Topological loops with solvable multiplication groups

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A set L with a binary operation $(x, y) \mapsto x \cdot y : L \times L \to L$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution, which we denote by $y = a \backslash b$ and $x = b/a$. A loop L is proper if it is not a group. The left and right translations $\lambda_a = y \mapsto a \cdot y : L \to L$ and $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$, are permutations of L. The permutation group $Mult(L) = \langle \lambda_a, \rho_a;\ a \in L \rangle$ is called the multiplication group of L. The stabilizer of the identity element $e \in L$ in $Mult(L)$ is called the inner mapping group $Inn(L)$ of L. T. Kepka and M. Niemenmaa gave a purely group theoretical criterion for a group K to be the group $Mult(L)$ of a loop L (cf. [2]): A group K is isomorphic to the multiplication group of a loop L if and only if there exist a subgroup S such that the core of S in K is trivial and left transversals A, B to S in K such that for every $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in S$ and K is generated by $A \cup B$. In this case the subgroup S is the group $Inn(L)$ of L and the transversals A and B correspond to the sets of left and right translations of L, respectively.

A loop L is called topological if L is a topological space, the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \backslash y, (x, y) \mapsto y/x : L \times L \to L$ are continuous. There is a bijection between connected topological loops L having a Lie group G topologically generated by the left translations of L and the triples (G, H, σ), where G is a connected Lie group, H is a closed subgroup of G such that the core of H in G is trivial and $\sigma : G/H \to G$ is a continuous sharply transitive section such that $\sigma(H) = 1 \in G$ and the set $\sigma(G/H)$ generates G. A section $\sigma : G/H \to G$ is called sharply transitive, if the set $\sigma(G/H)$ operates sharply transitively on G/H, i.e. for any xH and yH there exists precisely one $z \in \sigma(G/H)$ with $zxH = yH$. The loop L is defined on the homogeneous space G/H with the multiplication $xH \cdot yH = \sigma(xH)yH$ (cf. [1]).

If L is a connected topological loop having a Lie group as the group of its left translations, then in general the multiplication group $Mult(L)$ of L is a differentiable transformation group of infinite dimension. The condition that the group $Mult(L)$ is a (finite dimensional) Lie group gives a strong restriction for the group $Mult(L)$ and also for the loop L: For every proper 1-dimensional topological loop L the multiplication group $Mult(L)$ has infinite dimension. Only the elementary filiform Lie groups $\mathcal{F}_n$, $n \geq 4$, are the multiplication groups $Mult(L)$ of 2-dimensional connected simply connected topological loops L.

In this talk we wish to describe the structure of the solvable Lie groups which are the multiplication groups $Mult(L)$ for three-dimensional topological loops L. We use this result for the determination of solvable Lie groups of dimension ≤ 6 which occur as the group $Mult(L)$ of a three-dimensional loop L.

[1] P. T. Nagy, K. Strambach, *Loops in group theory and Lie theory*, de Gruyter Expositions in Mathematics. 35. Berlin, New York, 2002.

[2] M. Niemenmaa, T. Kepka, On Multiplication groups of loops, *J. Algebra* **135** (1990), 112-122.