① $[\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}] = 2 \cdot 2 = 4$

$\boxed{\begin{array}{l}[K(\alpha):K] = \deg m_{\alpha,K} = n \\ \text{bázis: } 1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\end{array}}$

$$\underbrace{\mathbb{Q} \leq \mathbb{Q}(\sqrt{3})}_{2} \underbrace{\leq \mathbb{Q}(\sqrt{3})(i)}_{2}$$

$m_{\sqrt{3},\mathbb{Q}} = x^2 - 3 \in \mathbb{Q}[x]$ irred. $\quad \sqrt{3}, -\sqrt{3} \notin \mathbb{Q}$

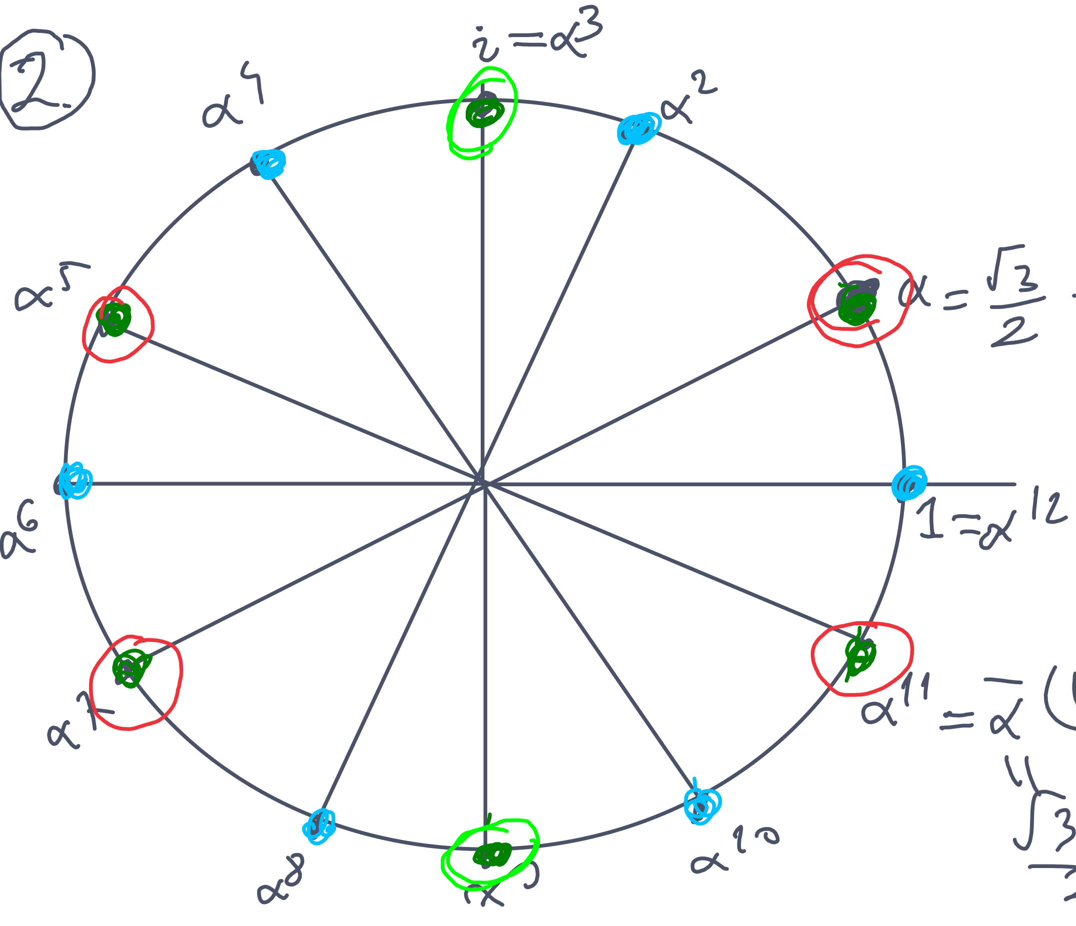
$m_{i,\mathbb{Q}(\sqrt{3})} = x^2 + 1 \in \mathbb{Q}(\sqrt{3})[x]$ irred. $\quad i, -i \notin \mathbb{Q}(\sqrt{3}) \leq \mathbb{R}$

$$\underbrace{\mathbb{Q} \leq \mathbb{Q}(\sqrt{3})}_{\text{bázis: } 1,\, \sqrt{3}} \underbrace{\leq \mathbb{Q}(\sqrt{3})(i)}_{\text{bázis: } 1,\, i}$$

bázis: $1, \sqrt{3}, i, \sqrt{3}i$

$\mathbb{Q}(\sqrt{3}, i) = \{a + b\sqrt{3} + ci + d\sqrt{3}i \mid a, b, c, d \in \mathbb{Q}\}$

②



$\alpha = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\alpha^{11} = \overline{\alpha}$ (MOIVRE-KÉPLET)

$\overline{\alpha} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$\alpha = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$2\alpha - \sqrt{3} = i$

$4\alpha^2 - 4\sqrt{3}\alpha + 3 = -1$

$4\alpha^2 - 4\sqrt{3}\alpha + 4 = 0$

$\alpha^2 - \sqrt{3}\alpha + 1 = 0$

$\alpha^2 + 1 = \sqrt{3}\alpha$

$\alpha^4 + 2\alpha^2 + 1 = 3\alpha^2$

$\alpha^4 - \alpha^2 + 1 = 0$

③ $[\mathbb{R}(\alpha) : \mathbb{R}] = 2$

FUN FACT: $\mathbb{R}(\alpha) = \mathbb{C}$

$\overbrace{x^2 - \sqrt{3}x + 1}^{m_{\alpha,\mathbb{R}}} \in \mathbb{R}[x]$

$m_{\alpha,\mathbb{R}} \mid m_{\alpha,\mathbb{Q}}$

$x^4 - x^2 + 1 \in \mathbb{Q}[x]$ irred.?

$\parallel$

$(x^2 - \sqrt{3}x + 1) \cdot (x^2 + \sqrt{3}x + 1)$

$m_{\alpha,\mathbb{Q}} = x^4 - x^2 + 1 \Rightarrow [\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$

bázis: $1, \alpha, \alpha^2, \alpha^3$

<u>Másik mo.</u>

$\alpha^{12} = 1$

$\alpha^{12} - 1 = 0$

$x^{12} - 1 \in \mathbb{Q}[x]$ irred.?

$\parallel$

$(x^6 - 1) \cdot (x^6 + 1)$

gyöke α

$x^6+1\in\mathbb{Q}[x]$ ired. ? $a^3+b^3=(a+b)\cdot(a^2-ab+b^2)$

$$(x^2)^3+1^3=\underline{(x^2+1)}\cdot\underline{(x^4-x^2+1)}$$

ničle α

$x^4-x^2+1\in\mathbb{Q}[x]$ ired nad $\mathbb{Q}$. sledi? ...

(4) $\mathbb{Q}(\sqrt{3},i)\overset{?}{=}\mathbb{Q}(\alpha)$

$\boxed{\supseteq}$ $\alpha\in\mathbb{Q}(\sqrt{3},i)$ ✓

$$\alpha=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$\boxed{\subseteq}$ $\sqrt{3},i\overset{?}{\in}\mathbb{Q}(\alpha)$

$$\sqrt{3}=\frac{\alpha^2+1}{\alpha}\in\mathbb{Q}(\alpha)\ ✓$$

$$i=2\alpha-\sqrt{3}=2\alpha-\frac{\alpha^2+1}{\alpha}\in\mathbb{Q}(\alpha)$$

<u>Misilno</u> $\sqrt{3}=\alpha+\overline{\alpha}=\alpha+\frac{1}{\alpha}\in\mathbb{Q}(\alpha)$

$i=\alpha-\overline{\alpha}=\alpha-\frac{1}{\alpha}\in\mathbb{Q}(\alpha)$

$$\underbrace{\mathbb{Q}\underbrace{\subseteq}_{4}\mathbb{Q}(\alpha)\underbrace{\subseteq}_{1}\mathbb{Q}(\sqrt{3},i)}_{4}$$

$\rightarrow \mathbb{Q}(\alpha)=\mathbb{Q}(\sqrt{3},i)$

(5) $\alpha = i\sqrt{2-\sqrt{2}}$ $\quad \sqrt{2}, \sqrt[3]{2} \overset{?}{\in} \mathbb{Q}(\alpha)$

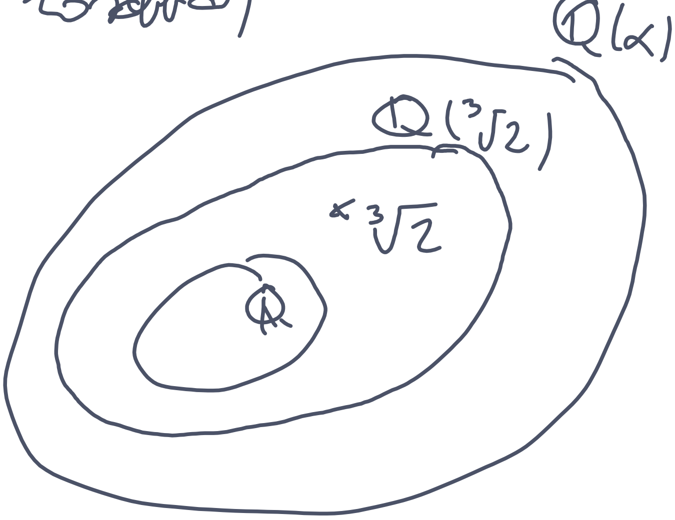
$$\alpha^2 = (-1)\cdot(2-\sqrt{2})$$

$$\alpha^2 + 2 = \sqrt{2} \implies \sqrt{2} \in \mathbb{Q}(\alpha)$$

$m_{\alpha,\mathbb{Q}} = x^4 + 4x^2 + 2$ (volt korábban)

$$[\mathbb{Q}(\alpha):\mathbb{Q}] = 4$$

Tfh $\sqrt[3]{2} \in \mathbb{Q}(\alpha)$



$$\underbrace{\underbrace{\mathbb{Q} \le \mathbb{Q}(\sqrt[3]{2})}_{3} \le \underbrace{\mathbb{Q}(\alpha)}_{\cancel{4/3}}}_{4}$$

↯

$3 = \deg m_{\sqrt[3]{2},\mathbb{Q}} \nmid 4 = [\mathbb{Q}(\alpha):\mathbb{Q}]$

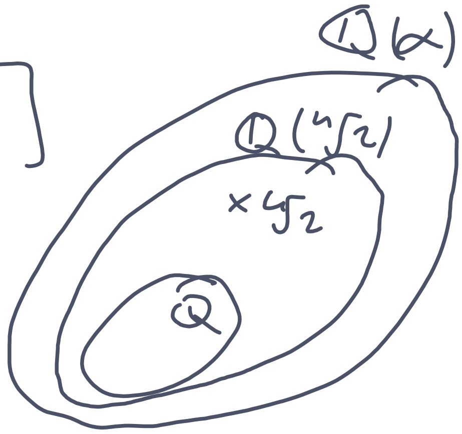
$\implies \sqrt[3]{2} \notin \mathbb{Q}(\alpha)$

<u>TÉTEL</u> $K \le L,\ \beta \in L \implies \deg m_{\beta,K} \mid [L:K]$

(6) $\sqrt[4]{2} \notin \mathbb{Q}(\alpha)$

$$\deg m_{\sqrt[4]{2},\mathbb{Q}} = 4 \mid 4 = [\mathbb{Q}(\alpha):\mathbb{Q}]$$

Tfh. $\sqrt[4]{2} \in \mathbb{Q}(\alpha)$



$$\underbrace{\mathbb{Q} \overset{4}{\le} \mathbb{Q}(\sqrt[4]{2}) \overset{1}{\le} \mathbb{Q}(\alpha)}_{4}$$

$$\Longrightarrow \underbrace{\mathbb{Q}(\sqrt[4]{2})}_{\subseteq \mathbb{R}} = \underbrace{\mathbb{Q}(\alpha)}_{\ni\ \alpha = i\sqrt{2-\sqrt{2}}}$$

$$\Downarrow\ \text{↯}$$

Tehát $\sqrt[4]{2} \notin \mathbb{Q}(\alpha)$.

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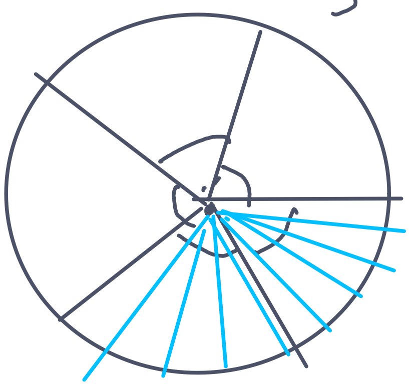
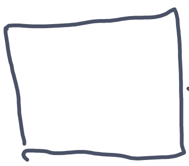
① Ha tudnék szöget ötödölni, akkor

$360° \rightsquigarrow \frac{360°}{5} \rightsquigarrow \frac{360°}{25}$, azaz 25-szög szerk. ↯ G-W



②  $\rightsquigarrow$  $\sqrt{\pi} \Leftarrow$ transzcendens

$\sqrt{\pi}$

$$\frac{a \cdot m}{2} = \pi$$

$\rightsquigarrow a, m, 2 \rightsquigarrow \frac{a \cdot m}{2} = \pi \Leftarrow$ transzc.

③ $1° \rightsquigarrow 360$-szög

$$\overset{\parallel}{10 \cdot 36} = 2 \cdot 5 \cdot 6 \cdot 6 = 2 \cdot 5 \cdot 2 \cdot 3 \cdot 2 \cdot 3 = 2^3 \cdot 3^2 \cdot 5$$

↯ G-W

$2° \rightsquigarrow 180$-szög

$$\overset{\parallel}{2^2 \cdot 3^2 \cdot 5}$$ ↯ G-W

$3° \rightsquigarrow 120$-szög

$$\overset{\parallel}{2^3 \cdot 3 \cdot 5}$$ szerk. G-W

④ $\alpha = \sqrt[13]{\frac{\cos\pi}{13}} = \sqrt[13]{\frac{-1}{13}} = \frac{-1}{\sqrt[13]{13}} \rightsquigarrow \sqrt[13]{13} = \alpha'$

$m_{\alpha',\mathbb{Q}} = x^{13} - 13 \in \mathbb{Q}[x]$ irred. S-E $p = 13$

$\deg m_{\alpha',\mathbb{Q}} = 13 \neq 2^k \Rightarrow \alpha'$ nem sz. $\Rightarrow \alpha$ sem szerk.

$\beta = \frac{\sqrt[13]{\cos\pi}}{13} = \frac{\sqrt[13]{-1}}{13} = \frac{-1}{13} \in \mathbb{Q}$ szerk.

$\gamma = \frac{\cos\pi}{\sqrt[13]{13}} = \frac{-1}{\sqrt[13]{13}} = \alpha$ nem szerk.

$\delta = \cos\left(\frac{\pi}{13}\right) \rightsquigarrow \underset{\overset{\parallel}{2 \cdot 13}}{26}$-szög G-W $\Rightarrow$ nem szerk.