8 + (-1136\,\alpha + 484\,\alpha^3)\,\#1^9 + (-280 - 42\,\alpha^2)\,\#1^{10} - 28\,\alpha\,\#1^{11} + 7\,\#1^{12}\ \&,\ 6\big]\Big)$$

```
(* Thus Formula (35) yields as possible solutions for β three parametric root
 objects (with parameter α) with indexes (marked in red) 4 or 5 or 6 respectively *)

(* Hence the lowest index is λ7 = 4; call the corresponding solution β =
 β(α,λ7) = β(α,4) = Root[-4096+2048 α^2+8448 α^4-7424 α^6+1040 α^8-
                        Nullstelle
   184 α^10-α^12+(-12288 α+17408 α^3-1536 α^5-4992 α^7+144 α^9-12 α^11) #1+
   (10240-25088 α^2+8448 α^4+4288 α^6-1976 α^8+118 α^10) #1^2+
   (13312 α-33792 α^3+10624 α^5+4032 α^7-364 α^9) #1^3+
   (-14080+29952 α^2-23712 α^4+3472 α^6+441 α^8) #1^4+
   (-5632 α+19328 α^3-11680 α^5+168 α^7) #1^5+(9984-10560 α^2+7632 α^4-1260 α^6) #1^6+
   (5760 α-3648 α^3+1800 α^5) #1^7+(-1776+3624 α^2-1311 α^4) #1^8+
   (-1136 α+484 α^3) #1^9+(-280-42 α^2) #1^10-28 α #1^11+7 #1^12&,4] *)

(* Next we need the given value s = s0 = 2 and the Formula for s7 = s7 = s7(α,β) *)
```

```
In[◦]:= s7 = -((10 α^9 - 25 α^8 β - 16 α^7 (11 + 2 β^2) + 4 α^6 β (-100 + 41 β^2) + 10 α^4 β (80 - 8 β^2 + β^4) - 4 α^5
        (32 - 148 β^2 + 43 β^4) + 16 α^3 (48 - 56 β^2 + 23 β^4 + 5 β^6) - 4 α^2 β (64 + 16 β^2 - 20 β^4 + 7 β^6) +
        β (-256 + 256 β^2 + 32 β^4 - 112 β^6 + 7 β^8) - 2 α (256 - 128 β^2 - 128 β^4 + 136 β^6 + 7 β^8)) /
      (7 (256 - 3 α^8 + 8 α^7 β - 512 β^2 + 160 β^4 + 64 β^6 - 3 β^8 - 24 α^5 β (-8 + 3 β^2) +
        4 α^6 (16 + 3 β^2) + 8 α β^3 (-48 + 24 β^2 + β^4) - 8 α^3 β (48 - 16 β^2 + 9 β^4) +
        10 α^4 (16 - 32 β^2 + 11 β^4) + 4 α^2 (-128 + 112 β^2 - 80 β^4 + 3 β^6))));
```

```
(* Formula (34) in [6] then reads  *)
```

```
In[◦]:= RootReduce[
        reduziere auf Nullstelle
         Reduce[s7 - 2 == 0 ∧ β == Root[-4096 + 2048 α^2 + 8448 α^4 - 7424 α^6 + 1040 α^8 - 184 α^10 - α^12 +
         reduziere             Nullstelle
             (-12 288 α + 17 408 α^3 - 1536 α^5 - 4992 α^7 + 144 α^9 - 12 α^11) #1 +
             (10 240 - 25 088 α^2 + 8448 α^4 + 4288 α^6 - 1976 α^8 + 118 α^10) #1^2 +
             (13 312 α - 33 792 α^3 + 10 624 α^5 + 4032 α^7 - 364 α^9) #1^3 +
             (-14 080 + 29 952 α^2 - 23 712 α^4 + 3472 α^6 + 441 α^8) #1^4 +
             (-5632 α + 19 328 α^3 - 11 680 α^5 + 168 α^7) #1^5 + (9984 - 10 560 α^2 + 7632 α^4 - 1260 α^6) #1^6 +
             (5760 α - 3648 α^3 + 1800 α^5) #1^7 + (-1776 + 3624 α^2 - 1311 α^4) #1^8 +
             (-1136 α + 484 α^3) #1^9 + (-280 - 42 α^2) #1^10 - 28 α #1^11 + 7 #1^12 &, 4] ∧
           1 < α, {α, β}, Reals, Backsubstitution → True]]
                          Menge reeller Zahlen         wahr

Out[◦]=
   α == 14.0… && β == 14.0…
```

```
(* or, in the traditional form *)

α = Root[1 596 640 261 888 + 1 382 185 254 912 z - 2 943 026 759 680 z^2 + 1 110 163 950 336 z^3 +
     Nullstelle
   433 952 450 240 z^4 - 579 218 113 600 z^5 + 268 614 534 928 z^6 - 73 851 260 224 z^7 +
   13 191 160 848 z^8 - 1 542 461 200 z^9 + 113 418 000 z^10 - 4 725 000 z^11 + 84 375 z^12, 6]

(* and *)
```

$$\beta = \text{Root}\left[614\,128\,742\,778\,112 - 378\,933\,592\,095\,744\, z - 107\,722\,313\,416\,704\, z^2 + 365\,630\,011\,835\,648\, z^3 - 320\,300\,886\,075\,520\, z^4 + 150\,022\,279\,642\,816\, z^5 - 39\,835\,757\,959\,760\, z^6 + 5\,296\,218\,682\,048\, z^7 - 1\,384\,872\,496\, z^8 - 114\,339\,324\,400\, z^9 + 17\,574\,291\,000\, z^{10} - 1\,157\,625\,000\, z^{11} + 28\,940\,625\, z^{12}, 6\right]$$

Nullstelle

(* Compare with Formula (33) in [6] *)

(* Consider now the limiting behaviour of the three parametric root objects β above when α tends to the left boundary of the interval $(1,\infty)$ *)

In[∘]:=

$$\text{Limit}\Big[\text{Root}\big[-4096 + 2048\,\alpha^2 + 8448\,\alpha^4 - 7424\,\alpha^6 + 1040\,\alpha^8 - 184\,\alpha^{10} - \alpha^{12} + (-12\,288\,\alpha + 17\,408\,\alpha^3 - 1536\,\alpha^5 - 4992\,\alpha^7 + 144\,\alpha^9 - 12\,\alpha^{11})\,\#1 + (10\,240 - 25\,088\,\alpha^2 + 8448\,\alpha^4 + 4288\,\alpha^6 - 1976\,\alpha^8 + 118\,\alpha^{10})\,\#1^2 + (13\,312\,\alpha - 33\,792\,\alpha^3 + 10\,624\,\alpha^5 + 4032\,\alpha^7 - 364\,\alpha^9)\,\#1^3 + (-14\,080 + 29\,952\,\alpha^2 - 23\,712\,\alpha^4 + 3472\,\alpha^6 + 441\,\alpha^8)\,\#1^4 + (-5632\,\alpha + 19\,328\,\alpha^3 - 11\,680\,\alpha^5 + 168\,\alpha^7)\,\#1^5 + (9984 - 10\,560\,\alpha^2 + 7632\,\alpha^4 - 1260\,\alpha^6)\,\#1^6 + (5760\,\alpha - 3648\,\alpha^3 + 1800\,\alpha^5)\,\#1^7 + (-1776 + 3624\,\alpha^2 - 1311\,\alpha^4)\,\#1^8 + (-1136\,\alpha + 484\,\alpha^3)\,\#1^9 + (-280 - 42\,\alpha^2)\,\#1^{10} - 28\,\alpha\,\#1^{11} + 7\,\#1^{12}\,\&, 4\big], \alpha \to 1\Big]$$

Gren··· Nullstelle

Out[∘]=

1.10...

(* or *)

$$\text{Root}\left[-169 + 245\,z - 91\,z^2 + 7\,z^3, 1\right]$$

Nullstelle

(* or *)

1.104190167203374061295309543958639216307467987430470069039684958853843179823239744164247184390967144484857597950055 4 ...

(* with *)

$$\text{RootReduce}\left[\text{Root}\left[-169 + 245\,z - 91\,z^2 + 7\,z^3, 1\right] - \left(1 + 2\left(\text{Tan}\left[\frac{\pi}{14}\right]\right)^2\right)\right]$$

reduziere au··· Nullstelle

Out[∘]=

0

(* On the other hand,

schalte Benachrichtigung ein

for the parametric root objects $\beta = \beta(\alpha,5)$ and $\beta = \beta(\alpha,6)$ one gets *)

In[•]:=
```
Limit[Root[-4096 + 2048 α^2 + 8448 α^4 - 7424 α^6 + 1040 α^8 -
    184 α^10 - α^12 + (-12 288 α + 17 408 α^3 - 1536 α^5 - 4992 α^7 + 144 α^9 - 12 α^11) #1 +
    (10 240 - 25 088 α^2 + 8448 α^4 + 4288 α^6 - 1976 α^8 + 118 α^10) #1^2 +
    (13 312 α - 33 792 α^3 + 10 624 α^5 + 4032 α^7 - 364 α^9) #1^3 +
    (-14 080 + 29 952 α^2 - 23 712 α^4 + 3472 α^6 + 441 α^8) #1^4 +
    (-5632 α + 19 328 α^3 - 11 680 α^5 + 168 α^7) #1^5 + (9984 - 10 560 α^2 + 7632 α^4 - 1260 α^6) #1^6 +
    (5760 α - 3648 α^3 + 1800 α^5) #1^7 + (-1776 + 3624 α^2 - 1311 α^4) #1^8 +
    (-1136 α + 484 α^3) #1^9 + (-280 - 42 α^2) #1^10 - 28 α #1^11 + 7 #1^12 &, 5], α → 1]
```
Gren··· Nullstelle

Out[•]=
```
2.27...

(* or *)

Root[-169 + 245 z - 91 z^2 + 7 z^3, 2]
```
Nullstelle

```
(* or *)

2.27192761195117179633643425600496153242288944359887539202394635224755173368324131948
  54751713308999214524517762510322 ...

(* respectively *)
```

In[•]:=
```
Limit[Root[-4096 + 2048 α^2 + 8448 α^4 - 7424 α^6 + 1040 α^8 -
    184 α^10 - α^12 + (-12 288 α + 17 408 α^3 - 1536 α^5 - 4992 α^7 + 144 α^9 - 12 α^11) #1 +
    (10 240 - 25 088 α^2 + 8448 α^4 + 4288 α^6 - 1976 α^8 + 118 α^10) #1^2 +
    (13 312 α - 33 792 α^3 + 10 624 α^5 + 4032 α^7 - 364 α^9) #1^3 +
    (-14 080 + 29 952 α^2 - 23 712 α^4 + 3472 α^6 + 441 α^8) #1^4 +
    (-5632 α + 19 328 α^3 - 11 680 α^5 + 168 α^7) #1^5 + (9984 - 10 560 α^2 + 7632 α^4 - 1260 α^6) #1^6 +
    (5760 α - 3648 α^3 + 1800 α^5) #1^7 + (-1776 + 3624 α^2 - 1311 α^4) #1^8 +
    (-1136 α + 484 α^3) #1^9 + (-280 - 42 α^2) #1^10 - 28 α #1^11 + 7 #1^12 &, 6], α → 1]
```
Gren··· Nullstelle

Out[•]=
```
9.62...

(* or *)

Root[-169 + 245 z - 91 z^2 + 7 z^3, 3]
```
Nullstelle

```
(* or *)

9.62388222084545414236825620003639925126964256897065453893636868889860508649351893635
  02776442781329340626906257989124 ...

(* That is, only the parametric root object β =
 β(α,λ7) = β(α,4) yields for the limiting value α =
   1 the corresponding limiting value β = ν7 = 1+2 (Tan[π/14])^2 *)

(* Observe that for n = n0 = 7 and the given value s =
   s0 = 1/10 (see Example 2.1 in [6]) the Formula (34) would yield *)
```

In[∘]:=
```
RootReduce[
reduziere auf Nullstelle
 Reduce[s7 - 1 / 10 == 0 ∧ β == Root[-4096 + 2048 α^2 + 8448 α^4 - 7424 α^6 + 1040 α^8 - 184 α^10 -
 reduziere                      Nullstelle
      α^12 + (-12 288 α + 17 408 α^3 - 1536 α^5 - 4992 α^7 + 144 α^9 - 12 α^11) #1 +
      (10 240 - 25 088 α^2 + 8448 α^4 + 4288 α^6 - 1976 α^8 + 118 α^10) #1^2 +
      (13 312 α - 33 792 α^3 + 10 624 α^5 + 4032 α^7 - 364 α^9) #1^3 +
      (-14 080 + 29 952 α^2 - 23 712 α^4 + 3472 α^6 + 441 α^8) #1^4 +
      (-5632 α + 19 328 α^3 - 11 680 α^5 + 168 α^7) #1^5 + (9984 - 10 560 α^2 + 7632 α^4 - 1260 α^6) #1^6 +
      (5760 α - 3648 α^3 + 1800 α^5) #1^7 + (-1776 + 3624 α^2 - 1311 α^4) #1^8 +
      (-1136 α + 484 α^3) #1^9 + (-280 - 42 α^2) #1^10 - 28 α #1^11 + 7 #1^12 &, 4] ∧
   1 < α, {α, β}, Reals, Backsubstitution → True]]
                  Menge reeller Zahlen          wahr
```

Out[∘]=
```
α == 1.20… && β == 1.24…
```

```
(* with *)
```

In[∘]:=
```
N[1.20…, 99]
numerischer Wert
```

Out[∘]=
```
1.19866143766542254932939884701978141296580144417618473722028375167019738380883552338·
 261653152864119
```

In[∘]:=
```
N[1.24…, 99]
numerischer Wert
```

Out[∘]=
```
1.23981047745568101332620951741403068757993433344465394732522907834912048074673524557·
 699952276657319
```

```
(* The traditional form of these two root objects is,
compare with Example 2.1 in [6] *)
```

```
α = Root[3 914 891 670 199 951 + 8 878 474 066 953 960 z -
    Nullstelle
   19 010 729 822 984 200 z^2 - 39 116 783 113 716 000 z^3 - 18 506 663 976 550 000 z^4 +
   110 183 085 863 200 000 z^5 + 240 031 304 644 000 000 z^6 - 381 749 266 640 000 000 z^7 -
   275 006 257 500 000 000 z^8 + 534 908 500 000 000 000 z^9 - 12 712 500 000 000 000 z^10 -
   236 250 000 000 000 000 z^11 + 84 375 000 000 000 000 z^12, 4]
```

```
β = Root[39 002 486 124 484 801 - 1 096 930 547 763 975 960 z +
    Nullstelle
   31 622 160 919 837 021 800 z^2 - 55 221 804 013 946 948 000 z^3 - 93 502 902 628 078 090 000 z^4 +
   233 928 978 169 863 200 000 z^5 + 67 781 581 288 876 000 000 z^6 - 352 068 831 333 520 000 000 z^7 +
   51 271 674 177 500 000 000 z^8 + 233 097 140 500 000 000 000 z^9 - 85 394 137 500 000 000 000 z^10 -
   57 881 250 000 000 000 000 z^11 + 28 940 625 000 000 000 000 z^12, 3]
```