Solution sets of systems of equations over finite algebras

Endre Tóth, Tamás Waldhauser
University of Szeged, Szeged, Hungary

Let K be a field. If we have a system of linear equations (with n unknowns) over K, then the set $T \subseteq K^n$ of all solutions is an affine subspace in K^n, that is, T is closed under affine combinations. On the other hand, if $T \subseteq K^n$ is closed under affine combinations, then there exists a system of linear equations (with n unknowns) such that T is the set of solutions of this system of equations. A similar statement holds for systems of homogeneous linear equations and subspaces.

We would like to generalize these results to arbitrary algebras by characterizing solution sets of systems of equations via closure conditions. Let A be a nonempty set and let F be a set of operations on A. Assume that we are allowed to use operations from F in our equations, like the operations ax, $x + y$ and the constants in linear equations. Since we can use these operations several times, we can write composite operations like $a_1x_1 + \cdots + a_nx_n + c$. This means that every equation can be rewritten as $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$, where f and g are obtained as compositions of operations from F. The set of all such composite functions is denoted by $[F]$, and it is called the *clone* generated by F. Then we say that our equations are over the clone $C = [F]$, and the system of equations is a system of equations over C. This definition coincides with the definition of systems of equations over arbitrary algebras: notice, that if we have an algebra $\mathbf{A} = (A, F)$ with carrier set A and operations F, then the systems of equations over $\mathbf{A}$ are exactly the systems of equations over $[F]$.

Our main problem is the following: given a clone $C = [F]$, characterize sets $T \subseteq A^n$ that can appear as the set of all solutions of a system of equations over C. For every clone C there exists a clone, called the *centralizer* of C (denoted as C^*), such that the set of solutions of each system of equations over C is closed under C^*. It is natural to ask the following question. If a set $T \subseteq A^n$ is closed under C^* (for some clone C over A), then is there a system of equations such that T is its set of solutions? We say that C has Property (SSCC) if the answer to this question is positive. (SSCC stands for "Solution Set $\equiv$ Closed under Centralizer".) We found an example, which shows that Property (SSCC) does not hold in general, and it seems that it is not a trivial problem to characterize clones with Property (SSCC).

Our main results are the following. We proved that the only clone that can characterize sets of solutions of systems of equations over a clone C is the centralizer of C. Therefore if Property (SSCC) does not hold for some clone, then the set of solutions of systems of equations over this clone can not be characterized via closure conditions. Then we proved that for clones of Boolean functions (i.e., for clones over the two-element set), Property (SSCC) always holds. After this, we determined which lattices and semilattices have Property (SSCC). For lattices Property (SSCC) holds if and only if the lattice is a Boolean lattice, and for semilattices, Property (SSCC) holds if and only if the semilattice with the partial order defined by its meet operation is a distributive lattice.