írásbelik: dec 20, 15:00, jan 4, 14:00, jan 17, 14:00

DEF. $X \subseteq \mathbb{Q}$ Dedekind-szelet ($X \in \mathcal{R}$), ha

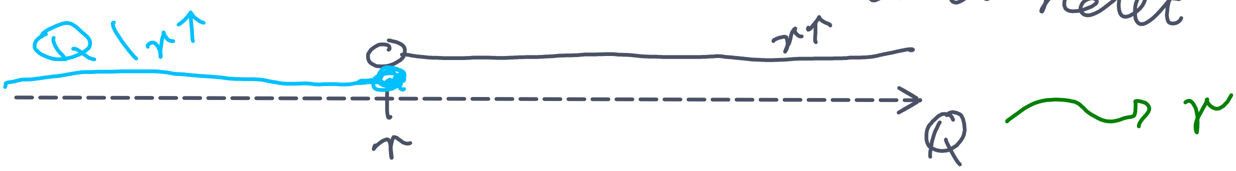
(NURH) $X \neq \emptyset$ és $X \neq \mathbb{Q}$

(FSZ) $\forall x \in X \; \forall r \in \mathbb{Q}: \; r > x \Rightarrow r \in X$

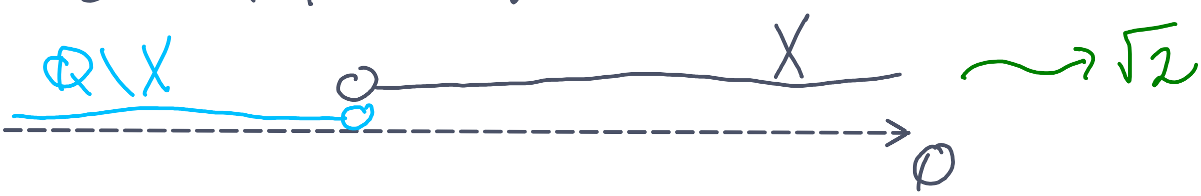
(NLK) $\forall x \in X \; \exists x' \in X: \; x' < x$

PÉLDÁK

1) $r \in \mathbb{Q}$ $\quad r^{\uparrow} = \{x \in \mathbb{Q} \mid x > r\}$ racionális szelet



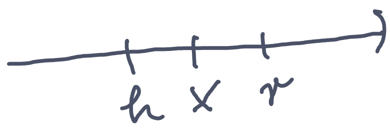
2) $X = \{x \in \mathbb{Q}^{+} \mid x^2 > 2\}$

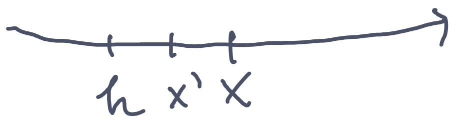


DEF $H \subseteq \mathbb{Q}$ $\qquad H^{\uparrow} = \{x \in \mathbb{Q} \mid \exists h \in H: \; x > h\} =$

$= \{h + \varepsilon \mid h \in H, \; \varepsilon \in \mathbb{Q}^{+}\}$

HF $\forall H \subseteq \mathbb{Q}: H^{\uparrow}$ (FSZ) és (NLK)





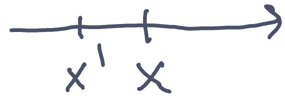
$H^{\uparrow} = \emptyset \iff H = \emptyset$

$H^{\uparrow} = \mathbb{Q} \iff H$ nem korlátos alulról

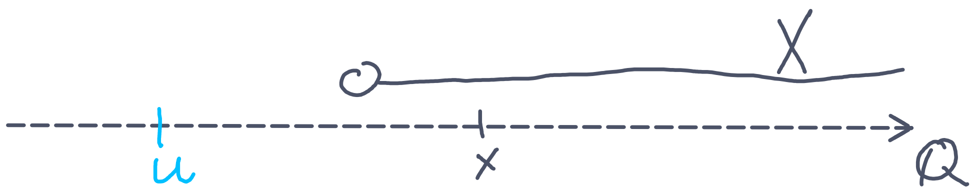
HF $\forall X \subsetneq \mathbb{Q},\ X \neq \emptyset : X \in \mathbb{R} \Longleftrightarrow X = X^{\uparrow}$

$\Longleftarrow$ ✓

$\Longrightarrow$

$X \subseteq X^{\uparrow}$  (NLK)

$X^{\uparrow} \subseteq X$  (FSZ)



$x \in X,\ u \notin X \Longrightarrow u < x$ (FSZ)

DEF $X, Y \in \mathbb{R}$ $\quad X + Y = \{x + y \mid x \in X, y \in Y\} \subseteq \mathbb{Q}$

ÁLL $\forall X, Y \in \mathbb{R} : X + Y \in \mathbb{R}$.

Biz (NVRH) $X + Y \neq \emptyset$ ✓

$X + Y \overset{?}{\neq} \mathbb{Q}$

$X \neq \mathbb{Q} \Longrightarrow \exists u \in \mathbb{Q} : u \notin X$

$Y \neq \mathbb{Q} \Longrightarrow \exists v \in \mathbb{Q} : v \notin Y$

Tfh. $u + v = \underset{X}{\underset{\cap}{x}} + \underset{Y}{\underset{\cap}{y}}$

$\left.\begin{array}{l} X \text{ (FSZ)}: u < x \\ Y \text{ (FSZ)}: v < y \end{array}\right\} \Longrightarrow u + v < x + y$ ↯

$u + v \notin X + Y$

(FSZ) $x+y \in X+Y \quad r > x+y \overset{?}{\Longrightarrow} r \in X+Y$

$$r = \underbrace{x}_{\in X} + \underbrace{(y + \underset{\underset{0}{\vee}}{\varepsilon})}_{\in Y\,(FSZ)} \in X+Y$$

(NLK) $x+y \in X+Y$

$X\ (NLK) \Longrightarrow \exists\, x' \in X : x' < x$

$$\underbrace{\underset{X}{\overset{\cap}{x'}} + \underset{Y}{\overset{\cap}{y}}}_{\in X+Y} < x+y$$

$\square$

<u>ÁLL</u> $(\mathbb{R};+)$ Abel-csop.

<u>Biz</u> $+$ asszoc., komm. ✓

egységelem: $0^{\uparrow} = \mathbb{Q}^{+}$

$$\underline{X + \mathbb{Q}^{+}} = \{x+\varepsilon \mid x \in X, \varepsilon > 0\} = X^{\uparrow} \overset{HF}{=\!=} \underline{X}$$

additív inverz: $Y := \{-u \mid u \notin X\}^{\uparrow} = \{-u+\varepsilon \mid u \notin X, \varepsilon > 0\}$

Y (NVRH): $X \neq \mathbb{R} \Longrightarrow \exists\, u \notin X \quad -u+1 \in Y \Longrightarrow Y \neq \emptyset$

$X \neq \emptyset \Longrightarrow \exists\, x \in X : -x \overset{?}{\notin} Y \quad \Longrightarrow Y \neq \mathbb{R}$

$-x = -u+\varepsilon$

$u = x+\varepsilon > x$ ↯

Y (FSZ): $Y = \{\ldots\}^{\uparrow}$ HF

Y (NLK): $Y = \{\ldots\}^{\uparrow}$ HF

Tehát $Y \in \mathbb{R}$

$X+Y \stackrel{?}{=} \mathbb{Q}^+$

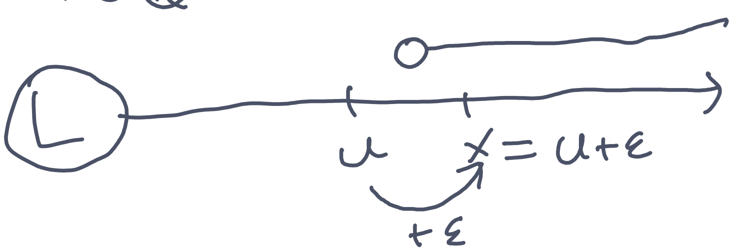
$X+Y=\{x+y \mid x\in X,\ y\in Y\}=$

$=\{x-u+\varepsilon \mid x\in X,\ u\notin X,\ \varepsilon>0\}$

$\boxed{\stackrel{?}{=}}\ \mathbb{Q}^+$

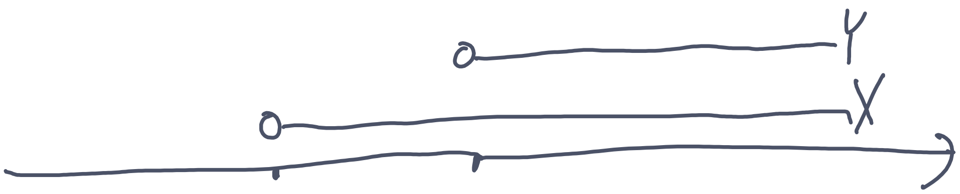
$\boxed{\subseteq}$ $\quad \underbrace{x-u}_{>0}+\underbrace{\varepsilon}_{>0}>0$

$\boxed{\supseteq}$ $\quad r\in\mathbb{Q}^+$ tetsz. legy $\varepsilon=r/2$



$X+Y \ni x-u+\varepsilon=\varepsilon+\varepsilon=2\varepsilon=r$ $\square$

$2^\uparrow\cdot(-3)^\uparrow \neq \{x\cdot y \mid x>2,\ y>-3\}=\mathbb{Q}\notin\mathbb{R}$



<u>DEF</u> $\quad X\le Y \iff Y-X\in P=\mathbb{R}^+\cup\{0^\uparrow\}$

<u>TÉTEL</u> $\quad X\le Y\iff X\supseteq Y$

<u>Biz:</u> $(\Rightarrow)$ Tfh. $X\le Y \iff Y-X=Z\in\mathbb{R}^+\cup\{0^\uparrow\}$

$Y=X+Z=\{\overbrace{x+\underbrace{z}_{>0}}^{X} \mid x\in X,\ z\in Z\}\subseteq X$

Tehát $Y\subseteq X$.

($\Leftarrow$) Tfh. $X \supseteq Y$

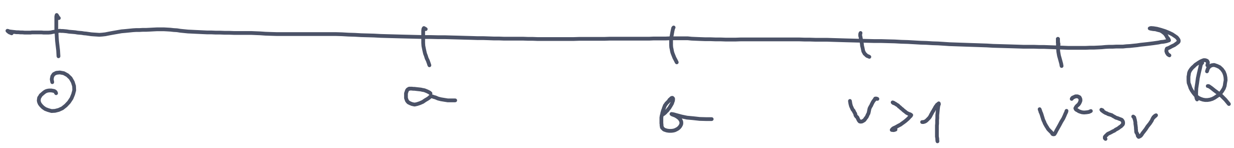
$\leq$ dichotóm: 1) $X \leq Y$ ✓

2) $Y \leq X \Rightarrow Y \supseteq X \Rightarrow X = Y$ ✓ □

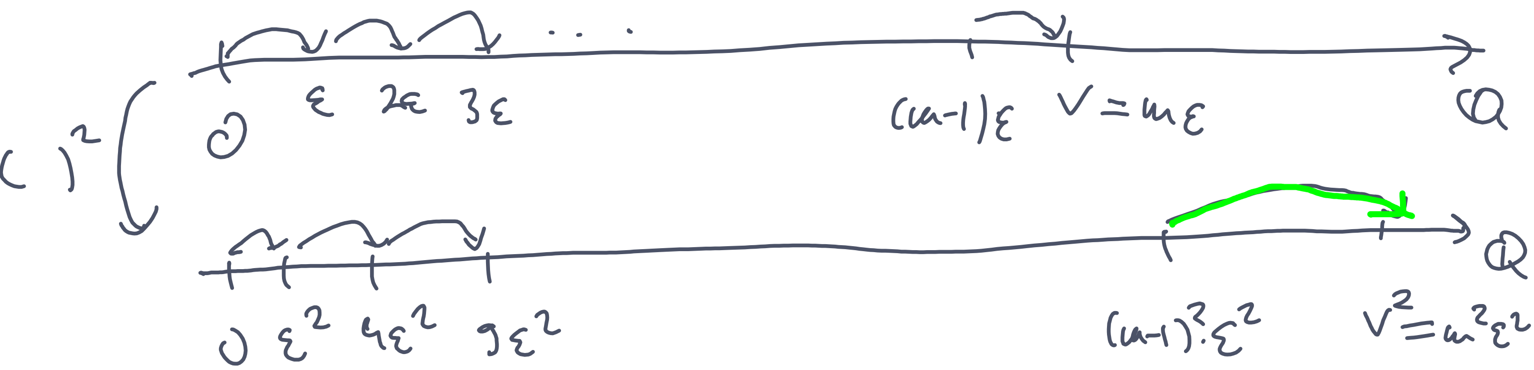
$X = \{x \in \mathbb{Q}^+ \mid x^2 > 2\}$ Cél: $X \in \mathbb{R}$ és $X \cdot X = 2^{\uparrow}$

LEMMA $\forall a, b \in \mathbb{Q}^+,\ a < b \Longrightarrow \exists\, r \in \mathbb{Q}^+ : a < r^2 < b.$

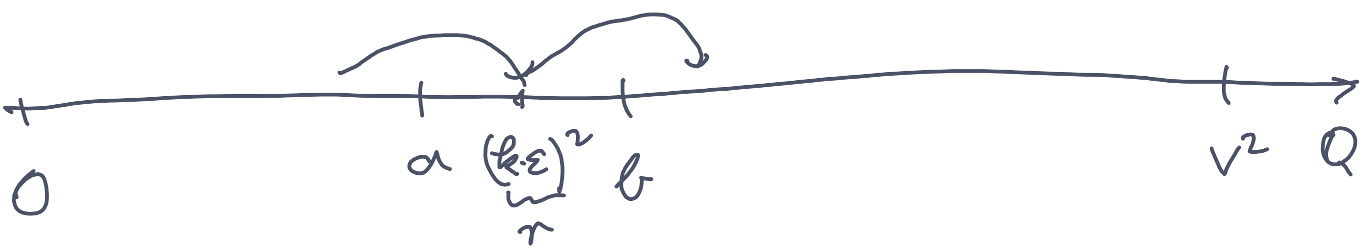
Biz:



$\varepsilon := \dfrac{v}{n}$



$: n^2 \cdot \varepsilon^2 - (n-1)^2 \cdot \varepsilon^2 = (2n-1) \cdot \varepsilon^2$



$$(2n-1) \cdot \varepsilon^2 < 2n \cdot \varepsilon^2 = 2n \frac{v^2}{n^2} = \frac{2v^2}{n} < b - a$$

$\mathbb{Q}$ ARKHIMÉDESZI $\Rightarrow \exists\, n \in \mathbb{N} : n > \dfrac{2v^2}{b-a} \in \mathbb{Q}$ □

$X = \{x \in \mathbb{Q}^+ \mid x^2 > 2\}$ Cél: $X \in \mathcal{R}$ és $X \cdot X = 2^{\uparrow}$

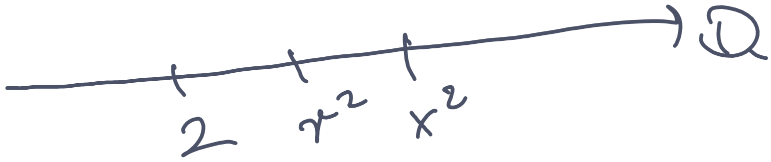
Áll $X \in \mathcal{R}^+$

Biz (NVRH): $3^2 > 2 \Rightarrow 3 \in X \Rightarrow X \neq \emptyset$

$1^2 \not> 2 \Rightarrow 1 \notin X \Rightarrow X \neq \mathbb{Q} \Rightarrow X \in \mathcal{R}^+$

(FZA): $\left.\begin{array}{l} x^2 > 2 \\ r > x > 0 \end{array}\right\} \Rightarrow r^2 > x^2 > 2 \Rightarrow r^2 > 2 \Rightarrow r \in X$

(NLK): $x^2 > 2$



(L) $\exists r \in \mathbb{Q}^+ : \underbrace{2 < r^2}_{r \in X} \underbrace{< x^2}_{r < x}$

$x' := r$ jó lesz. □

Áll $X \cdot X = 2^{\uparrow}$.

Biz $\boxed{\subseteq}$ $X \cdot X = \{x_1 \cdot x_2 \mid x_1, x_2 \in X\}$

AÁMNTPH: $x_1 \leq x_2$

$x_1 \cdot x_2 \geq x_1 \cdot x_1 = x_1^2 \underset{x_1 \in X}{>} 2 \Rightarrow x_1 x_2 \in 2^{\uparrow}$

$\boxed{2}$ $s \in 2^{r}$ $\quad s > 2$

$r^2 > 2 \Rightarrow r \in X \Rightarrow r \cdot r \in X \cdot X$

$$\left.\begin{array}{l} r^2 \in X \cdot X \\ s > r^2 \end{array}\right\} \xRightarrow{X \cdot X\,(\text{FCA})} s \in X \cdot X . \quad \square$$