Maximal non-affine reducts of simple affine algebras

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1. Introduction

In tame congruence theory the strongest result revealing the general structure of finite simple algebras of type **2** is the following representation theorem (cf. Theorem 6.1): every finite simple algebra $\mathbf{S}$ of type **2** can be embedded in a reduct $\mathbf{A}$ of a finite simple affine algebra; in more detail,

$$\mathbf{S} \cong \mathbf{S}' \subseteq \mathbf{A} = (A; F) \quad \text{with} \quad F \subseteq \mathrm{Pol}\,({}_{(\mathrm{End}\,_K \widehat{A})}\widehat{A})$$

where ${}_{(\mathrm{End}\,_K \widehat{A})}\widehat{A}$ is a finite simple module arising from a vector space ${}_K\widehat{A} = (A; +, K)$ by considering it a module over its own endomorphism ring.

To see how far finite simple algebras $\mathbf{S}$ of type **2** are from being affine, under what conditions they are affine, it is natural to ask the same questions for the algebras $\mathbf{A}$ first. The aim of this paper is to answer this question (Theorem 2.1). Essentially, we determine the maximal possible clones for non-affine algebras $\mathbf{A} = (A; F)$ with $F \subseteq \mathrm{Pol}\,({}_{(\mathrm{End}\,_K \widehat{A})}\widehat{A})$.

Selecting appropriate representing algebras $\mathbf{A}$ for finite simple algebras $\mathbf{S}$ of type **2**, one can ensure that certain properties of $\mathbf{S}$ are reflected in $\mathbf{A}$; for example, $\mathbf{S}$ and $\mathbf{A}$ generate the same variety, or if $\mathbf{S}$ has surjective fundamental operations, then $\mathbf{A}$ has the same property. This enables us to give an easy proof for the facts, known earlier, that every finite simple algebra of type **2** with surjective fundamental operations is affine [16,17] (Theorem 6.2), and every finite simple algebra of type **2** generating a minimal variety is affine [3] (Theorem 6.5).

The result in Theorem 2.1 yields also a Rosenberg-type description for the maximal subclones of the clones $\mathcal{P} = \mathrm{Pol}\,({}_{(\mathrm{End}\,_K \widehat{A})}\widehat{A})$ (Theorem 6.11). We note that a Słupecki-type theorem for these clones was proved in [12].

2. Preliminaries and main results

If not stated otherwise, algebras are denoted by boldface capitals and their universes by the corresponding letters in italics. The clone of term operations [the set of n-ary term operations] of an algebra $\mathbf{A}$ is denoted by $\mathrm{Clo}\,\mathbf{A}$ [resp., $\mathrm{Clo}_n\,\mathbf{A}$]. Similarly, the clone of polynomial operations [the set of n-ary polynomial operations] of $\mathbf{A}$ is denoted by $\mathrm{Pol}\,\mathbf{A}$ [resp., $\mathrm{Pol}_n\,\mathbf{A}$].

We will call an algebra $\mathbf{A}$ *surjective* if every fundamental operation of $\mathbf{A}$ is surjective. For algebras $\mathbf{A} = (A; F)$ and $\mathbf{A}' = (A'; F')$, we say that $\mathbf{A}$ is a *reduct* [*polynomial reduct*] of $\mathbf{A}'$ if $A = A'$ and $F \subseteq \mathrm{Clo}\,\mathbf{A}'$ [$F \subseteq \mathrm{Pol}\,\mathbf{A}'$]. The algebras $\mathbf{A} = (A; F)$ and $\mathbf{A}' = (A'; F')$ are called *term equivalent* [*polynomially equivalent*] if $A = A'$ and $\mathrm{Clo}\,\mathbf{A} = \mathrm{Clo}\,\mathbf{A}'$ [$\mathrm{Pol}\,\mathbf{A} = \mathrm{Pol}\,\mathbf{A}'$].

For a set N, let T_N, S_N, and C_N denote the full transformation monoid on N, the full symmetric group on N and the set of (unary) constant operations on N, respectively. The identity mapping and the equality relation on N are denoted by id and Δ, respectively (N will be clear from the context). For convenience we identify every natural number n with the set $n = \{0, 1, \dots, n-1\}$.

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For a set A and for $k \geq 1$, the nonvoid subsets of A^k will also be called *k-ary relations* (on A), and for an algebra $\mathbf{A}$ the universes of subalgebras of $\mathbf{A}^k$ will be called *compatible relations* of $\mathbf{A}$. An operation f on A is said to *preserve* a relation ρ if ρ is a compatible relation of the algebra $(A; f)$.

We say that an algebra $\mathbf{A}$ is *semi-affine with respect to an Abelian group* $\widehat{A} = (A; +)$ if $\mathbf{A}$ and $\widehat{A}$ have the same universe and

$$Q_{\widehat{A}} = \{(a, b, c, d) \in A^4 \colon\ a - b + c = d\}$$

is a compatible relation of $\mathbf{A}$ (or equivalently, the operations of $\mathbf{A}$ commute with $x - y + z$). Furthermore, $\mathbf{A}$ is said to be *affine with respect to* $\widehat{A}$ if it is semi-affine with respect to $\widehat{A}$ and, in addition, $x - y + z$ is a term operation of $\mathbf{A}$. It is well known (cf. [14; 2.1, 2.7– 2.8]) that

- — an algebra $\mathbf{A}$ is semi-affine with respect to an Abelian group $\widehat{A}$ if and only if $\mathbf{A}$ is a polynomial reduct of the module ${}_{(\mathrm{End}\,\widehat{A})}\widehat{A}$ (i.e. $\widehat{A}$ considered as a module over its endomorphism ring $\mathrm{End}\,\widehat{A}$), and
- — $\mathbf{A}$ is affine with respect to $\widehat{A}$ if and only if $\mathbf{A}$ is polynomially equivalent to a module ${}_R\widehat{A}$ for some subring R of $\mathrm{End}\,\widehat{A}$; this ring R, which is generated by all coefficients of term (or polynomial) operations of $\mathbf{A}$, is called the *ring of* $\mathbf{A}$, and is denoted by $R_{\mathbf{A}}$.

In the representation theorem for finite simple algebras of type **2** important role is played by the modules ${}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A}$ where ${}_K\widehat{A}$ is a finite vector space (these modules are essentially all finite simple modules with trivial annihilator ideals, cf. [2]). In analogy with the concept of semi-affineness and affineness with respect to an Abelian group we introduce the following notions.

Definition. Let ${}_K\widehat{A} = (A; +, K)$ be a vector space over a field K, and $\mathbf{A}$ an algebra. We will say that

- — $\mathbf{A}$ is *semi-affine with respect to* ${}_K\widehat{A}$ if $\mathbf{A}$ is a polynomial reduct of the module ${}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A}$, and
- — $\mathbf{A}$ is *affine with respect to* ${}_K\widehat{A}$ if it is semi-affine with respect to ${}_K\widehat{A}$ and, in addition, $x - y + z$ is a term operation of $\mathbf{A}$ (or, equivalently, if $\mathbf{A}$ is polynomially equivalent to a module ${}_R\widehat{A}$ for some subring R of $\mathrm{End}\,{}_K\widehat{A}$).

Clearly, if $\mathbf{A}$ is semi-affine [resp., affine] with respect to a vector space ${}_K\widehat{A}$, then it is semi-affine [resp., affine] with respect to the Abelian group $\widehat{A}$. Conversely, an elementary Abelian p-group $\widehat{A}$ can naturally be regarded as a vector space ${}_{\mathbf{Z}_p}\widehat{A}$, and it is obvious that semi-affineness [resp. affineness] with respect to $\widehat{A}$ and ${}_{\mathbf{Z}_p}\widehat{A}$ are the same.

It is easy to see that if an algebra $\mathbf{A}$ is semi-affine with respect to an Abelian group $\widehat{A}$, then $x - y + z$ is the only Mal'tsev operation that can be a term operation of $\mathbf{A}$; furthermore, the group $\widehat{A}$ is uniquely determined by the operation $x - y + z$, up to the choice of the element 0. Hence, if an algebra that is semi-affine with respect to an Abelian group $\widehat{A}$ [or vector space ${}_K\widehat{A}$] is affine for *some* Abelian group [or vector space]), then it is affine with respect to $\widehat{A}$ [resp., ${}_K\widehat{A}$] as well. We will use this fact without further reference, and omit to mention $\widehat{A}$ [resp., ${}_K\widehat{A}$] in such a situation.

For an Abelian group $\widehat{A} = (A; +)$ the group $\{x + a\colon\ a \in A\}$ of all translations of $\widehat{A}$ will be denoted by $T(\widehat{A})$. For a vector space ${}_K\widehat{A} = (A; +, K)$ we will also need the family

$$P({}_K\widehat{A}) = \{cx + a\colon\ c \in K - \{0\}, a \in A\}$$

of nonconstant unary polynomial operations of ${}_K\widehat{A}$. Clearly, $P({}_K\widehat{A})$ is a permutation group on A and $T(\widehat{A}) \subseteq P({}_K\widehat{A})$. For an algebra $\mathbf{A} = (A; F)$ that is semi-affine with respect to $\widehat{A}$, $\mathbf{A}^*$ will stand for the algebra $(A; F, T(\widehat{A}))$ arising from $\mathbf{A}$ by adding all translations of $\widehat{A}$ as unary operations. Analogously, for an algebra $\mathbf{A} = (A; F)$ that is semi-affine with respect to ${}_K\widehat{A}$, ${}_K\mathbf{A}^*$ will stand for the algebra $(A; F, P({}_K\widehat{A}))$ arising from $\mathbf{A}$ by adding all nonconstant unary polynomial operations of ${}_K\widehat{A}$ to $\mathbf{A}$.

Let $q \geq 3$. A family $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$ $(m \geq 1)$ of equivalence relations on A is called *q-regular* if each Θ_i $(0 \leq i \leq m-1)$ has exactly q blocks and $\Theta_T = \Theta_0 \cap \ldots \cap \Theta_{m-1}$ has exactly q^m blocks. A relation on A is called *q-regular* if it is of the form

$$\lambda_T = \{(a_0, \ldots, a_{q-1}) \in A^q\colon \text{ for all } i\ (0 \leq i \leq m-1),\ a_0, \ldots, a_{q-1} \text{ are not pairwise incongruent modulo } \Theta_i\}$$

for a q-regular family T of equivalence relations on A.

The *mth matrix power* of any unary algebra $\mathbf{U} = (U; F)$ is the algebra $\mathbf{U}^{[m]}$ whose base set is U^m, and its operations are exactly all operations $h^\sigma_\mu[g_0, \dots, g_{m-1}]$ defined for arbitrary mappings $\sigma: m \to m$, $\mu: m \to n$ and $g_0, \dots, g_{m-1} \in \mathrm{Clo}_1\, \mathbf{U}$ as follows: for $x_i = (x_i^0, \dots, x_i^{m-1}) \in U^m$ $(0 \le i \le n-1)$,

$$h^\sigma_\mu[g_0, \dots, g_{m-1}](x_0, \dots, x_{n-1}) = (g_0(x^{0\sigma}_{0\mu}), \dots, g_{m-1}(x^{(m-1)\sigma}_{(m-1)\mu})).$$

The mappings σ, μ will be called the *component mapping* and the *variable mapping* of $h^\sigma_\mu[g_0, \dots, g_{m-1}]$, respectively. For unary operations the subscript indicating the variable mapping $m \to 1$ will be omitted.

An algebra $\mathbf{A}$ is called *Abelian* if $\mathbf{A}$ satisfies the so-called *term condition* (or *TC*): for all $n \ge k \ge 1$, for every n-ary term operation f of $\mathbf{A}$ and for arbitrary $\bar{u}, \bar{v} \in A^k$, $\bar{a}, \bar{b} \in A^{n-k}$,

$$f(\bar{u}, \bar{a}) = f(\bar{u}, \bar{b}) \quad \Leftrightarrow \quad f(\bar{v}, \bar{a}) = f(\bar{v}, \bar{b}).$$

Furthermore, $\mathbf{A}$ is *strongly Abelian* if it satisfies the *strong term condition* (or *TC**): for all $n \ge k \ge 1$, for every n-ary term operation f of $\mathbf{A}$ and for arbitrary $\bar{u}, \bar{v} \in A^k$, $\bar{a}, \bar{b}, \bar{c} \in A^{n-k}$,

$$f(\bar{u}, \bar{a}) = f(\bar{v}, \bar{b}) \quad \Rightarrow \quad f(\bar{u}, \bar{c}) = f(\bar{v}, \bar{c}).$$

It is not hard to see that every strongly Abelian algebra is Abelian, and it is obvious from the definitions that both properties are inherited for subalgebras. Affine algebras are Abelian and not strongly Abelian, while matrix powers of unary algebras are strongly Abelian.

Our main result is

Theorem 2.1. *For arbitrary finite algebra* $\mathbf{A}$ *that is semi-affine with respect to a vector space* ${}_K\widehat{A} = (A; +, K)$*, one of the following conditions holds:*

(2.1.a) $\mathbf{A}$ *is affine with respect to* ${}_K\widehat{A}$*;*

(2.1.b) $\mathbf{A}$ *has a nontrivial congruence which is a congruence of* ${}_K\widehat{A}$*;*

(2.1.c) *there is a vector space isomorphism* ${}_K\widehat{A} \to ({}_KK)^m$ *which is simultaneously an isomorphism between* $\mathbf{A}$ *and a reduct of* $(K; \mathrm{Pol}_1\,({}_KK))^{[m]}$*;*

(2.1.d) $\mathbf{A}$ *has a compatible relation* λ_T *for some* q*-regular family* T *of congruences of* ${}_K\widehat{A}$ *with* $q > |K|$*.*

Clearly, if for an algebra $\mathbf{A}$ as in Theorem 2.1 condition (2.1.c) or (2.1.d) holds, then $\mathbf{A}$ cannot be affine. Thus Theorem 2.1 yields a necessary and sufficient condition for simple semi-affine algebras to be affine.

Corollary 2.2. *Let* $\mathbf{A}$ *be a finite simple algebra that is semi-affine with respect to a vector space* ${}_K\widehat{A} = (A; +, K)$*. Then* $\mathbf{A}$ *is affine with respect to* ${}_K\widehat{A}$ *if and only if both of conditions (2.1.c) and (2.1.d) fail for* $\mathbf{A}$.

The special case and weaker form of Theorem 2.1 and Corollary 2.2 for finite algebras that are semi-affine with respect to elementary Abelian groups (that is, vector spaces over prime fields) was proved in [18].

The proof of Theorem 2.1 is based on a strong version of Rosenberg's primal algebra characterization theorem (see [7]) stated in Theorem 2.3 below. Recall that a finite algebra $\mathbf{A}$ is called *quasiprimal* ([5], [6]) if every operation on A preserving the internal isomorphisms (i.e. isomorphisms between subalgebras) of $\mathbf{A}$ is a term operation of $\mathbf{A}$. Further, a k-ary relation ρ on A is said to be *central* if $\rho \ne A^k$, ρ is totally reflexive, totally symmetric, and there exists a $c \in A$ such that $(c, a_1, \dots, a_{k-1}) \in \rho$ for all $a_1, \dots, a_{k-1} \in A$.

Theorem 2.3. [15] *Let* $\mathbf{A}$ *be a finite simple algebra having no proper subalgebra. Then one of the following conditions holds:*

(2.3.a) $\mathbf{A}$ *is quasiprimal;*

(2.3.b) $\mathbf{A}$ *is affine with respect to an elementary Abelian* p*-group (*p *prime);*

(2.3.c) $\mathbf{A}$ *is isomorphic to a reduct of* $(2; T_2)^{[m]}$ *for some integer* $m \ge 1$*;*

(2.3.d) $\mathbf{A}$ *has a compatible* q*-regular relation for some* $q \ge 3$*;*

(2.3.e) $\mathbf{A}$ *has a compatible* k*-ary central relation for some* $k \ge 2$*;*

(2.3.f) $\mathbf{A}$ *has a compatible bounded partial order.*

Given a finite algebra $\mathbf{A}$ that is semi-affine with respect to a vector space ${}_K\widehat{A} = (A; +, K)$, Theorem 2.3 will not be applied directly to $\mathbf{A}$; it will be applied to the extended algebra ${}_K\mathbf{A}^*$. Therefore it is a crucial step to show beforehand that under mild restrictions on $\mathbf{A}$, ${}_K\mathbf{A}^*$ is affine if and only $\mathbf{A}$ has this property. We have

Theorem 2.4. *Let $\mathbf{A}$ be a finite algebra such that $\mathbf{A}$ is semi-affine with respect to a vector space ${}_K\widehat{A} = (A; +, K)$, and $\mathbf{A}$ has no nontrivial congruence which is a congruence of ${}_K\widehat{A}$. Then $\mathbf{A}$ is affine if and only if ${}_K\mathbf{A}^*$ is affine.*

The proof of Theorem 2.4 is presented in Section 3. Interestingly, the argument also requires an application of Theorem 2.3.

It will turn out that considering ${}_K\mathbf{A}^*$ instead of $\mathbf{A}$ when applying Theorem 2.3 has the effect that

— the congruences of ${}_K\mathbf{A}^*$ are automatically vector space congruences (cf. Lemma 3.4),

— (2.3.e), (2.3.f) cannot hold for ${}_K\mathbf{A}^*$ ((2.3.a) does not hold either, in view of semi-affineness), and

— even if (2.3.c) or (2.3.d) holds for ${}_K\mathbf{A}^*$, the presence of $P({}_K\widehat{A})$ in the set of operations forces that the matrix power, resp. compatible q-regular relation is nicely related to the vector space ${}_K\widehat{A}$.

The latter is the difficult part of the proof; it requires some group theoretical results, which are developed in Section 4. The proof of Theorem 2.1 is finished in Section 5, and finally the applications of Theorem 2.1 mentioned in the Introduction are presented in Section 6.

3. Proof of Theorem 2.4

Let $\widehat{A}$ be an Abelian group. For arbitrary polynomial operation $f = \sum_{i=0}^{n-1} r_i x_i + a$ of ${}_{(\mathrm{End}\,\widehat{A})}\widehat{A}$ we define

$$f^{\nabla} = f - f(0, \ldots, 0) = \sum_{i=0}^{n-1} r_i x_i,$$

and for a set F of such operations we put $F^{\nabla} = \{f^{\nabla}\colon\ f \in F\}$. If $\mathbf{A} = (A; F)$ is an algebra that is semi-affine with respect to $\widehat{A}$, then $\mathbf{A}^{\nabla}$ will denote the algebra $(A; F^{\nabla})$. Clearly, $\mathbf{A}^{\nabla}$ is also semi-affine with respect to $\widehat{A}$ (in fact, $\mathbf{A}^{\nabla}$ is a reduct of $\mathbf{A}^*$) and $\{0\}$ is a subalgebra of $\mathbf{A}^{\nabla}$. Moreover, if $\mathbf{A}$ is an algebra that is semi-affine with respect to a vector space ${}_K\widehat{A}$, then $\mathbf{A}^{\nabla}$ is also semi-affine with respect to ${}_K\widehat{A}$.

The next lemma clarifies how the clones of the algebras $\mathbf{A}$ and $\mathbf{A}^*$ are related if $\mathbf{A}$ is semi-affine with respect to $\widehat{A}$.

Lemma 3.1. *For arbitrary algebra $\mathbf{A}$ that is semi-affine with respect to an Abelian group $\widehat{A} = (A; +)$,*

$$\mathrm{Clo}\,\mathbf{A}^* = \{\sum_{i=0}^{n-1} r_i x_i + a\colon\ n \geq 1,\ a \in A,\ \textit{and}\ \sum_{i=0}^{n-1} r_i x_i + a_0 \in \mathrm{Clo}\,\mathbf{A}\ \textit{for some}\ a_0 \in A\}.$$

The proof if straightforward. Lemma 3.1 has the immediate consequence that the 'Abelian group analogue' of Theorem 2.4 is valid without any restriction on $\mathbf{A}$:

Corollary 3.2. *For a finite algebra $\mathbf{A}$ that is semi-affine with respect to an Abelian group $\widehat{A} = (A; +)$, the algebra $\mathbf{A}^*$ is affine if and only if $\mathbf{A}$ is affine.*

Proof. If $\mathbf{A}^*$ is affine, that is, $x - y + z \in \mathrm{Clo}\,\mathbf{A}^*$, then $x - y + z + a_0 \in \mathrm{Clo}\,\mathbf{A}$ for some $a_0 \in A$. By identifying variables we get that $x + a_0 \in \mathrm{Clo}\,\mathbf{A}$. Hence, in view of the finiteness, it follows that $x - y + z \in \mathrm{Clo}\,\mathbf{A}$, that is, $\mathbf{A}$ is affine. The converse implication is obvious. ⋄

The claims in the following two lemmas are well-known and easy to check.

Lemma 3.3. *For an Abelian group $\widehat{A} = (A; +)$, if Θ is an equivalence relation on A such that Θ is preserved by all permutations in $T(\widehat{A})$, then Θ is a congruence of $\widehat{A}$.*

Lemma 3.4. *For a vector space ${}_K\widehat{A} = (A;+,K)$, if Θ is an equivalence relation on A such that Θ is preserved by all permutations in $P({}_K\widehat{A})$, then Θ is a congruence of ${}_K\widehat{A}$.*

We will need a characterization of strongly Abelian algebras in terms of compatible relations.

Lemma 3.5 *An algebra $\mathbf{A}$ is strongly Abelian if and only if $\mathbf{A}$ has a 4-ary compatible relation ρ such that*
$(\mathrm{SA1})_\rho$ *$(a,b,a,b)\in\rho$ and $(a,b,c,c)\in\rho$ for all $a,b,c\in A$, and*
$(\mathrm{SA2})_\rho$ *for any elements $a,x,y\in A$, $(a,a,x,y)\in\rho$ implies $x=y$.*

Proof. The subalgebra of $\mathbf{A}^4$ generated by the quadruples of the form (a,b,a,b) and (a,b,c,c) with $a,b,c\in A$ is

$$\sigma=\{(f(\bar u,\bar a),f(\bar v,\bar b),f(\bar u,\bar c),f(\bar v,\bar c))\colon\ n\geq 2,\ 1\leq k\leq n,\ \bar u,\bar v\in A^k,\ \bar a,\bar b,\bar c\in A^{n-k},\ f\in \mathrm{Clo}_n\,\mathbf{A}\}.$$

Clearly, σ is the least 4-ary compatible relation of $\mathbf{A}$ satisfying condition $(\mathrm{SA1})_\sigma$, moreover, $\mathbf{A}$ is strongly Abelian if and only if σ has property $(\mathrm{SA2})_\sigma$. This implies the claim. ◇

Proof of Theorem 2.4. Suppose $\mathbf{A}$ satisfies the assumptions of the theorem. If $\mathbf{A}$ is affine then clearly ${}_K\mathbf{A}^*$ is also affine. Assume from now on that $\mathbf{A}$ is not affine, and hence by Corollary 3.2 $\mathbf{A}^*$ and its reduct $\mathbf{A}^\nabla$ are not affine either. It is clear from Lemma 3.1 that $(\mathbf{A}^\nabla)^*$ is term equivalent to $\mathbf{A}^*$, whence also ${}_K(\mathbf{A}^\nabla)^*$ is term equivalent to ${}_K\mathbf{A}^*$. Therefore, to prove that ${}_K\mathbf{A}^*$ is not affine, there is no loss of generality in replacing $\mathbf{A}$ with $\mathbf{A}^\nabla$, or equivalently, in assuming that $\{0\}$ is a subalgebra of $\mathbf{A}$.

Because of the translations, $\mathbf{A}^*$ has neither proper subalgebras, nor compatible central relations, nor compatible bounded partial orders. Obviously, $\mathbf{A}^*$ is not quasiprimal. Now it follows from Theorem 2.3 that either $\mathbf{A}^*$ is not simple, or one of conditions (2.3.c) or (2.3.d) holds for $\mathbf{A}^*$ (in place of $\mathbf{A}$). Making use of Lemma 3.5 we conclude that $\mathbf{A}^*$ has a compatible relation ρ where
(I) ρ satisfies conditions $(\mathrm{SA1})_\rho$ and $(\mathrm{SA2})_\rho$, or
(II) ρ is an at least ternary totally reflexive, totally symmetric relation distinct from the full relation, or
(III) ρ is a nontrivial equivalence relation.

The assumption that $\{0\}$ is a subalgebra of $\mathbf{A}$ ensures that multiplication with each element of $K-\{0\}$ is an automorphism of ${}_K\widehat{A}$ as well as of $\mathbf{A}$.

For ρ as above, say ρ is q-ary, and for any element $c\in K-\{0\}$, put

$$\rho_c=\{(a_0,\dots,a_{q-1})\in A^q\colon\ (ca_0,\dots,ca_{q-1})\in\rho\}.$$

It is easy to check that each ρ_c, and hence their intersection $\tau=\bigcap_{c\in K-\{0\}}\rho_c$ as well, inherits the following properties of ρ:
— it is a compatible relation of $\mathbf{A}$;
— it is closed under the translations in $T(\widehat{A})$ (acting componentwise), i.e., it is a compatible relation of the unary algebra $(A;T(\widehat{A}))$;
— it is of the same kind (I), (II), resp. (III) as ρ, allowing the possibility $\tau=\Delta$ in case (III).

In addition, τ is closed under the componentwise action of each multiplication with an element $c\in K-\{0\}$. Thus τ is a compatible relation of ${}_K\mathbf{A}^*$.

In cases (I) and (II) this shows that ${}_K\mathbf{A}^*$ is not affine, as was to be proved.

In case (III) each ρ_c with $c\in K-\{0\}$ is a nontrivial congruence of $\mathbf{A}^*$, and τ is a congruence of ${}_K\mathbf{A}^*$. By Lemma 3.3 this means that each ρ_c with $c\in K-\{0\}$ is a nontrivial simultaneous congruence of $\mathbf{A}$ and $\widehat{A}$, while τ is a simultaneous congruence of $\mathbf{A}$ and ${}_K\widehat{A}$. By the assumptions of the theorem, τ must be a trivial congruence, hence $\tau=\Delta$.

Let $\mathbf{B}=\mathbf{A}/\rho$, $\widehat{B}=\widehat{A}/\rho$. It is straightforward to check that $\mathbf{B}$ is semi-affine with respect to $\widehat{B}$.

For any element $c\in K-\{0\}$, the mapping

$$\varphi_c\colon\ A/\rho_c\to A/\rho=B,\quad a/\rho_c\mapsto ca/\rho$$

is an isomorphism $\mathbf{A}/\rho_c \to \mathbf{A}/\rho = \mathbf{B}$ as well as an isomorphism $\widehat{A}/\rho_c \to \widehat{A}/\rho = \widehat{B}$. Now let d be a generating element of the cyclic group $K - \{0\}$. The family $\{\rho_c\colon\ c \in K - \{0\}\} = \{\rho_{d^k}\colon\ 0 \le k \le |K| - 2\}$ of congruences yields a subdirect representation

$$\mathbf{A} \to \prod_{k=0}^{|K|-2} \mathbf{A}/\rho_{d^k}, \quad a \mapsto (a/\rho, a/\rho_d, a/\rho_{d^2}, \ldots, a/\rho_{d^{|K|-2}}),$$

and the same mapping embeds also $\widehat{A}$ into $\prod_{k=0}^{|K|-2} \widehat{A}/\rho_{d^k}$. Using the isomorphisms φ_{d^k} $(0 \le k \le |K| - 2)$ we can replace each component with $\mathbf{B}$ (resp. $\widehat{B}$), to get an embedding

$$\varphi\colon\ \mathbf{A} \to \mathbf{B}^{|K|-2}, \quad a \mapsto (a/\rho, da/\rho, d^2a/\rho, \ldots, d^{|K|-3}a/\rho, d^{|K|-2}a/\rho),$$

which is simultaneously an embedding of $\widehat{A}$ into $\widehat{B}^{|K|-2}$. Note that the $(|K| - 1)$-tuple assigned by φ to the element da is $(da/\rho, d^2a/\rho, d^3a/\rho, \ldots, d^{|K|-2}a/\rho, a/\rho)$, and this $(|K| - 1)$-tuple arises from the image of a by a cyclic permutation of the components.

For convenience, we will identify $\mathbf{A}$ with its image under φ.

Suppose now that the previous argument was carried out for a ρ which is maximal among the nontrivial congruences of $\mathbf{A}$ that are also congruences of $\widehat{A}$. Hence $\mathbf{B}$ has no nontrivial congruence which is a congruence of $\widehat{B}$. Since $\mathbf{A}$ is a subdirect power of $\mathbf{B}$ and is not affine, therefore $\mathbf{B}$ cannot be affine. Repeating the first half of this proof for $\mathbf{B}$ in place of $\mathbf{A}$ we see that $\mathbf{B}^*$ has a compatible relation ρ' of type (I) or (II). Let q denote the arity of ρ' ($q = 4$ in case (I), $q \ge 3$ in case (II)), and consider the q-ary relation $\xi = \rho^{|K|-1} \cap A^q$ on A (here $\rho^{|K|-1}$ denotes the relation defined componentwise by ρ on the set $B^{|K|-1}$). Clearly, in case (I) ξ is of type (I), while in case (II) ξ is of type (II). The operations of $\mathbf{A}^*$ act componentwise, therefore ξ is a compatible relation of $\mathbf{A}^*$. Since the operation 'multiplication by d' on A is reflected as a cyclic permutation of the components, therefore ξ is compatible with this additional operation as well. Thus ξ is a compatible relation of ${}_K\mathbf{A}^*$, hence ${}_K\mathbf{A}^*$ is not affine. ◇

4. Some subgroups of general wreath products

Let $G \subseteq S_A$ be a permutation group acting on a set A. The *orbits* of G are the minimal nonvoid subsets of A that are closed under all permutations in G. Clearly, the orbits of G yield a partition of A. We say that G is *transitive* on A if A is an orbit of G, and G acts *regularly* on A if it is transitive and the non-identity permutation in G have no fixed points.

Let k and m be arbitrary positive integers, and let P be a subgroup of S_m. Clearly, the unary term operations $h^\sigma[g_0, \ldots, g_{m-1}]$ of $(k; S_k)^{[m]}$ with $\sigma \in P$ form a permutation group acting on the set k^m. In group theory this group is called the *general wreath product* of S_k and P, and is denoted by $S_k \operatorname{Wr} P$ (cf. [11; p. 272]). In $S_k \operatorname{Wr} P$ the elements $h^{\mathrm{id}}[g_0, \ldots, g_{m-1}]$ form a normal subgroup (isomorphic to the mth direct power of S_k), which will be denoted by $(S_k)^m$, while the elements $h^\pi[\mathrm{id}, \ldots, \mathrm{id}]$ form a subgroup (isomorphic to P), which will be denoted by $\tilde{P}$. Obviously, $\tilde{P}$ is a *complement* of $(S_k)^m$ in $S_k \operatorname{Wr} P$ in the sense that $(S_k)^m \cap \tilde{P} = \{\mathrm{id}\}$ and $(S_k)^m \tilde{P} = S_k \operatorname{Wr} P$.

If P is a regular permutation group on m, then $S_k \operatorname{Wr} P$ essentially coincides with the so-called *complete wreath product* of S_k and P (cf. [11; pp. 270, 272]).

The following proposition was used already in [18], however, for the readers' convenience the proof is included here. Reference [11] which makes the proof relatively short was pointed out to me by P. P. Pálfy.

Proposition 4.1. *Let H be a subgroup of the permutation group $S_q \operatorname{Wr} S_m$ where q is a power of a prime number p and m is an arbitrary positive integer. If H is an elementary Abelian p-group which acts regularly on q^m, then H is a subgroup of $(S_q)^m$.*

Proof. Let H be a subgroup of $S_q \operatorname{Wr} S_m$ satisfying the assumptions of the lemma, and let P denote the group of component mappings of permutations in H. Thus H is an elementary Abelian p-subgroup of $S_q \operatorname{Wr} P$ acting regularly on q^m. Let $I_0, \ldots, I_{t-1}$ denote the orbits of P. Then each member $h^\sigma[g_0, \ldots, g_{m-1}]$ of H acts componentwise, via $h^{\sigma|_{I_l}}[g_i\colon\ i \in I_l]$ $(l = 0, \ldots, t-1)$ on the set $q^m = q^{I_0} \times \ldots \times q^{I_{t-1}}$. For

$0 \leq l \leq t-1$ let $H^{(l)}$ consist of the restrictions of all members of H to the component q^{I_l} of the base set. Clearly, $H^{(l)}$ is an elementary Abelian p-subgroup of $S_q \operatorname{Wr} S_{I_l}$, which acts transitively on q^{I_l}. By the well-known fact that every commutative, transitive permutation group is regular, it follows that each $H^{(l)}$ $(l = 0, \ldots, t-1)$ is a regular permutation group. Consequently, for cardinality reasons, H splits into a direct product of these groups $H^{(l)}$ $(l = 0, \ldots, t-1)$. Hence it suffices to prove that if P is transitive, then $m = 1$.

Assume that P is transitive. Since P is a homomorphic image of H, therefore P is an elementary Abelian p-group. From the transitivity and commutativity of P it follows that P is regular as well.

Consider the subgroup $\bar{H} = H \cap (S_q)^m$ of H. Since H is finite and Abelian, it has a subgroup $\bar{P}$ that is a complement of $\bar{H}$ in H (that is, $\bar{H} \cap \bar{P} = \{\mathrm{id}\}$ and $\bar{H}\bar{P} = H$). Clearly, for each $\sigma \in P$, $\bar{P}$ contains exactly one permutation with component mapping σ. Thus $\bar{P}$ is a complement of $(S_q)^m$ in the complete wreath product $S_q \operatorname{Wr} P$. It is known (cf. [11; 10.7 in Chapter 2]) that any two complements of $(S_q)^m$ in $S_q \operatorname{Wr} P$ — specifically $\tilde{P}$ and $\bar{P}$ — are conjugate. Since all assumptions on H and the required conclusion as well are invariant under conjugation, we may assume without loss of generality that $\tilde{P} \subseteq H$. However, as H is Abelian, H is contained in the centralizer of $\tilde{P}$ in $S_q \operatorname{Wr} P$, which is easily seen to be equal to

$$\{h^\sigma[g, \ldots, g] \colon\ g \in S_q,\ \sigma \in P\}$$

(cf. [11; Exercise 2 on p. 277]). Obviously, this group is transitive only if $m = 1$, completing the proof. ◇

A permutation group G acting on a set A will be called (for the lack of a better name) a *vector space group* if $G = P({}_K\widehat{A})$ for some vector space ${}_K\widehat{A}$ on A. Clearly, a group of this form is transitive, moreover, it is non-regular unless $|K| = 2$, and every nonidentity permutation in G has at most one fixed point. For $a \in A$ the *stabilizer of* a in G is the subgroup of G consisting of all permutations in G fixing a, and is denoted by G_a. Clearly, in a vector space group $G = P({}_K\widehat{A})$

— the subgroup $T(\widehat{A})$ of all translations is the normal subgroup of G consisting exactly of the identity and all fixed point free permutations, and this subgroup uniquely determines the $+$ of the Abelian group $\widehat{A}$ once the neutral element 0 is fixed;

— the multiplications by nonzero scalars in K are exactly the members of the stabilizer G_0 of the element 0.

Thus, up to the choice of 0, the vector space ${}_K\widehat{A}$ can uniquely be reconstructed from the group $G = P({}_K\widehat{A})$.

Proposition 4.2. *Let G be a subgroup of the permutation group $S_q \operatorname{Wr} S_m$ where q is a power of a prime number p and m is an arbitrary positive integer. If G is a vector space group on q^m, then G is a subgroup of $(S_q)^m$.*

For the proof we need two lemmas.

Lemma 4.3. *If the minimal polynomial of an element a of the Galois field $GF(p^n)$ is of the form $\bar{f}(x^r)$ for some integer $r > 1$ and polynomial $\bar{f} \in \mathbf{Z}_p[x]$, then a does not generate the multiplicative group of $GF(p^n)$.*

Proof. Suppose the assumption of the claim holds. If the degree of the element a over $\mathbf{Z}_p$ (i.e. the degree of its minimal polynomial $\bar{f}(x^r)$) is less than n, then the conclusion is obvious. So assume the degree of a is n, and hence $r|n$, say $n = rs$. Since $\bar{f}(x^r)$ is irreducible over $\mathbf{Z}_p$, so is $\bar{f}$. Thus $\bar{f}$ is the minimal polynomial of the element a^r. Consequently the degree of a^r is s, i.e. the subfield $\mathbf{Z}_p(a^r)$ of $GF(p^{rs})$ has order p^s. Hence for the multiplicative order of a^r we have $o(a^r)|p^s - 1$. Thus

$$p^s - 1 \geq o(a^r) = \frac{o(a)}{\gcd(o(a), r)} \geq \frac{o(a)}{r},$$

yielding that

$$o(a) \leq (p^s - 1)r < (p^s - 1)(p^{s(r-1)} + p^{s(r-2)} + \ldots + p^s + 1) = p^{rs} - 1 = p^n - 1.$$

This proves that a cannot be a generating element of the multiplicative group of $GF(p^n)$. ◇

Lemma 4.4 *Let $(q;+)$ be an elementary Abelian p-group for some prime p, and let $\widehat{A}=(q;+)^m$ for some integer $m \geq 1$. Consider an automorphism of $\widehat{A}$ of the form $h = h^{\sigma}[g_0, \dots, g_{m-1}]$ for some $g_0, \dots, g_{m-1} \in$ $\mathrm{Aut}\,(q;+)$, $\sigma \in S_m$. If h belongs to a subfield of $\mathrm{End}\,\widehat{A}$ and $\sigma \neq \mathrm{id}$, then the minimal polynomial of h over $\mathbf{Z}_p$ is of the form $\bar{f}(x^r)$ for some integer $r > 1$ and polynomial $\bar{f} \in \mathbf{Z}_p[x]$.*

Proof. Suppose h belongs to a subfield K of $\mathrm{End}\,\widehat{A}$ (K is of characteristic p), and let, say, $(0\ 1\ \dots\ r-1)$ with $r > 1$ be one of the disjoint cycles of σ. Consider an element of q^m of the form $\xi = (0, \dots, 0, u, 0, \dots, 0)$ with u in the $(r-1)$st coordinate, and apply repeatedly h to it. It is straightforward to check by induction that for $t \geq 0$ and $0 \leq i \leq r-1$ we have

$$(\dagger) \qquad h^{rt+i}(\xi) = (0, \dots, 0, \overbrace{(g_{r-i} \cdots g_{r-1} g^t)(u)}^{(r-1-i)\text{th coordinate}}, 0, \dots, 0) \quad \text{where} \quad g = g_0 g_1 \dots g_{r-1}.$$

Let us denote the minimal polynomial of h over $\mathbf{Z}_p$ by

$$f(x) = \sum_{0 \leq j \leq n} c_j x^j \ \in \ \mathbf{Z}_p[x],$$

and for $0 \leq i \leq r-1$ put

$$f_i(x) = \sum_{0 \leq j \leq n,\ j \equiv i \pmod{r}} c_j x^j \ \in \ \mathbf{Z}_p[x].$$

Clearly, $f_0(h), \dots f_{r-1}(h)$ belong to K, and we have

$$\sum_{0 \leq i \leq r-1} f_i(h) = f(h) = 0 \quad (\in K).$$

Thus

$$(\ddagger) \qquad \sum_{0 \leq i \leq r-1} f_i(h)(\xi) = f(h)(\xi) = 0 \quad (\in q^m).$$

In view of $(\dagger)$, for $0 \leq i \leq r-1$, all coordinates of the m-tuple $f_i(h)(\xi)$, except possibly the $(r-1-i)$th coordinate, equal 0. Hence $(\ddagger)$ yields that

$$f_0(h)(\xi) = 0 \quad \text{for all} \quad \xi = (0, \dots, 0, \overbrace{u}^{(r-1)\text{st}}, 0, \dots, 0) \quad \text{with} \quad u \in q.$$

Since $f_0(h) \in K$ and all nonzero members of K are automorphisms of $\widehat{A}$, we conclude that $f_0(h) = 0$. Here f_0 is not the zero polynomial (as $c_0 \neq 0$), and its degree does not exceed the degree of f, therefore $f_0 = f$, completing the proof. ⋄

Proof of Proposition 4.2. Let G be a subgroup of $S_q \,\mathrm{Wr}\, S_m$ which is a vector space group, say $G = P({}_K\widehat{A})$ for some vector space ${}_K\widehat{A} = (q^m; +, K)$, and let $H = T(\widehat{A})$. Clearly, the assumptions of Proposition 4.1 hold for H, therefore H is a subgroup of $(S_q)^m$. For $0 \leq l \leq m-1$ let $H^{(l)}$ consist of the restrictions of all members of H to the lth component of the base set q^m. As we have seen in the first paragraph of the proof of Proposition 4.1, each $H^{(l)}$ $(l = 0, \dots, m-1)$ is an elementary Abelian p-group acting regularly on q^m, and H splits into a direct product of these groups.

Let $(q;+,0)$ be a fixed elementary Abelian p-group on q, and consider the zero element of $\widehat{A}$: $0 = (o_0, \dots, o_{m-1})$. For $0 \leq i \leq m-1$ there exist elementary Abelian p-groups $(q;+_i,o_i)$ such that $H^{(i)} = T((q;+_i,o_i))$. Fixing any isomorphisms $\pi_i\colon (q;+_i,o_i) \to (q;+,0)$ we get that the permutation $h^{\mathrm{id}}[\pi_0, \dots, \pi_{m-1}] \in S_q \,\mathrm{Wr}\, S_m$ conjugates $H = T(\widehat{A})$ into $T((q;+,0)^m)$.

Since the assumptions on G as well as the conclusion of the lemma are invariant under conjugation, we may assume that the additive group of the vector space ${}_K\widehat{A}$ with $G = P({}_K\widehat{A})$ is $\widehat{A} = (q;+)^m$ for a fixed elementary Abelian p-group $(q;+)$ on q. The multiplications by nonzero scalars in K are exactly the members of the stabilizer G_0 of the zero element $0=(0,\dots,0)$ of $\widehat{A}=(q;+)^m$. Thus $G_0 \subseteq \operatorname{End}\widehat{A}$, and G_0 together with the zero endomorphism forms a field isomorphic to K. Consequently G_0 is a cyclic group. On the other hand, $G_0 \subseteq S_q \operatorname{Wr} S_m$. Let $h=h^\sigma[g_0,\dots,g_{m-1}]$ ($\sigma \in S_m$, $g_0,\dots,g_{m-1}\in S_q$) be a generating element in G_0. It is straightforward to check that $g_0,\dots,g_{m-1}\in \operatorname{Aut}(q;+)$. Now Lemmas 4.3 and 4.4 show that $\sigma=\mathrm{id}$, whence $G_0\subseteq (S_q)^m$. Since $H\cup G_0$ generates G, it follows that $G\subseteq (S_q)^m$. ⋄

5. Proof of Theorem 2.1

Let U be a q-element set ($q\geq 3$), and let $m\geq 1$. The kernels of the m distinct projections $U^m\to U$ form a q-regular family of equivalences on U^m, which will be called the *standard q-regular family of equivalences on U^m*; the corresponding q-regular relation is called the *standard q-regular relation on U^m*. It is well known that the mth matrix power $\mathbf{U}^{[m]}$ of any unary algebra $\mathbf{U}=(U;F)$ admits the standard q-regular relation as a compatible relation.

In the lemma below we collect some well-known facts on finite algebras admitting q-regular compatible relations.

Lemma 5.1. *Let $\mathbf{A}=(A;F)$ be a finite algebra, and let $T=\{\Theta_0,\dots,\Theta_{m-1}\}$ be a q-regular family of equivalence relations on A such that λ_T is a compatible relation of $\mathbf{A}$.*

(5.1.i) *$T/\Theta_T=\{\Theta_0/\Theta_T,\dots,\Theta_{m-1}/\Theta_T\}$ is a q-regular family of equivalences on A/Θ_T, and there exists a bijection $\varphi\colon A/\Theta_T\to q^m$ carrying T/Θ_T into the standard q-regular family of equivalences on q^m.*

(5.1.ii) *If $f\in F$ is an n-ary operation whose range meets each block of some Θ_i, then there exist j,l ($0\leq j\leq m-1$, $0\leq l\leq n-1$) such that for all $x_0,\dots,x_{n-1},y_0,\dots,y_{n-1}\in A$ we have*

$$(*)\qquad f(x_0,\dots,x_{n-1})\,\Theta_i\,f(y_0,\dots,y_{n-1}) \quad\Leftrightarrow\quad x_l\,\Theta_j\,y_l.$$

(5.1.iii) *If $\mathbf{A}$ is a surjective algebra, then*

(1) *Θ_T is a congruence of $\mathbf{A}$,*

(2) *the relation λ_{T/Θ_T} is a compatible relation of $\mathbf{A}/\Theta_T$, and*

(3) *the bijection φ yields an isomorphism between $\mathbf{A}/\Theta_T$ and a reduct of the matrix power $(q;S_q)^{[m]}$.*

The proof of (5.1.ii) can be found, e.g. in [9; Lemma 7.3]. In fact, what is proved there is the implication $\Leftarrow$ in $(*)$ for the case when the equivalences in T are assumed to have at least q blocks, rather than exactly q blocks, each; however, if the equivalences have the same number of blocks, then $\Rightarrow$ cannot fail in $(*)$. The claims in (5.1.iii) are well-known consequences of (5.1.i) and (5.1.ii); see [10], [8]. We note that Rousseau [10] (cf. also [8]) proved (5.1.iii)(3) for the case $\Theta_T=\Delta$, however, in view of (5.1.iii)(1)–(2) the more general claim follows immediately from this special case.

Lemma 5.2. *Let ${}_K\widehat{A}$ be a finite vector space, and let $T=\{\Theta_0,\dots,\Theta_{m-1}\}$ be a q-regular family of equivalences on A such that λ_T is preserved by all permutations in $P({}_K\widehat{A})$. Then*

(5.2.i) *$\Theta_0,\dots,\Theta_{m-1}$, and hence their intersection Θ_T as well, are congruences of ${}_K\widehat{A}$, and*

(5.2.ii) *for any vector space $(q;+,K)$, there exists an isomorphism ${}_K\widehat{A}/\Theta_T\to(q;+,K)^m$ carrying T/Θ_T into the standard q-regular family of equivalences on q^m.*

Proof. Consider the unary algebra $\mathbf{A}=(A;P({}_K\widehat{A}))$. By our assumption λ_T is a compatible relation of $\mathbf{A}$. Since $\mathbf{A}$ is surjective, we get from Lemma 5.1 (5.1.iii)(1) that Θ_T is a congruence of $\mathbf{A}$. So by Lemma 3.4 Θ_T is a congruence of ${}_K\widehat{A}$. Applying Lemma 5.1 (5.1.i) and (5.1.iii)(3) we get also that there exists an isomorphism φ between the algebra $\mathbf{A}/\Theta_T=(A/\Theta;P({}_K\widehat{A}/\Theta))$ and a reduct of the matrix power $(q;S_q)^{[m]}$ such that φ carries T/Θ_T into the standard q-regular family $\{\Phi_0,\dots,\Phi_{m-1}\}$ of equivalences on q^m. Let G denote the subgroup of S_{q^m} corresponding to the group $P({}_K\widehat{A}/\Theta)$ under φ. Clearly, G is a subgroup of $S_q\operatorname{Wr}S_m$. Furthermore, by construction, G is a vector space group on q^m. Now Proposition 4.2 states that $G\subseteq(S_q)^m$, whence it follows that $\Phi_0,\dots,\Phi_{m-1}$ are congruences of $(q^m;G)$. Via the isomorphism φ we get that $\Theta_0/\Theta_T,\dots,\Theta_{m-1}/\Theta_T$ are congruences of $\mathbf{A}/\Theta_T$, and hence $\Theta_0,\dots,\Theta_{m-1}$ are congruences of $\mathbf{A}$. Now by Lemma 3.4 we conclude that (5.2.i) holds.

Since the family T of congruences of ${}_K\widehat{A}$ is q-regular, the natural embedding

$$ {}_K\widehat{A}/\Theta_T \to {}_K\widehat{A}/\Theta_0 \times \ldots \times {}_K\widehat{A}/\Theta_{m-1} $$

is an isomorphism, and all quotient spaces on the right are q-element vector spaces over K. Up to isomorphism, we can replace them with the given space $(q;+,K)$, and the requirements in (5.2.ii) obviously hold. $\diamond$

Lemma 5.3. *Let* $\mathbf{A}$ *be a finite algebra that is semi-affine with respect to a vector space* ${}_K\widehat{A} = (A;+,K)$, *and let* T *be a* $|K|$*-regular family of congruences of* ${}_K\widehat{A}$ *such that* λ_T *is a compatible relation of* ${}_K\mathbf{A}^*$. *Then*

(5.3.i) Θ_T *is a congruence of* $\mathbf{A}$, *and*

(5.3.ii) *if* $\Theta_T = \Delta$, *then there exists a vector space isomorphism* ${}_K\widehat{A} \to ({}_KK)^m$ *which is simultaneously an isomorphism between* $\mathbf{A}$ *and a reduct of* $(K;\mathrm{Pol}_1\,({}_KK))^{[m]}$.

Proof. Let $T = \{\Theta_0,\ldots,\Theta_{m-1}\}$. By the previous lemma these equivalences are congruences of ${}_K\widehat{A}$, and so is their intersection Θ_T.

To prove (5.3.i) let f be an n-ary operation of $\mathbf{A}$, and let $x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1} \in A$ be arbitrary elements of $\mathbf{A}$ such that $x_k\,\Theta_T\,y_k$ for all $0 \le k \le n-1$. Let $0 \le i \le m-1$. Assume first that the range of f meets at least two blocks of Θ_i. Since ${}_K\widehat{A}/\Theta_i$ is a one-dimensional vector space and $\mathbf{A}$ is semi-affine with respect to ${}_K\widehat{A}$, it is clear that the range of f meets each block of Θ_i. Thus we get from Lemma 5.1 (5.1.ii) that $f(x_0,\ldots,x_{n-1})\,\Theta_i\,f(y_0,\ldots,y_{n-1})$. The same conclusion is obvious, if the range of f meets only one block of Θ_i. Since i was arbitrary, we conclude that $f(x_0,\ldots,x_{n-1})\,\Theta_T\,f(y_0,\ldots,y_{n-1})$, as required.

Now let $\Theta_T = \Delta$. By Lemma 5.2 (5.2.ii) there exists an isomorphism ${}_K\widehat{A} \to ({}_KK)^m$ carrying T into the standard $|K|$-regular family of equivalences on K^m. Let $\mathbf{B} = (K^m;F)$ be the algebra corresponding to $\mathbf{A}$ under this isomorphism. Notice that the standard $|K|$-regular relation on K^m is a compatible relation of $\mathbf{B}$, and apply Lemma 5.1 (5.1.ii) to each operation f of $\mathbf{B}$. Let, say, f be n-ary. For $b \in q^m$ the components of b will be denoted by $b^0,\ldots,b^{m-1}$. Let $0 \le i \le m-1$ be arbitrary. As in the previous paragraph, we see that the set of ith components of $f(b_0,\ldots,b_{n-1})$ as the arguments run over all elements of K^m is either K or a one-element set. In the first case we get from (5.1.ii) that there exist indices j_i, l_i ($0 \le j_i \le m-1$, $0 \le l_i \le n-1$) and a permutation $g_i \in S_K$ such that the ith component of $f(b_0,\ldots,b_{n-1})$ equals $g_i(b_{l_i}^{j_i})$ for all $b_0,\ldots,b_{n-1} \in K^m$. In the second case the same holds with g_i constant (and j_i, l_i arbitrary). Thus $f = h^\sigma_\mu[g_0,\ldots,g_{m-1}]$ where σ and μ are the mappings $\sigma\colon m \to m$, $i \mapsto j_i$ and $\mu\colon m \to n$, $i \mapsto l_i$. Hence $\mathbf{B}$ is a reduct of $(K; S_K \cup C_K)^{[m]}$. Taking into consideration that $\mathbf{B}$ is semi-affine with respect to $({}_KK)^m$, one can easily derive that $\mathbf{B}$ is a reduct of $(K;\mathrm{Pol}_1\,({}_KK))^{[m]}$, completing the proof of (5.3.ii). $\diamond$

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathbf{A}$ be a finite algebra that is semi-affine with respect to a vector space ${}_K\widehat{A} = (A;+,K)$, and consider the algebra ${}_K\mathbf{A}^*$. Because of the translations, ${}_K\mathbf{A}^*$ has no proper subalgebra, no compatible bounded partial order and no compatible central relation. If ${}_K\mathbf{A}^*$ is not simple, then by Lemma 3.4 (2.1.b) trivially holds, so assume ${}_K\mathbf{A}^*$ is simple. Now we can apply Theorem 2.3 for ${}_K\mathbf{A}^*$. Since a semi-affine algebra cannot be quasiprimal, condition (2.3.b), (2.3.c) or (2.3.d) in Theorem 2.3 holds for ${}_K\mathbf{A}^*$.

Assume first that (2.3.b) holds for ${}_K\mathbf{A}^*$, that is ${}_K\mathbf{A}^*$ is affine. Since ${}_K\mathbf{A}^*$ is simple, $\mathbf{A}$ has no nontrivial congruence which is a congruence of ${}_K\widehat{A}$. Hence, by Theorem 2.4, (2.1.a) holds for $\mathbf{A}$.

Now let us consider the case when (2.3.c) holds for ${}_K\mathbf{A}^*$, that is, there exists an isomorphism φ between ${}_K\mathbf{A}^*$ and a reduct of the matrix power $(2;T_2)^{[m]}$. Let G denote the subgroup of S_{2^m} corresponding to the group $P({}_K\widehat{A})$ under φ. Clearly, G is a subgroup of $S_2 \,\mathrm{Wr}\, S_m$, and G is a vector space group on 2^m. By Proposition 4.1 we have $G \subseteq (S_2)^m$, so for cardinality reasons $G = (S_2)^m$ and $|K| = 2$. Let ω be the image of $0 \in A$ under φ, and let τ be the translation $x + \omega$ of the Abelian group $(2;+)^m$. It is straightforward to check that the mapping $\varphi\tau$ is a vector space isomorphism ${}_K\widehat{A} \to (2;+,K)^m$ which is simultaneously an isomorphism between $\mathbf{A}$ and a reduct of $(2;T_2)^{[m]}$. Identifying the set 2 with K in the natural way we get ${}_KK$ from $(2;+,K)$, and $(K;\mathrm{Pol}_1\,({}_KK))$ from $(2;T_2)$. Hence (2.1.c) holds with $|K| = 2$.

Finally, suppose condition (2.3.d) holds for ${}_K\mathbf{A}^*$, and let T be a q-regular family of equivalences on A such that λ_T is a compatible relation of ${}_K\mathbf{A}^*$. Obviously, λ_T is preserved by all permutations in $P({}_K\widehat{A})$, so by Lemma 5.2 T consists of congruences of ${}_K\widehat{A}$. It follows now that q is a power of $|K|$. If $q > |K|$, then (2.1.d) trivially holds, while if $q = |K|$, then by Lemma 5.3 and by the simplicity of $\mathbf{A}$ we have $\Theta_T = \Delta$ and condition (2.1.c) holds for $\mathbf{A}$. ◇

6. Applications of Theorem 2.1

First we discuss two results on finite simple algebras of type **2**.

By the basics of tame congruence theory ([1]), a finite simple algebra is of type **2** if and only if it is Abelian but not strongly Abelian. A remarkable result in tame congruence theory is that every finite simple algebra of type **2** is representable as a subalgebra of a finite semi-affine algebra:

Theorem 6.1. ([1; Theorem 13.5]) *For every finite simple algebra* $\mathbf{S}$ *of type* **2** *there exists a finite algebra* $\mathbf{A}$ *such that* $\mathbf{A}$ *is semi-affine with respect to a vector space* ${}_K\widehat{A}$*, and* $\mathbf{S}$ *is isomorphic to a subalgebra* $\mathbf{S}'$ *of* $\mathbf{A}$ *(such that* $0 \in S'$ *and* S' *generates the vector space* ${}_K\widehat{A}$*).*

Such a triple $(\mathbf{A}, {}_K\widehat{A}, \mathbf{S}')$ will be called a *representation of* $\mathbf{S}$.

(A) *Surjective finite simple algebras of type* **2**

The fact that all surjective algebras among the finite simple algebras of type **2** are affine was proved earlier separately for those algebras which do not have, and for those which have a trivial subalgebra (cf. [16] and [17]). Now Theorem 2.1, combined with Theorem 6.1 allows a unified treatment.

Theorem 6.2. [16, 17] *Every finite, simple, surjective algebra of type* **2** *is affine.*

Let $\mathbf{S}$ be a finite simple algebra of type **2**. We want to show that if $\mathbf{S}$ is surjective, then it has a representation where the semi-affine algebra $\mathbf{A}$ is also surjective. To this end we need a slightly stronger condition than the one in the parentheses in Theorem 6.1. Moreover, when applying Theorem 2.1 for $\mathbf{A}$ it will be useful if (2.1.b) fails for $\mathbf{A}$. Therefore we start with a modification of Theorem 6.1.

Theorem 6.3. *Every finite simple algebra* $\mathbf{S}$ *of type* **2** *has a representation* $(\mathbf{A}, {}_K\widehat{A}, \mathbf{S}')$ *such that*

(6.3.i) *for every element* $a \in S'$ *the set* $S' - a$ *generates the vector space* ${}_K\widehat{A}$*, and*

(6.3.ii) $\mathbf{A}$ *has no nontrivial congruence which is a congruence of* ${}_K\widehat{A}$*.*

Proof. In the proof of [1; Theorem 13.5] the construction yields a representation $(\mathbf{A}, {}_K\widehat{A}, \mathbf{S}')$ where ${}_K\widehat{A} = ({}_K\widehat{V})^k$, ${}_K\widehat{V}$ is a 1-dimensional vector space, $k \geq 1$, and S' is a subset of V^k containing $(0, \ldots, 0)$ such that the projection mappings

$$\delta_i\colon\ S' \to V, \quad (v^0, \ldots, v^{k-1}) \mapsto v^i \quad (i = 0, \ldots, k-1)$$

are linearly independent, as members of the vector space $({}_K\widehat{V})^{S'}$.

(6.3.i) can be verified in the same way as its special case $a = (0, \ldots, 0)$ in [1]. Indeed, let $a = (a^0, \ldots, a^{k-1})$. Suppose $S' - a$ is contained in a proper subspace of $({}_K\widehat{V})^k$. Then there exist elements $c_0, \ldots, c_{k-1} \in K$, not all 0, such that for all $s = (s^0, \ldots, s^{k-1}) \in S'$ we have $\sum_{i=0}^{k-1} c_i(s^i - a^i) = 0$. For $s = (0, \ldots, 0)$ this implies $\sum_{i=0}^{k-1} c_i a^i = 0$. Hence

$$\sum_{i=0}^{k-1} c_i s^i = 0 \quad \text{for all} \quad s = (s^0, \ldots, s^{k-1}) \in S'.$$

This means that in the vector space $({}_K\widehat{V})^{S'}$ the equality $\sum_{i=0}^{k-1} c_i \delta_i = 0$ holds, contradicting the construction of S'. This proves (6.3.i).

Consider now arbitrary representation $(\mathbf{A}, {}_K\widehat{A}, \mathbf{S}')$ of $\mathbf{S}$, and assume $\mathbf{A}$ has a nontrivial congruence Θ which is a congruence of ${}_K\widehat{A}$. It is straightforward to check that in this case the algebra $\mathbf{A}/\Theta$ is semi-affine with respect to the vector space ${}_K\widehat{A}/\Theta$. Obviously, the restriction $\Theta|_{S'}$ of Θ to S' is a congruence of $\mathbf{S}'$. Since $0 \in S'$ and S' generates ${}_K\widehat{A}$, therefore S' is not contained in a single block of Θ; i.e., $\Theta|_{S'}$ is not the full relation. As $\mathbf{S}'$ is simple, $\Theta|_{S'} = \Delta$. Hence $(\mathbf{A}/\Theta,\ {}_K\widehat{A}/\Theta, \mathbf{S}'/\Theta|_{S'})$ is again a representation of $\mathbf{S}$.

Clearly, whenever the representation $(\mathbf{A},\ {}_K\widehat{A}, \mathbf{S}')$ of $\mathbf{S}$ satisfies (6.3.i), so does $(\mathbf{A}/\Theta,\ {}_K\widehat{A}/\Theta, \mathbf{S}'/\Theta|_{S'})$. Consequently, among all representations $(\mathbf{A},\ {}_K\widehat{A}, \mathbf{S}')$ of $\mathbf{S}$ satisfying (6.3.i), every representation for which $|A|$ is minimal possesses property (6.3.ii) as well. ◇

Lemma 6.4 *Let* $\mathbf{S}$ *be a finite simple algebra of type* $\mathbf{2}$*, and let* $(\mathbf{A},\ {}_K\widehat{A}, \mathbf{S}')$ *be a representation of* $\mathbf{S}$ *satisfying condition (6.3.i). If* $\mathbf{S}$ *is surjective, then so is* $\mathbf{A}$.

Proof. Let f be a fundamental operation of $\mathbf{S}$, say f is n-ary. The operation of $\mathbf{A}$ corresponding to f, denoted $f^{\mathbf{A}}$, is an n-ary polynomial operation of ${}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A}$, say $f^{\mathbf{A}}(x_0,\ldots,x_{n-1}) = \sum_{i=0}^{n-1} r_i x_i + a$ $(r_0,\ldots,r_{n-1} \in \mathrm{End}\,{}_K\widehat{A},\ a \in A)$. Since $0 \in S'$, we have $a = f^{\mathbf{A}}(0,\ldots,0) = f(0,\ldots,0) \in S'$. Let $\mathrm{Im}\, r_i$ denote the range of r_i; it is a subspace of ${}_K\widehat{A}$.

Suppose now that f is surjective. Then

$$S' = f^{\mathbf{A}}(S',\ldots,S') = \sum_{i=0}^{n-1} r_i S' + a,$$

implying that $S' - a$ is contained in the subspace $\sum_{i=0}^{n-1} \mathrm{Im}\, r_i$ of ${}_K\widehat{A}$. By assumption $S' - a$ generates ${}_K\widehat{A}$, therefore $\sum_{i=0}^{n-1} \mathrm{Im}\, r_i$ equals ${}_K\widehat{A}$. Hence $f^{\mathbf{A}}$ is surjective. ◇

Now we are in a position to prove Theorem 6.2.

Proof of Theorem 6.2. Let $\mathbf{S}$ be a finite, simple, surjective algebra of type $\mathbf{2}$, and consider a representation $(\mathbf{A},\ {}_K\widehat{A}, \mathbf{S}')$ of $\mathbf{S}$ satisfying both conditions in Theorem 6.3.

We show that conditions (2.1.b)–(2.1.d) in Theorem 2.1 fail for $\mathbf{A}$. (2.1.b) fails in view of (6.3.ii), and (2.1.c) fails because otherwise $\mathbf{A}$, and hence $\mathbf{S}$, too, would be strongly Abelian. Finally, assume (2.1.d) holds for $\mathbf{A}$. Clearly, the equivalence relation Θ_T is a congruence of ${}_K\widehat{A}$. Now we make use of the fact that by Lemma 6.4 $\mathbf{A}$ is a surjective algebra. It follows from Lemma 5.1 (5.1.iii) (1) that Θ_T is a congruence of $\mathbf{A}$ as well, hence by the assumption (6.3.ii) we conclude that $\Theta_T = \Delta$. Thus (5.1.iii) (3) yields that $\mathbf{A}$, and hence also $\mathbf{S}$, is strongly Abelian, which is impossible.

Consequently, by Theorem 2.1, $\mathbf{A}$ is affine with respect to ${}_K\widehat{A}$. Therefore the subalgebra $\mathbf{S}'$ of $\mathbf{A}$ is an affine algebra, and hence $\mathbf{S}$ is also affine. ◇

(B) *Finite simple algebras of type* $\mathbf{2}$ *generating minimal varieties*

A variety V is called *minimal* if it has exactly two subvarieties: V itself and the trivial variety. Obviously, every locally finite minimal variety is generated by a finite simple algebra having no nontrivial proper subalgebra. Recently, while investigating the problem which finite simple algebras of type $\mathbf{2}$ generate residually small varieties, K. Kearnes, E. W. Kiss, and M. Valeriote noticed the following interesting fact:

Theorem 6.5. [3] *Every finite simple algebra of type* $\mathbf{2}$ *that generates a minimal variety is affine.*

Here we derive this result from Theorems 2.1 and 6.1. For an algebra $\mathbf{A}$ the variety generated by $\mathbf{A}$ is denoted by $V(\mathbf{A})$.

Lemma 6.6. *Let* $\mathbf{S}$ *be a finite simple algebra of type* $\mathbf{2}$*, and let* $(\mathbf{A},\ {}_K\widehat{A}, \mathbf{S}')$ *be a representation of* $\mathbf{S}$*. The algebra* $\mathbf{A}$ *generates the same variety as* $\mathbf{S}$.

Proof. Since $\mathbf{S}$ is isomorphic to the subalgebra $\mathbf{S}'$ of $\mathbf{A}$, it suffices to prove that every identity that holds in $\mathbf{S}'$, holds in $\mathbf{A}$. Let t and $\bar{t}$ be arbitrary n-ary terms in the language of $\mathbf{A}$ $(n \geq 1)$; the corresponding term operations are denoted as $t^{\mathbf{A}}$, resp. $\bar{t}^{\mathbf{A}}$. Let, say, $t^{\mathbf{A}} = \sum_{i=0}^{n-1} r_i x_i + a$ and $\bar{t}^{\mathbf{A}} = \sum_{i=0}^{n-1} \bar{r}_i x_i + \bar{a}$.

Looking at the n-tuples $(0,\ldots,0)$ and $(0,\ldots,0,s,0,\ldots,0)$ from $(S')^n$ we see that if the identity $t = \bar{t}$ holds in $\mathbf{S}'$ then $a = \bar{a}$ and the endomorphisms $r_i, \bar{r}_i$ $(i = 0,\ldots,n-1)$ of ${}_K\widehat{A}$ coincide on S'. Since S' generates ${}_K\widehat{A}$, we get that $r_i = \bar{r}_i$ for all $i = 0,\ldots,n-1$, and hence the identity $t = \bar{t}$ holds in $\mathbf{A}$. ◇

It is well known that for an algebra $\mathbf{A}$ the subvarieties of $V(\mathbf{A})$ are in one-to-one correspondence with the congruences of the clone $\operatorname{Clo}\mathbf{A}$ of $\mathbf{A}$. Therefore $\mathbf{A}$ generates a minimal variety if and only if every homomorphism of $\operatorname{Clo}\mathbf{A}$ into the clone of a nontrivial algebra is one-to-one.

We will need a clone homomorphism which is implicit in [4].

Let A be a finite set, and let $T = \{\Theta_0, \dots, \Theta_{m-1}\}$ be a q-regular family of equivalence relations on A. The clone consisting of all operations on A preserving λ_T will be denoted by $\mathcal{C}_T$. The other clone playing a role is the clone of the mth matrix power of the two-element unary algebra $\mathbf{U} = (2; 0)$ whose only fundamental operation is the constant with value 0.

In [4] it is shown that $\mathbf{U}^{[m]}$, made into an indexed algebra in an appropriate way, generates one of the proper subvarieties of the variety generated by the algebra $(A; \mathcal{C}_T)$. Our aim is to describe the clone homomorphism witnessing this fact.

Let f be any operation in $\mathcal{C}_T$, say f is n-ary. We put

$$I_f = \{i\colon\ 0 \le i \le m-1,\ \text{the range of } f \text{ meets each block of } \Theta_i\}.$$

By Lemma 5.1 (5.1.ii), to each $i \in I_f$ there correspond indices $j = j_i$, $l = l_i$ $(0 \le j \le m-1,\ 0 \le l \le n-1)$ with property $(*)$. It is easy the see that these indices are uniquely determined. This yields two mappings

$$\sigma_f\colon I_f \to m, \quad i \mapsto j_i, \qquad \text{and} \qquad \mu_f\colon I_f \to n, \quad i \mapsto l_i$$

with the following property: for every $i \in I_f$,

$$f(x_0, \dots, x_{n-1})\, \Theta_i\, f(y_0, \dots, y_{n-1}) \qquad \Leftrightarrow \qquad x_{i\mu_f}\, \Theta_{i\sigma_f}\, y_{i\mu_f}.$$

Lemma 6.7. *The mapping*

$$\chi\colon \mathcal{C}_T \to \operatorname{Clo}\mathbf{U}^{[m]}, \quad f \mapsto h^{\sigma}_{\mu}[g_0, \dots, g_{m-1}] \quad \textit{with} \quad \sigma|_{I_f} = \sigma_f,\ \mu|_{I_f} = \mu_f,\ \textit{and}\ g_i = \begin{cases} \mathrm{id} & \textit{if } i \in I_f \\ 0 & \textit{otherwise} \end{cases}$$

is a clone homomorphism.

Proof. Notice first that χ is well-defined, since for each $i \in m$ outside I_f where the values of σ and μ are arbitrary, g_i is constant, and hence the operation $h^{\sigma}_{\mu}[g_0, \dots, g_{m-1}]$ is independent on these values.

It is straightforward to check that for each $n \ge 1$ and $0 \le i \le n-1$ the mapping χ sends the ith n-ary projection from $\mathcal{C}_T$ into the ith n-ary projection from $\operatorname{Clo}\mathbf{U}^{[m]}$. We verify that for each $k, n \ge 1$, χ commutes with the clone operation 'substituting k-ary operations into an n-ary operation'.

Let f be an n-ary, and f_j $(0 \le j \le n-1)$ be k-ary operations from $\mathcal{C}_T$, and consider the k-ary operation $\bar{f} = f(f_0, \dots, f_{n-1})$. Furthermore, let

$$f\chi = h^{\sigma}_{\mu}[g_0, \dots, g_{m-1}], \quad f_j\chi = h^{\tau_j}_{\nu_j}[g_{0,j}, \dots, g_{m-1,j}], \quad \bar{f}\chi = h^{\bar{\sigma}}_{\bar{\mu}}[\bar{g}_0, \dots, \bar{g}_{m-1}].$$

For $0 \le i \le k-1$, $\bar{f} = f(f_0, \dots, f_{n-1})$ meets each block of Θ_i if and only if

(1) f meets each block of Θ_i,

whence for all elements $x_0, \dots, x_{k-1}, y_0, \dots, y_{k-1} \in A$ we have

$$\begin{aligned} & \bar{f}(x_0, \dots, x_{k-1})\, \Theta_i\, \bar{f}(y_0, \dots, y_{k-1}) \\ \Leftrightarrow\ & f(f_0(x_0, \dots, x_{k-1}), \dots\ )\, \Theta_i\, f(f_0(y_0, \dots, y_{k-1}), \dots\ ) \\ \Leftrightarrow\ & f_{i\mu}(x_0, \dots, x_{k-1})\, \Theta_{i\sigma}\, f_{i\mu}(y_0, \dots, y_{k-1}), \end{aligned} \qquad (\bullet)$$

and

(2) $f_{i\mu}$ meets each block of $\Theta_{i\sigma}$,

whence $(\bullet)$ can be continued with

$$\Leftrightarrow\ x_{i\sigma\nu_{i\mu}}\, \Theta_{i\sigma\tau_{i\mu}}\, y_{i\sigma\nu_{i\mu}}.$$

Thus

$$I_{\bar f}=\{i\in I_f\colon\ i\sigma\in I_{f_{i\mu}}\},\quad \bar\sigma|_{I_{\bar f}}=\sigma_{\bar f}\colon\ I_{\bar f}\to m,\ i\mapsto i\sigma\tau_{i\mu},\quad \bar\mu|_{I_{\bar f}}=\mu_{\bar f}\colon\ I_{\bar f}\to k,\ i\mapsto i\sigma\nu_{i\mu}.$$

These data uniquely determine the operation $\bar f\chi$.

Now let us compute the operation $(f\chi)(f_0\chi,\dots,f_{n-1}\chi)$:

$$\begin{aligned}
&(f\chi)(f_0\chi,\dots,f_{n-1}\chi)(x_0,\dots,x_{k-1})\\
&\quad= h_\mu^\sigma[g_0,\dots,g_{m-1}](h_{\nu_0}^{\tau_0}[g_{00},\dots,g_{m-1,0}](x_0,\dots,x_{k-1}),\dots,h_{\nu_{n-1}}^{\tau_{n-1}}[g_{0,n-1},\dots,g_{m-1,n-1}](x_0,\dots,x_{k-1}))\\
&\quad= h_\mu^\sigma[g_0,\dots,g_{m-1}]((g_{00}(x_{0\nu_0}^{0\tau_0}),\dots,g_{m-1,0}(x_{(m-1)\nu_0}^{(m-1)\tau_0})),\dots,(g_{0,n-1}(x_{0\nu_{n-1}}^{0\tau_{n-1}}),\dots,g_{m-1,n-1}(x_{(m-1)\nu_{n-1}}^{(m-1)\tau_{n-1}})))\\
&\quad= (g_0g_{0\sigma,0\mu}(x_{0\sigma\nu_{0\mu}}^{0\sigma\tau_{0\mu}}),\dots,g_{m-1}g_{(m-1)\sigma,(m-1)\mu}(x_{(m-1)\sigma\nu_{(m-1)\mu}}^{(m-1)\sigma\tau_{(m-1)\mu}})).
\end{aligned}$$

Clearly, for $0\le i\le m-1$ the mapping $g_ig_{i\sigma,i\mu}$ is the identity if and only if $i\in I_f$ and $i\sigma\in I_{f_{i\mu}}$, or equivalently, if $i\in I_{\bar f}$. Therefore $g_ig_{i\sigma,i\mu}=\bar g_i$ for all $0\le i\le m-1$, implying that

$$(f\chi)(f_0\chi,\dots,f_{n-1}\chi)=h_{\bar\mu}^{\bar\sigma}[\bar g_0,\dots,\bar g_{m-1}]=\bar f\chi=(f(f_0,\dots,f_{n-1}))\chi,$$

as required. ◇

It is worth noting, though we will not need this fact later on, that the homomorphism χ in Lemma 6.7 is surjective.

After these preparations we prove Theorem 6.5.

Proof of Theorem 6.5. Let $\mathbf{S}$ be a finite simple algebra of type $\mathbf{2}$, and consider a representation $(\mathbf{A},\ {}_K\widehat{A},\mathbf{S}')$ of $\mathbf{S}$ satisfying both conditions in Theorem 6.3. By Lemma 6.6 $V(\mathbf{S})=V(\mathbf{A})$. Let us apply Theorem 2.1 for $\mathbf{A}$. In the same manner as in the proof of Theorem 6.2 we see that conditions (2.1.b) and (2.1.c) fail. Therefore it suffices to show that if (2.1.d) holds, then the variety $V(\mathbf{A})$ is not minimal.

Assume $\mathbf{A}$ has a compatible relation λ_T for some q-regular family $T=\{\Theta_0,\dots,\Theta_{m-1}\}$ of congruences of ${}_K\widehat{A}$ with $q>|K|$. Clearly, $\mathrm{Clo}\,\mathbf{A}$ is a subclone of $\mathcal{C}_T$, so the homomorphism χ in Lemma 6.7 restricts to a clone homomorphism $\mathrm{Clo}\,\mathbf{A}\to\mathrm{Clo}\,\mathbf{U}^{[m]}$. Thus, if the fundamental operations of $\mathbf{A}$ are f_κ ($\kappa<\alpha$), then the reduct $\mathbf{B}=(2^m;\ \{f_\kappa\chi\colon\ \kappa<\alpha\}$ of $\mathbf{U}^{[m]}$ is contained in the variety $V(\mathbf{A})$. To show that $V(\mathbf{B})$ is a proper subvariety of $V(\mathbf{A})$, we have to verify that χ is not one-to-one on $\mathrm{Clo}\,\mathbf{A}$.

Since $\mathbf{A}$ is a finite simple algebra of type $\mathbf{2}$, its induced minimal algebras are polynomially equivalent to one-dimensional vector spaces. Therefore $\mathbf{A}$ has a non-constant ternary polynomial operation d_0 such that for arbitrary elements u,v in the range U of d_0 we have $d_0(v,v,u)=u=d_0(u,v,v)$. Let us construct the following polynomial operations of $\mathbf{A}$: $\bar d_0(x)=d_0(x,x,x)$ and $d(x,y,z)=d_0(\bar d_0(x),\bar d_0(y),\bar d_0(z))$. Obviously, $\bar d_0(u)=u$ for all $u\in U$, hence d and d_0 coincide on U, moreover,

$$d(a,a,u)=u=d(u,a,a)\quad\text{for all}\quad u\in U,\ a\in A.$$

For some $n\ge 3$, $\mathbf{A}$ has an n-ary term operation f and elements $a_3,\dots,a_{n-1}$ such that

$$f(x_0,x_1,x_2,a_3,\dots,a_{n-1})=d(x_0,x_1,x_2)\quad\text{for all}\quad x_0,x_1,x_2\in A.$$

The properties of d ensure that $d(x,y,z)=ex-ey+ez+a$ for some $e\in\mathrm{End}\,{}_K\widehat{A}$ and $a\in A$ with $e^2=e$, $ea=0$, and hence

$$f(x_0,x_1,x_2,x_3,\dots,x_{n-1})=ex_0-ex_1+ex_2+r_3x_3+\ldots+r_{n-1}x_{n-1}+b$$

for some $r_3,\dots,r_{n-1}\in\mathrm{End}\,{}_K\widehat{A}$ and $b\in A$. Making use of Lemma 5.1 (5.1.ii) one can easily see that for each $i\in I_f$ (if any) we must have $ex\,\Theta_i\,ey$ for all $x,y\in A$, and $i\mu_f\in\{3,\dots,n-1\}$. Hence $f\chi$ does not depend on its variables x_0,x_1,x_2. Now it is clear that for the n-ary term operation $f'(x_0,x_1,x_2,x_3,\dots,x_{n-1})=ex_0-ex_0+ex_2+r_3x_3+\ldots+r_{n-1}x_{n-1}+b$ of $\mathbf{A}$ arising from f by identifying its variables x_0,x_1, we have $f'\ne f$ and $f'\chi=f\chi$, completing the proof. ◇

(C) *Maximal subclones of* $\mathrm{Pol}\,({}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A})$

Let K be a finite field, and ${}_K\widehat{A} = (A; +, K)$ an m-dimensional vector space over K. Our aim is to determine the maximal subclones of the clone $\mathcal{P}({}_K\widehat{A}) = \mathrm{Pol}\,({}_{(\mathrm{End}\,{}_K\widehat{A})}\widehat{A})$. For any coset S of a subspace of ${}_K\widehat{A}$, $\mathcal{R}_S({}_K\widehat{A})$ will denote the clone consisting of all operations $f \in \mathcal{P}({}_K\widehat{A})$ such that $f(s, \ldots, s) = s$ for all $s \in S$, and for any subspace U of ${}_K\widehat{A}$, $\mathcal{T}_U({}_K\widehat{A})$ will denote the clone consisting of all operations $f \in \mathcal{P}({}_K\widehat{A})$ which admit every translation $x + u$ with $u \in U$ as an automorphism. Clearly, $\mathcal{P}({}_K\widehat{A}) = \mathcal{T}_{\{0\}}({}_K\widehat{A})$.

We shall need a description of the clones of simple affine algebras. Note that if $\mathbf{A}$ is a finite simple algebra which is affine with respect to an Abelian group $\widehat{A}$, then $\widehat{A}$ is an elementary Abelian p-group for some prime p.

Theorem 6.8. [14] *For every finite simple algebra $\mathbf{A} = (A; F)$ which is affine with respect to an elementary Abelian p-group $\widehat{A} = (A; +)$ (p prime), there exist a finite field K and a vector space ${}_K\widehat{A} = (A; +, K)$ such that*

$$\mathrm{Clo}\,\mathbf{A} = \mathcal{R}_S({}_K\widehat{A}) \quad \textit{for a coset } S \textit{ of a subspace of } {}_K\widehat{A},$$

or

$$\mathrm{Clo}\,\mathbf{A} = \mathcal{T}_U({}_K\widehat{A}) \quad \textit{for a subspace } U \textit{ of } {}_K\widehat{A}.$$

K and ${}_K\widehat{A}$ are uniquely determined by the fact that the ring of $\mathbf{A}$ is $\mathrm{End}\,{}_K\widehat{A}$.

From now on, we keep K and ${}_K\widehat{A}$ fixed, and for brevity we write $\mathcal{P}$, $\mathcal{R}_S$, $\mathcal{T}_U$ instead of $\mathcal{P}({}_K\widehat{A})$, $\mathcal{R}_S({}_K\widehat{A})$, $\mathcal{T}_U({}_K\widehat{A})$, respectively. First we will look at the isomorphic copies of the clone $\mathcal{D} = \mathrm{Clo}\,(K; \mathrm{Pol}_1\,({}_KK))^{[m]}$, as suggested by (2.1.c) in Theorem 2.1. Let $\varphi\colon ({}_KK)^m \to {}_K\widehat{A}$ be an arbitrary vector space isomorphism. The clone on A corresponding to $\mathcal{D}$ under this mapping is

$$\varphi^{-1}\mathcal{D}\varphi = \{(f(x_0\varphi^{-1}, \ldots, x_{n-1}\varphi^{-1})\varphi\colon\ f(x_0, \ldots, x_{n-1}) \in \mathcal{D}\}.$$

Consider the unit base $e_0 = (1, 0, \ldots, 0), \ldots, e_{m-1} = (0, \ldots, 0, 1)$ in $({}_KK)^m$. Clearly, $({}_KK)^m$ is a direct sum of its one-dimensional subspaces Ke_i $(0 \le i \le m-1)$, and ${}_K\widehat{A}$ is a direct sum of the image spaces $(Ke_i)\varphi$ $(0 \le i \le m-1)$. The family $\{(Ke_i)\varphi\colon\ 0 \le i \le m-1\}$ will be called the direct decomposition of ${}_K\widehat{A}$ determined by φ.

Lemma 6.9. *Let ${}_K\widehat{A}$ be an m-dimensional vector space over a finite field K, and let $\varphi_1, \varphi_2\colon ({}_KK)^m \to {}_K\widehat{A}$ be arbitrary vector space isomorphisms. The clones $\varphi_1^{-1}\mathcal{D}\varphi_1$ and $\varphi_2^{-1}\mathcal{D}\varphi_2$ coincide if and only if φ_1 and φ_2 determine the same direct decomposition of ${}_K\widehat{A}$.*

Proof. It suffices to show that for arbitrary automorphism ψ of $({}_KK)^m$ the equality $\psi^{-1}\mathcal{D}\psi = \mathcal{D}$ holds if and only if ψ determines the direct decomposition $\{Ke_i\colon\ 0 \le i \le m-1\}$, that is, if and only if there exist scalars $c_i \in K$ $(0 \le i \le m-1)$ and a permutation $\pi \in S_m$ such that $e_i\psi = c_ie_{i\pi}$ for all $0 \le i \le m-1$.

Suppose ψ has this property, and let $\bar{\psi} = h^{\pi^{-1}}[g_0, \ldots, g_{m-1}]$ where $g_i(x) = c_ix$ $(0 \le i \le m-1)$. Then $\psi = \bar{\psi}$, because the two mappings agree on the vectors $e_0, \ldots, e_{m-1}$, and $\bar{\psi}$, too, is an automorphism of $({}_KK)^m$. Thus $\psi \in \mathcal{D}$, and the equality $\psi^{-1}\mathcal{D}\psi = \mathcal{D}$ is obvious.

Conversely, assume $\psi^{-1}\mathcal{D}\psi = \mathcal{D}$ holds. Since the mapping

$$\mathcal{D} \to \psi^{-1}\mathcal{D}\psi = \mathcal{D}, \quad f(x_0, \ldots, x_{n-1}) \mapsto (f(x_0\psi^{-1}, \ldots, x_{n-1}\psi^{-1}))\psi$$

is a clone isomorphism, it sends the m-ary diagonal operation $h^{\mathrm{id}}_{\mathrm{id}}[\mathrm{id}, \ldots, \mathrm{id}]$, which depends on all of its variables, into an m-ary idempotent operation depending on all of its variables. Hence (replacing $x_i\psi^{-1}$ with y_i) we get that for some $\mu \in S_m$,

$$(h^{\mathrm{id}}_{\mathrm{id}}[\mathrm{id}, \ldots, \mathrm{id}](y_0, \ldots, y_{m-1}))\psi = h^{\mathrm{id}}_{\mu}[\mathrm{id}, \ldots, \mathrm{id}](y_0\psi, \ldots, y_{m-1}\psi)$$

for all $y_0, \ldots, y_{m-1} \in ({}_KK)^m$. It follows now that for any $0 \le i \le m-1$

$$e_i\psi = (h^{\mathrm{id}}_{\mathrm{id}}[\mathrm{id}, \ldots, \mathrm{id}](0, \ldots, 0, \overbrace{e_i}^{i\text{th}}, 0, \ldots, 0))\psi = h^{\mathrm{id}}_{\mu}[\mathrm{id}, \ldots, \mathrm{id}](0, \ldots, 0, \overbrace{e_i\psi}^{i\text{th}}, 0, \ldots, 0) \in Ke_{i\mu^{-1}}.$$

This completes the proof. $\diamond$

By Lemma 6.9 the clone $\varphi^{-1}\mathcal{D}\varphi$, where $\varphi\colon ({}_KK)^m \to {}_K\widehat{A}$ is a vector space isomorphism, depends only on the corresponding direct decomposition $\{(Ke_i)\varphi\colon\ 0 \le i \le m-1\}$ of ${}_K\widehat{A}$, or equivalently, on the corresponding q-regular family $T = \{\Theta_0, \dots, \Theta_{m-1}\}$ of congruences of ${}_K\widehat{A}$ with $q = |K|$ and $\Theta_T = \Delta$. (Notice that here the case $|K| = 2$ is also allowed.) Therefore we will use the notation $\mathcal{D}_T$ for $\varphi^{-1}\mathcal{D}\varphi$.

For arbitrary q-regular family T of congruences of ${}_K\widehat{A}$ with $q \ge 3$, $\mathcal{E}_T$ will denote the clone consisting of all operations of $\mathcal{P}$ preserving λ_T (i.e., $\mathcal{E}_T = \mathcal{P} \cap \mathcal{C}_T$). We note that in view of Lemma 5.3, for the case $q = |K| > 2$ we have $\mathcal{D}_T = \mathcal{E}_T$.

To investigate inclusions among the clones $\mathcal{E}_T$ for various T's we need some facts on these clones. We will use the notation $\mathcal{E}_T^+$ for the set of all surjective operations in $\mathcal{E}_T$. If $r \in \operatorname{End}_K\widehat{A}$, then $\operatorname{Im} r$ will stand for the range of r, while if Θ is a congruence of ${}_K\widehat{A}$, then $\operatorname{Ker}\Theta$ will stand for the block of Θ containing 0 (both are subspaces of ${}_K\widehat{A}$). If for some elements $e_0, \dots, e_{k-1} \in \operatorname{End}_K\widehat{A}$ we have $\sum_{i=0}^{k-1} e_i = 1$ and $e_i^2 = e_i$, $e_ie_j = 0$ for all $0 \le i, j \le k-1$, $i \ne j$, then they will be called *pairwise orthogonal idempotents summing up to* 1.

Lemma 6.10. *Let ${}_K\widehat{A}$ be a finite dimensional vector space over a finite field K, and let $T = \{\Theta_0, \dots, \Theta_{d-1}\}$ be a q-regular family of congruences of ${}_K\widehat{A}$ ($q \ge 3$). For $0 \le i \le d-1$ let $U_i = \operatorname{Ker} \bigcap\{\Theta_j\colon\ 0 \le j \le d-1,\ j \ne i\}$, and put $U_T = \operatorname{Ker}\Theta_T$.*

(6.10.i) *$\mathcal{E}_T$ contains all operations from $\mathcal{P}$ whose range has dimension less than $\dim U_0 = \dots = \dim U_{d-1}$; however, for any element $r \in \operatorname{End}_K\widehat{A}$ with $\operatorname{Im} r = U_0$ we have $rx + ry \notin \mathcal{E}_T$.*

(6.10.ii) *For arbitrary elements $r \in \operatorname{End}_K\widehat{A}$, $x - ry + rz \in \mathcal{E}_T$ if and only if $\operatorname{Im} r \subseteq U_T$.*

(6.10.iii) *Θ_T is a maximal congruence of the algebra $(A; \mathcal{E}_T^+)$.*

(6.10.iv) *$\mathcal{E}_T$ contains an operation $\sum_{i=0}^{d} e_ix_i$ such that $e_0, \dots, e_d$ are pairwise orthogonal idempotents in $\operatorname{End}_K\widehat{A}$ summing up to 1, and $\operatorname{Im} e_d = U_T$, $\operatorname{Im} e_i + U_T = U_i$ for all $0 \le i \le d-1$.*

(6.10.v) *For every operation $\sum_{i=0}^{n} e_i'x_i \in \mathcal{E}_T$ such that $e_0', \dots, e_n'$ are pairwise orthogonal nonzero idempotents in $\operatorname{End}_K\widehat{A}$ summing up to 1, and $\operatorname{Im} e_n' = U_T$, we have $n \le d$; furthermore, if $n = d$, then there exists a permutation $\pi \in S_d$ with $\operatorname{Im} e_i' + U_T = U_{i\pi}$ for all $0 \le i \le d-1$.*

Proof. (6.10.i) Both parts of the claim are easy to check by the definition of $\mathcal{E}_T$.

(6.10.ii) The operations of the form $x - ry + rz$ are clearly surjective (in fact, idempotent), so the claim follows by a straightforward application of Lemma 5.1 (5.1.ii).

(6.10.iii) Clearly, T yields a direct decomposition

$$ {}_K\widehat{A}/\Theta_T = {}_K\widehat{A}/U_T = U_0/U_T \oplus \dots \oplus U_{d-1}/U_T. $$

Fixing a vector space ${}_K\widehat{V} = (q; +, K)$ we get an isomorphism $\varphi\colon {}_K\widehat{A}/U_T = {}_K\widehat{A}/\Theta_T \to {}_K\widehat{V}^d = (q; +, K)^d$ carrying T into the standard q-regular family of congruences on ${}_K\widehat{V}^d$. By Lemma 5.1 (5.1.iii)(a) Θ_T is a congruence of the surjective algebra $\mathbf{A} = (A; \mathcal{E}_T^+)$. Since the operations of $\mathbf{A}$ are exactly the surjective polynomial operations of ${}_{(\operatorname{End}_K\widehat{A})}\widehat{A}$ preserving λ_T, and since Θ_T is a congruence of ${}_K\widehat{A}$, it is not hard to see that the operations of the quotient algebra $\mathbf{A}/\Theta_T$ are exactly the surjective polynomial operations of ${}_{(\operatorname{End}({}_K\widehat{A}/\Theta_T))}(\widehat{A}/\Theta_T)$ preserving λ_{T/Θ_T}. Making use of Lemma 5.1 (5.1.iii)(c) for the vector space isomorphism φ we conclude that $\mathbf{A}/\Theta_T$ is isomorphic, under φ, to the reduct of $(q; S_q)^{[d]}$ whose operations are exactly all surjective term operations of $(q; S_q)^{[d]}$ which are simultaneously polynomial operations of ${}_{(\operatorname{End}_K\widehat{V}^d)}(\widehat{V}^d)$. Hence the operations of this algebra are exactly all surjective term operations of the algebra $(q; G)^{[d]}$ where G consists of all surjective unary polynomial operations of ${}_{(\operatorname{End}_K\widehat{V})}\widehat{V}$. It is well known that the d-ary operation $h_{\mathrm{id}}^{\mathrm{id}}[\mathrm{id}, \dots, \mathrm{id}]$ together with the unary operations $h^{(0\ 1\ \cdots\ d-1)}[\mathrm{id}, \dots, \mathrm{id}]$ and $h^{\mathrm{id}}[g, \dots, g]$ ($g \in G$) — all of them surjective term operations of $(q; G)^{[d]}$ — generate the clone of $(q; G)^{[d]}$. Hence $\mathbf{A}/\Theta_T$ is isomorphic to an algebra term equivalent to $(q; G)^{[d]}$. Since the unary algebra $(q; G)$ is simple, its matrix power $(q; G)^{[d]}$ is also simple, implying that $\mathbf{A}/\Theta_T$ is simple. Thus Θ_T is a maximal congruence of $\mathbf{A}$, as was to be proved.

(6.10.iv) Selecting subspaces V_i of U_i with $U_i = V_i \oplus U_T$ we get a direct decomposition ${}_K\widehat{A} = V_0 \oplus \ldots \oplus V_{d-1} \oplus U_T$. For the endomorphisms $e_0, \ldots, e_{d-1}, e_d$ projecting onto the subspaces $V_0, \ldots, V_{d-1}, U_T$ respectively, all requirements are satisfied.

(6.10.v) Let f be the $(d+1)$-ary operation constructed in (6.10.iv), and f' an $(n+1)$-ary operation $\sum_{i=0}^{n} e_i' x_i$ satisfying the requirements described in (6.10.v). Clearly, f, f' are operations of the algebra $\mathbf{A}$ considered in the proof of (6.10.iii). The corresponding operations of $\mathbf{A}/\Theta_T$ are

$$\bar{f}(y_0 + U_T, \ldots, y_d + U_T) = \sum_{i=0}^{d} e_i(y_i + U_T) \qquad \text{and} \qquad \bar{f}'(y_0 + U_T, \ldots, y_n + U_T) = \sum_{i=0}^{n} e_i'(y_i + U_T).$$

Obviously, $\bar{f}$ and $\bar{f}'$ are idempotent operation depending exactly on their first d, resp. n variables. In the isomorphic copy of $\mathbf{A}/\Theta_T$ under φ they are assigned to some term operations of $(q; G)^{[d]}$ with the same properties. Since every term operation of $(q; G)^{[d]}$ depends on at most d variables, we get that $n \leq d$. Suppose $n = d$, and omit the fictitious variable x_d from $\bar{f}, \bar{f}'$ and from the corresponding term operations of $(q; G)^{[d]}$. Since any two d-ary idempotent term operations of $(q; G)^{[d]}$ depending on all of their variables arise from each other by permuting variables, we get the same for $\bar{f}$ and $\bar{f}'$. This concludes the proof of the lemma. ◇

Now we are in a position to describe the maximal subclones of $\mathcal{P}$. Since $\mathcal{P}$ is finitely generated, every proper subclone of $\mathcal{P}$ is contained in a maximal one. This fact will follow also from the proof of the theorem below.

Theorem 6.11. *The maximal subclones of $\mathcal{P}$ are the following:*

(a) *$\mathcal{R}_{\{s\}}$ for an element $s \in A$, or*
$\mathcal{T}_U$ for a one-dimensional subspace U of ${}_K\widehat{A}$;

(b) *$\mathrm{Pol}\,({}_R\widehat{A})$ for a maximal subring R of $\mathrm{End}\,{}_K\widehat{A}$;*

(c) *$\mathcal{D}_T$ for a q-regular family T of congruences of ${}_K\widehat{A}$ with $q = |K|$ and $\Theta_T = \Delta$;*

(d) *$\mathcal{E}_T$ for a q-regular family T of congruences of ${}_K\widehat{A}$ with $q > |K|$.*

The clones listed above are pairwise distinct.

Proof. For a subclone $\mathcal{F}$ of $\mathcal{P}$ the ring of the algebra $(A; \mathcal{F})$ will also be denoted by $R_{\mathcal{F}}$.

First we prove that every proper subclone $\mathcal{F}$ of $\mathcal{P}$ is contained in one of the clones (a)–(d). Let us consider the algebra $\mathbf{A} = (A; \mathcal{F})$, which is obviously semi-affine with respect to ${}_K\widehat{A}$, and apply Theorem 2.1. Assume that (2.1.a) holds for $\mathbf{A}$, that is, $x - y + z \in \mathcal{F}$. It is not hard to see (cf. [13], [14]) that in this case all operations $\sum_{i=0}^{n-1} r_i x_i$ with $r_0, \ldots, r_{n-1} \in R_{\mathcal{F}}$ and $\sum_{i=0}^{n-1} r_i = 1$ belong to $\mathcal{F}$. Therefore, if $R_{\mathcal{F}} = \mathrm{End}\,{}_K\widehat{A}$, then $\mathbf{A}$ is simple, so $\mathcal{F} = \mathrm{Clo}\,\mathbf{A}$ is one of the clones occurring in Theorem 6.8. Looking at the inclusion relations among these clones we see that $\mathcal{F}$ is included in one of the clones in (a). If $R_{\mathcal{F}}$ is a proper subring of $\mathrm{End}\,{}_K\widehat{A}$, then $R_{\mathcal{F}} \subseteq R$ for some maximal unitary subring R of $\mathrm{End}\,{}_K\widehat{A}$, whence it is obvious that $\mathcal{F} \subseteq \mathrm{Pol}\,({}_R\widehat{A})$. Suppose that (2.1.b) holds for $\mathbf{A}$, say Θ is a nontrivial congruence of $\mathbf{A}$ which is a congruence of ${}_K\widehat{A}$. Clearly, Θ is a congruence of the algebra $(A; \mathcal{F}, x - y + z)$ as well. Hence the clone $\mathcal{F}'$ of this algebra is a proper subclone of $\mathcal{P}$ containing $\mathcal{F}$, and the same argument as before applies for $\mathcal{F}'$. Finally, if (2.1.c) or (2.1.d) holds for $\mathbf{A}$, then by Lemma 6.9 it is clear that $\mathcal{F}$ is contained in one of the clones in (c) or (d).

To prove the maximality and the distinctness of the clones listed in the theorem it suffices to verify that none of the clones are contained in any other one. It is straightforward to check that there is no inclusion among two clones if both are of type (a) or (b). By Lemma 6.9 the same holds if both clones are of type (c). Indeed, any two clones of type (c) are isomorphic, and hence contain for each n the same (finite!) number of n-ary operations; therefore an inclusion implies equality.

Let us consider now two clones of type (d), say $\mathcal{E}_T$ and $\mathcal{E}_{T'}$, and assume that $\mathcal{E}_T \subseteq \mathcal{E}_{T'}$. Here $T = \{\Theta_0, \ldots, \Theta_{d-1}\}$ is a q-regular, while $T' = \{\Theta_0', \ldots, \Theta_{d'-1}'\}$ is a q'-regular family of congruences of ${}_K\widehat{A}$. As in Lemma 6.10, the corresponding subspaces are $U_0, \ldots, U_{d-1}, U_T$ and $U_0', \ldots, U_{d'-1}', U_{T'}$, respectively. Applying Lemma 6.10 we verify that $T = T'$. Firstly, from (6.10.ii) we get that $U_T \subseteq U_{T'}$, and then from (6.10.iii) that $U_T = U_{T'}$. Now, by (6.10.i), $\dim U_0 \leq \dim U_0'$, which implies that $q = |U_0/U_T| \leq |U_0'/U_{T'}| = q'$. Furthermore, in view of (6.10.iv) and (6.10.v), we have $d \leq d'$. Since $q^d |U_T| = p^m = (q')^{d'} |U_{T'}|$, we conclude that $q = q'$ and $d = d'$. Thus, again making use of (6.10.iv) and (6.10.v) we see that $\{U_0, \ldots, U_{d-1}\} = \{U'_0, \ldots, U'_{d'-1}\}$, or equivalently, $T = T'$.

It remains to consider inclusions where the two clones are of different types. The clones in (b)–(d) contain all constants and all translations, therefore a clone of type (a) never contains a clone of different type. Since the ring of each clone in (b) is R and the ring of every other clone in the list is $\mathrm{End}\,{}_K\widehat{A}$, a clone of type (b) never contains a clone of different type. For a clone $\mathcal{F}$ of type (c) the algebra $(A;\mathcal{F})$ is strongly Abelian, while for other clones the corresponding algebras are not strongly Abelian (cf. (6.10.i)); therefore a clone of type (c) never contains a clone of different type. Finally, a clone of type (d) does not contain the operation $x - y + z$, and hence any clone of type (a) or (b). Observing that the analogue of (6.10.iii) and (6.10.iv) holds for the clone $\mathcal{D}_T$ for every q-regular family T of congruences of ${}_K\widehat{A}$ with $q = |K|$ and $\Theta_T = \Delta$, a similar argument as in the previous paragraph shows that $\mathcal{D}_T \not\subseteq \mathcal{E}_{T'}$ for any q'-regular family T' of congruences of ${}_K\widehat{A}$ with $q' > |K|$. $\diamond$

One might pose the problem of determining all maximal subclones of each clone $\mathcal{A}$ which is the clone of a simple affine algebra, see the description in Theorem 6.8 (not just the maximal subclones of $\mathcal{P}$, which are in some sense the 'largest' clones of these kinds). As at the beginning of the proof of Theorem 6.11, one can easily derive from Theorem 2.1 and Theorem 6.8 that every proper subclone of $\mathcal{A}$ where

(1) $\mathcal{A} = \mathcal{R}_S$ for a coset S of a subspace of ${}_K\widehat{A}$, or

(2) $\mathcal{A} = \mathcal{T}_U$ for a subspace U of ${}_K\widehat{A}$

is contained in one of the following clones:

- (a1) $\mathcal{R}_{S'}$ for a coset S' of a subspace of ${}_K\widehat{A}$ with $\dim(S' - S') = 1 + \dim(S - S)$ in case (1);
- (a2) $\mathcal{R}_S$ for a coset S of a subspace of ${}_K\widehat{A}$ with $S - S = U$, or
  $\mathcal{T}_{U'}$ for a subspace U' of ${}_K\widehat{A}$ with $\dim U' = 1 + \dim U$ in case (2);
- (b) $\mathcal{A} \cap \mathrm{Pol}\,({}_R\widehat{A})$ for a maximal subring R of $\mathrm{End}\,{}_K\widehat{A}$;
- (c) $\mathcal{A} \cap \mathcal{D}_T$ for a q-regular family T of congruences of ${}_K\widehat{A}$ with $q = |K|$ and $\Theta_T = \Delta$;
- (d) $\mathcal{A} \cap \mathcal{E}_T$ for a q-regular family T of congruences of ${}_K\widehat{A}$ with $q > |K|$.

However, the clones listed here are not necessarily pairwise incomparable. An analysis of inclusions among these clones my lead to an explicit description of the maximal subclones of $\mathcal{A}$.

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