Additional Problems

1. Let $\mathcal{P}_n$ be a regular polygon with n edges. By restricting the isometries in the dihedral group D_{2n} (see p. 6) to the set of vertices of $\mathcal{P}_n$ and to the set of edges of $\mathcal{P}_n$ we get two permutation representations of D_{2n} on n-element sets. Find all integers $n \geq 3$ for which these representations are equivalent.

2. Prove that the following conditions on a group G are equivalent:
(a) $\zeta G = 1$ and $\operatorname{Aut} G = \operatorname{Inn} G$;
(b) whenever $G \lhd K$ for some group K, then K is an internal direct product of G and another normal subgroup of K.

Definition. Let $H \leq \operatorname{Sym} X$ and $K \leq \operatorname{Sym} Y$. The *product action* of the complete wreath product[1] $H \,\bar{\wr}\, K$ on the set $X^Y = \{(x_y)_{y \in Y} : x_y \in X\}$ (the set of all functions $Y \to X$) is defined as follows:: for arbitrary $(x_y)_{y\in Y} \in X^Y$ and $\big(\kappa, (\gamma_y)_{y\in Y}\big) \in H \,\bar{\wr}\, K$,

$$(x_y)_{y\in Y}\big(\kappa, (\gamma_y)_{y\in Y}\big) \overset{\text{def}}{=} (x_{y\kappa^{-1}}\gamma_y)_{y\in Y}.$$

3. Prove that the product action of $H \wr K$ is primitive if
(a) H acts primitively but not regularly on X,
(b) Y is finite, and
(c) K acts transitively on Y.
Hint: Consider the stabilizer of $\mathbf{a} = (a, \ldots, a) \in X^Y$ in $H \wr K$ (in its product action).

4. Let $q = p^m$ where p is a prime number and m is a positive integer. Prove that
(1) $L(q)$ is isomorphic to $\mathrm{PGL}(2, q)$;
(2) If p is odd and m is even, then

$$T(q) = \left\{ \frac{xa+b}{xc+d} \ : ad - bc \text{ is a square in } \mathbb{F}_q \right\}$$

is a subgroup of index 2 in both $L(q)$ and $M(q)$, and $T(q)$ is isomorphic to $\mathrm{PSL}(2, q)$.

5. Let $G \leq \operatorname{Sym} X$ be a primitive permutation group, and let $x \in X$. Assume that G_x has an abelian normal subgroup K such that the conjugates K^g $(g \in G)$ of K generate G. Show that for arbitrary normal subgroup N of G either $N = 1$ or $G' \leq N$.
Hint: Show that if $1 \neq N \lhd G$, then $G = KN$.

The statement in Problem 5 yields the following sufficient condition for the simplicity of a primitive permutation group, which can be applied, for example, to prove the simplicity of $\mathrm{PSL}(2, F)$ for arbitrary field F with more than 3 elements.

Theorem. *If $G \leq \operatorname{Sym} X$ is a primitive permutation group such that $G = G'$ and a stabilizer G_x $(x \in X)$ has an abelian normal subgroup whose conjugates generate G, then G is simple.*

6. Find $\operatorname{Fit} G$ and $\operatorname{Frat} G$ for the following groups G:

$$G \quad = \quad S_n, \quad A_n, \quad D_{2n}, \quad \operatorname{Aff}(V), \quad \text{where } n \geq 2 \text{ and } V \text{ is a finite vector space.}$$

[1] The same definition also yields an action of the subgroup $H \wr K$ of $H \,\bar{\wr}\, K$ on X^Y.

7. Let G be a finite group. Prove that if $(xy)^3 = x^3y^3$ holds for all elements $x, y \in G$, then G is nilpotent, and all Sylow p-subgroups of G with $p \neq 3$ are abelian.

Hint: Show first that $(xy)^{-2} = x^{-2}y^{-2}$ and $[x, y]^6 = 1$ also hold for all $x, y \in G$. Derive that G has normal Sylow 2- and 3-subgroups, and consider the subgroups $G_n = \langle g^n : g \in G \rangle$ of G where n is a power of 2 or 3.

8. Let H and K be subnormal subgroups of G.
(1) Use Zassenhaus's Lemma to show that $H \cap K$ is a subnormal subgroup of G.
(2) Prove that if G is finite and H is a Sylow subgroup of G, then $H \lhd G$.

9. Use the theorem stated after Problem 5 to prove the simplicity of $\mathrm{PSL}(2, F)$ where F is a field with more than 3 elements.

10. Find and correct a gap in the proof of statement 3.2.8 in Robinson's book.

11. Let $H \leq \operatorname{Sym} X$ be a permutation group acting primitively on a finite set X, and let N, L be nontrivial normal subgroups of H such that $N \cap L = 1$. Prove that
(1) N and L are isomorphic centerless groups that act regularly on X, and the action of NL on X is similar to the following action of $N \times N$ on N:

$$N \times N \to \operatorname{Sym} N, \quad (\lambda, \nu) \mapsto \rho_{\lambda,\nu} \colon x \mapsto \lambda^{-1} x \nu = x^{\lambda}(\lambda^{-1}\nu).$$

(2) N and L are minimal normal subgroups of H, H has no other minimal normal subgroups, and NL acts primitively on X if and only if N is a nonabelian simple group.

Hint: (1) First prove that N acts regularly on X. To set up the similarity use the same bijection $\Phi \colon N \to X$ as in Lemma 3 on Handout 6.
(2) To prove the claim on primitivity, replace NL by the similar permutation group on N obtained via the similarity in part (1). (See the first paragraph of the proof of Theorem 5 (3) in the handout "Burnside's Theorem on 2-Transitive Permutation Groups".)

12. Let $M = \mathrm{PSL}(n+1, q)$ with its usual permutation action on the set X of points of the projective geometry $\mathcal{PG}(n, q)$. Prove the equality $N_{\operatorname{Sym} X}(M) = \mathrm{P\Gamma L}(n+1, q)$ for $n \geq 2$.

Hint: Show that every $\pi \in N_{\operatorname{Sym} X}(M)$ preserves collinearity, and use the Fundamental Theorem of Projective Geometry.

13. Prove that every group of order $3 \cdot 5 \cdot 11^2$ is isomorphic to a semidirect product $N \rtimes H$ where $|N| = 11^2$, and classify all such groups, up to isomorphism.

14. Let H be an abelian subgroup of finite index $|G : H| = n$ in G. Prove that
(1) if $\varphi \colon G \to H$ is a homomorphism and $\varphi|_H \colon H \to H$ is its restriction to H, then the transfer of $\varphi|_H$ is

$$G \to H, \quad x \mapsto (x^{\varphi})^n;$$

(2) if H has a normal complement K and $\psi \colon G = HK \to H$ is the natural homomorphism $hk \mapsto h$ ($h \in H$, $k \in K$), then the transfer of G into H is

$$G \to H, \quad x \mapsto (x^{\psi})^n.$$

(3) Use part (2) to determine all integers $m \geq 2$ such that the transfer of S_m into its subgroup $H = \langle (1\,2) \rangle$ is the trivial homomorphism.

15. Let G be a torsion-free group that has a cyclic subgroup of finite index. Prove that G is cyclic.

Hint: First use the transfer to show that

($*$) if $\tilde{G}$ is torsion-free and its center contains a cyclic subgroup that is of finite index in $\tilde{G}$, then $\tilde{G}$ is cyclic.

To prove the statement in the problem, consider a torsion-free group G that has a proper cyclic subgroup H of finite index, and use statement ($*$) to find a cyclic subgroup K of G such that $H < K \leq G$.