MINIMAL ABELIAN VARIETIES OF ALGEBRAS, II

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Abstract. We prove that the category of affine varieties, up to term equivalence, is categorically equivalent to the category of modules, considered as 2-sorted structures, with an additional unary operation from the module sort to the ring sort. In particular, this yields a description for the minimal affine varieties.

1. Introduction sec-intro

Minimal, locally finite, abelian varieties of algebras have been classified, and proofs can be found in the papers [5, 6, 9, 10]. This paper is part of a series investigating the structure of minimal, nonlocally finite, abelian varieties.

In the first paper of this series, [3], we proved that any minimal abelian variety of algebras must be affine or strongly abelian. In this paper, we classify the minimal affine varieties. The classification is deduced from a more general result showing that the category of affine varieties, up to term equivalence (or equivalently, the category of affine clones), is categorically equivalent to the category of modules, considered as 2-sorted structures, with an additional unary operation from the module sort to the ring sort. In the third paper of the series, [4], we will show that a minimal strongly abelian variety with a finite bound on the essential arity of its terms is categorically equivalent to a minimal unary variety, and we will classify the possibilities for such varieties. In that paper we also show that there exist minimal strongly abelian varieties that do not have have a finite bound on the essential arities of their terms.

2. Background on Varieties and Clones sec-prelim

We refer the reader to [2, Section 2.1] for the precise definitions of:

- *algebraic signature*, σ,
- *terms* of the signature σ in the variables $\{x_1, \ldots, x_k\}$, also called *k-ary terms*,
- the *language* L associated to the signature σ,
- *algebras* and *varieties* in the language L,

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- the axioms for *clones*,
- a *homomorphism* $\mathbf{\Phi}\colon \mathcal{C} \to \mathcal{D}$ between clones $\mathcal{C}$ and $\mathcal{D}$,
- the *clone of term operations* of an algebra,
- the *clone of* L*-terms* for a language L, and
- the *clone of a variety* in the language L.

We treat these concepts as known. Nevertheless, in this subsection we briefly sketch some of the meanings of these definitions and several basic consequences that we will need later on.

A set F of operation symbols, along with a function $\alpha\colon F \to \omega$ assigning arity, determines an algebraic language, $\mathsf{L} = \mathsf{L}_\alpha$, and a k-ary term $t(\mathbf{x}) = t(x_1, \dots, x_k)$ in this language is any member of the smallest set T_k containing the first k variables $x_1, \dots, x_k$ that is closed under the implication

$$t_1, \dots, t_m \in T_k, \quad f \in F, \quad f\ m\text{-ary} \quad \Longrightarrow \quad f(t_1, \dots, t_m) \in T_k.$$

Here we assume that $F \cap \{x_1, \dots, x_k\} = \emptyset$ for all $k < \omega$, and we call the members of $T := \bigcup_{k<\omega} T_k$ *terms* in the language L or briefly L-terms.

An *algebra* in the language L, or briefly an L-algebra, is a pair $\mathbf{A} = \langle A; F\rangle$ where A is a nonempty set and for each symbol $f \in F$ there is a fixed interpretation $f^{\mathbf{A}}\colon A^{\alpha(f)} \to A$ of that symbol as an $\alpha(f)$-ary operation on A. The assignment $f \mapsto f^{\mathbf{A}}$ can be extended to k-ary terms $t \mapsto t^{\mathbf{A}}$ ($t \in T_k$) for all $k < \omega$ by defining $x_i^{\mathbf{A}}$ to be the i-th k-ary projection operation $A^k \to A$, $(a_1, \dots, a_k) \mapsto a_i$, and by requiring

$$(f(t_1, \dots, t_m))^{\mathbf{A}} := f^{\mathbf{A}}(t_1^{\mathbf{A}}, \dots, t_m^{\mathbf{A}}) \quad \text{for all } f \in F,\ f\ m\text{-ary, and } t_1, \dots, t_m \in T_k.$$

The interpretation $t^{\mathbf{A}}$ of t is called the *term operation* associated to the term t.

An identity in the language L is an atomic formula $s \approx t$ where s, t are L-terms. The identity $s \approx t$ is *satisfied* by an L-algebra $\mathbf{A}$, written $\mathbf{A} \models s \approx t$, if $s^{\mathbf{A}} = t^{\mathbf{A}}$. Given a set Σ of identities in the language L, the class $\mathcal{V}$ of all L-algebras satisfying Σ is called the *variety* axiomatized by Σ. A *subvariety* of a variety $\mathcal{V}$ is a subclass of $\mathcal{V}$ that is a variety. A variety is *trivial* if it consists of 1-element algebras only. A variety is *minimal* if it is not trivial, but any proper subvariety is trivial.

For (abstract) clones we follow W. Taylor's approach of defining clones as multisorted abstract algebras (see [11, Definitions 2.8]), and adopt the definition and notation used in [2, Section 2.1] except that we won't allow 0-ary sorts.

In this paper we will often refer to clones of varieties and clones of algebras, therefore we will discuss these special cases now. The *clone of a variety* $\mathcal{V}$ is the multisorted algebraic structure

$$\mathrm{Clo}(\mathcal{V}) = \langle \mathrm{Clo}_1(\mathcal{V}), \mathrm{Clo}_2(\mathcal{V}), \dots; \{\mathrm{comp}_k^m \mid m, k \geq 1\},\ \{\pi_i^k \mid 1 \leq i \leq k\}\rangle$$

where for each $k \geq 1$, $\mathrm{Clo}_k(\mathcal{V})$ is the set of k-ary terms of $\mathcal{V}$ modulo $\mathcal{V}$-equivalence. By 'k-ary terms modulo $\mathcal{V}$-equivalence' we mean that if $s(\mathbf{x})$ and $t(\mathbf{x})$ are k-ary terms in the language of $\mathcal{V}$, we identify them in $\mathrm{Clo}_k(\mathcal{V})$ if $\mathcal{V} \models s(\mathbf{x}) \approx (\mathbf{x})$. The operation comp^m_k is composition of one m-ary term t with m k-ary terms $t_1, \dots, t_m$:

$$\mathrm{comp}^m_k(t, t_1, \dots, t_m) = t(t_1, \dots, t_m).$$

The nullary operation 'projection π^k_i' of $\mathrm{Clo}(\mathcal{V})$ selects the element $\pi^k_i := x_i$ of $\mathrm{Clo}_k(\mathcal{V})$.

Similarly, the clone of an algebra $\mathbf{A}$ is the multisorted algebraic structure

$$\mathrm{Clo}(\mathbf{A}) = \langle \mathrm{Clo}_1(\mathbf{A}), \mathrm{Clo}_2(\mathbf{A}), \dots ; \{\mathrm{comp}^m_k \mid m, k \geq 1\}, \{\pi^k_i \mid 1 \leq i \leq k\}\rangle$$

where for each $k \geq 1$, $\mathrm{Clo}_k(\mathcal{V})$ is the set of k-ary term operations of $\mathbf{A}$. The operation comp^m_k is composition of one m-ary term operation $t^{\mathbf{A}}$ with m k-ary term operations $t^{\mathbf{A}}_1, \dots, t^{\mathbf{A}}_m$:

$$\mathrm{comp}^m_k(t^{\mathbf{A}}, t^{\mathbf{A}}_1, \dots, t^{\mathbf{A}}_m) = t^{\mathbf{A}}(t^{\mathbf{A}}_1, \dots, t^{\mathbf{A}}_m).$$

The nullary operation 'projection π^k_i' of $\mathrm{Clo}(\mathbf{A})$ selects the projection operation $\pi^k_i \colon A^k \to A$, $(a_1, \dots, a_k) \mapsto a_i$ from $\mathrm{Clo}_k(\mathbf{A})$.

Another clone associated to an algebra $\mathbf{A}$ is the clone $\mathrm{Pol}(\mathbf{A})$ of *polynomial operations* of $\mathbf{A}$, which is defined as the clone of the full constant expansion $\mathbf{A}_A$ of $\mathbf{A}$. If $\mathbf{A} = \langle A; F\rangle$, it *full constant expansion* is the algebra $\mathbf{A}_A = \langle A; F \cup \{c_a \mid a \in A\}\rangle$ obtained from $\mathbf{A}$ by adding a new 0-ary (constant) symbol c_a for each element $a \in A$.

For arbitrary clones $\mathcal{C} = \langle C_1, C_2, \dots ; \{\mathrm{comp}^m_k \mid m, k \geq 1\}, \{\pi^k_i \mid 1 \leq i \leq k\}\rangle$ and $\mathcal{D} = \langle D_1, D_2, \dots ; \{\mathrm{comp}^m_k \mid m, k \geq 1\}, \{\pi^k_i \mid 1 \leq i \leq k\}\rangle$, a homomorphism $\mathbf{\Phi} \colon \mathcal{C} \to \mathcal{D}$ is a multisorted function $\mathbf{\Phi} = (\Phi_1, \Phi_2, \dots)$, where $\Phi_k \colon C_k \to D_k$ are functions, and $\mathbf{\Phi}$ respects the composition operations and the nullary operations π^k_i.

These definitions easily imply the following fact.

lm-cloVcloAiso

Lemma 2.1. *For any variety $\mathcal{V}$ and algebra $\mathbf{A} \in \mathcal{V}$, the map $t \mapsto t^{\mathbf{A}}$ defines a surjective clone homomorphism* $\mathrm{Clo}(\mathcal{V}) \to \mathrm{Clo}(\mathbf{A})$*; moreover, this map is a clone isomorphism if and only if $\mathbf{A}$ generates $\mathcal{V}$.* □

The class of all (abstract) clones under clone homomorphisms forms a category of multisorted algebraic structures. The concepts of isomorphism of clones, embedding of clones, subclones, quotients of clones, and products of clones have their expected meanings.

Next we state two result of Taylor [11] on the category of all (abstract) clones. The first one shows that every abstract clone can be represented as the clone of a variety, and hence also as the clone of any algebra generating that variety (e.g., the clone of the countably generated free algebra in the variety).

Lemma 2.2. [11, Lemma 2.12] *Every (abstract) clone is isomorphic to* lm-cloRepThm

- *the clone of some variety, and also to*
- *the clone of some algebra.* □

cor-catEquiv

Corollary 2.3. *The category of (abstract) clones is equivalent to both of the following full subcategories:*

- *the category of all clones of varieties under clone homomorphisms, and*
- *the category of all clones of algebras under clone homomorphisms.* □

The following results of [11] relate properties of varieties to properties of their clones.

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Lemma 2.4. [11] *Let $\mathcal{V}$ and $\mathcal{W}$ be arbitrary varieties (not necessarily in the same language).*

1. *$\mathcal{V}$ and $\mathcal{W}$ are term equivalent (= definitionally equivalent) varieties if and only if* $\mathrm{Clo}(\mathcal{V}) \cong \mathrm{Clo}(\mathcal{W})$. (Cf. [11, Lemma 2.11].)
2. *The subvarieties of $\mathcal{V}$ are in one-to-one correspondence with the congruences of* $\mathrm{Clo}(\mathcal{V})$. *In fact, for every subvariety, $\mathcal{U}$, of $\mathcal{V}$ the congruence of* $\mathrm{Clo}(\mathcal{V})$ *corresponding to $\mathcal{U}$ is the kernel, $\Theta_{\mathcal{U}}$, of the clone homomorphism* $\mathrm{Clo}(\mathcal{V}) \to \mathrm{Clo}(\mathcal{U})$ *assigning to each term modulo the identities of $\mathcal{V}$ the same term modulo the identities of $\mathcal{U}$; hence,* $\mathrm{Clo}(\mathcal{U}) \cong \mathrm{Clo}(\mathcal{V})/\Theta_{\mathcal{U}}$. (Cf. [11, Lemma 2.14].) □

We call a clone *simple* if it has exactly two congruences: the equality congruence (the equality relation in each sort) and the full congruence (the full relation in each sort). Therefore, the *trivial clone* in which every sort is a singleton is not considered simple. Thus, Lemma 2.4 yields the following characterization of minimal varieties.

Corollary 2.5. *A variety $\mathcal{V}$ is minimal if and only if its clone* $\mathrm{Clo}(\mathcal{V})$ *is simple. Equivalently, an algebra* $\mathbf{A}$ *generates a minimal variety of and only if its clone* $\mathrm{Clo}(\mathbf{A})$ *is simple.*

We conclude this section by introducing some terminology and notation on clones and clone homomorphisms that will be used later on in the paper. Let

$$\mathcal{C} = \langle C_1, C_2, \dots ; \{\mathrm{comp}_k^m : m, k \geq 1\}, \{\pi_i^k : 1 \leq i \leq k\}\rangle$$

be the clone $\mathrm{Clo}(\mathcal{V})$ of a variety or the clone $\mathrm{Clo}(\mathbf{A})$ of an algebra, and let $f, g, h, \ldots \in \mathcal{C}$, that is, $f, g, h, \ldots$ are terms of $\mathcal{V}$ (modulo $\mathcal{V}$-equivalence) or term operations of $\mathbf{A}$, respectively. Identities true in $\mathcal{V}$ (or in $\mathbf{A}$, respectively) that can be expressed using variables and $f, g, h, \ldots$ can be rewritten as equalities in $\mathcal{C}$, namely as equalities of the form

$$\mathbf{S}(f, g, h, \ldots, \boldsymbol{\pi}) = \mathbf{T}(f, g, h, \ldots, \boldsymbol{\pi}) \quad \text{in } \mathcal{C}$$

where $\mathbf{S}, \mathbf{T}$ are clone terms built up from composition operations comp_k^m, and $\boldsymbol{\pi}$ is a tuple of nullary operations π_i^k.

For example, if $f \in C_n$, then the condition that f is *idempotent*, i.e.,

$$\mathcal{V} \text{ (resp., } \mathbf{A}) \models f(x_1, \ldots, x_1) \approx x_1,$$

can be expressed in clone theoretic terms by saying that the equality

$$\mathrm{comp}_1^n(f, \pi_1^1, \dots, \pi_1^1) = \pi_1^1 \quad \text{holds in } \mathcal{C}. \tag{2.1}$$

Similarly, when $d \in C_3$, the condition that d is *Maltsev*, i.e.,

$$\mathcal{V} \text{ (resp., } \mathbf{A}) \models d(x_1, x_2, x_2) \approx x_1 \approx d(x_2, x_2, x_1),$$

can be rewritten in clone theoretic terms as follows: the equalities

$$\mathrm{comp}_2^3(d, \pi_1^2, \pi_2^2, \pi_2^2) = \pi_1^2 = \mathrm{comp}_2^3(d, \pi_2^2, \pi_2^2, \pi_1^2) \quad \text{hold in } \mathcal{C}. \tag{2.2}$$

We say that $f \in C_n$ *does not depend on its i-th variable* $(1 \le i \le n)$ if

$$\begin{aligned} \mathcal{V} \text{ (resp., } \mathbf{A}) \models f(x_1, \dots, &x_{i-1}, x_i, x_{i+1} \dots, x_n) \\ &\approx f(x_1, \dots, x_{i-1}, x_{n+1}, x_{i+1} \dots, x_n), \end{aligned}$$

which in clone theoretic language is the equality

$$\begin{aligned} &\mathrm{comp}_2^{n+1}(f, \pi_1^{n+1}, \dots, \pi_{i-1}^{n+1}, \pi_i^{n+1}, \pi_{i+1}^{n+1}, \dots, \pi_n^{n+1}) \\ &\qquad = \mathrm{comp}_2^{n+1}(f, \pi_1^{n+1}, \dots, \pi_{i-1}^{n+1}, \pi_{n+1}^{n+1}, \pi_{i+1}^{n+1}, \dots, \pi_n^{n+1}) \quad \text{in } \mathcal{C}. \end{aligned} \tag{2.3}$$

lm-clohom

Lemma 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be clones of varieties or clones of algebras, and let $\mathbf{\Phi}\colon \mathcal{C} \to \mathcal{D}$ be a clone homomorphism. If $f, g, h, \dots \in \mathcal{C}$ satisfy an equality*

$$\mathbf{S}(f, g, h, \dots, \boldsymbol{\pi}) = \mathbf{T}(f, g, h, \dots, \boldsymbol{\pi}) \quad \text{in } \mathcal{C},$$

where $\mathbf{S}, \mathbf{T}$ are clone terms built up from composition operations comp_k^m, and $\boldsymbol{\pi}$ is a tuple of nullary symbols π_i^k, then their $\mathbf{\Phi}$-images $\mathbf{\Phi}(f), \mathbf{\Phi}(g), \mathbf{\Phi}(h), \dots \in \mathcal{D}$ satisfy the analogous equality

$$\mathbf{S}(\mathbf{\Phi}(f), \mathbf{\Phi}(g), \mathbf{\Phi}(h) \dots, \boldsymbol{\pi}) = \mathbf{T}(\mathbf{\Phi}(f), \mathbf{\Phi}(g), \mathbf{\Phi}(h) \dots, \boldsymbol{\pi}) \quad \text{in } \mathcal{D}.$$

In particular,

(i) *if f is idempotent, then so is $\mathbf{\Phi}(f)$;*
(ii) *if f is Maltsev, then so is $\mathbf{\Phi}(f)$; and*
(iii) *if f is n-ary $(n \ge 1)$ and for some $1 \le i \le n$, f does not depend on its i-th variable, then $\mathbf{\Phi}(f)$ does not depend on its i-th variable either.*

Proof. The first statement follows directly from the definition of a clone homomorphism and the fact that $\mathbf{\Phi}(\boldsymbol{\pi}) = \boldsymbol{\pi}$, since $\boldsymbol{\pi}$ is a tuple of nullary symbols π_i^k. Statements (i)–(iii) are special cases that rely on the equalities (2.1)–(2.3). $\square$

Notice that in Lemma 2.6 we used the informal notation $\mathbf{\Phi}(f)$ for the image of a clone element $f \in \mathcal{C}$ under a clone homomorphism $\mathbf{\Phi}\colon \mathcal{C} \to \mathcal{D}$. For a formally correct notation we should have specified the sort (= arity) n of f, and should have written $\Phi_n(f)$ in place of $\mathbf{\Phi}(f)$. We want to continue using this informal notation, but before doing so, we need to make sure that it causes no ambiguity.

If $\mathcal{C}$ is the clone $\mathrm{Clo}(\mathbf{A})$ of an algebra $\mathbf{A}$, then the arity of a term operation $f \in \mathrm{Clo}(\mathbf{A})$ is well-defined, since for $n < \bar{n}$ an n-ary operation $A^n \to A$ and an

$\bar{n}$-ary operation $A^{\bar{n}} \to A$ are different sets. Nevertheless, it is not uncommon to (informally) view an n-ary operation as an $\bar{n}$-ary operation that fails to depend on its last $\bar{n}-n$ variables $x_{n+1}, \dots, x_{\bar{n}}$.

However, if $\mathcal{C}$ is the clone $\mathrm{Clo}(\mathcal{V})$ of a variety $\mathcal{V}$, then the arities of the elements of $\mathrm{Clo}(\mathcal{V})$ are not well-determined. Indeed, if $f = f(x_1, \dots, x_n)$ is an n-ary term of $\mathcal{V}$, then the same expression f is also an $\bar{n}$-ary term for every $\bar{n} > n$ (although the variables $x_{n+1}, \dots, x_{\bar{n}}$ do not occur in f). Hence f represents an element of $\mathrm{Clo}_{\bar{n}}(\mathcal{V})$ for all $\bar{n} \geq n$. Therefore, the informal notation $\mathbf{\Phi}(f)$ is justified only if for every clone homomorphism $\mathbf{\Phi}\colon \mathrm{Clo}(\mathcal{V}) \to \mathrm{Clo}(\mathcal{W})$ we have that $\Phi_n(f)$ (n-ary) and $\Phi_{\bar{n}}(f)$ ($\bar{n}$-ary) are essentially the same terms in $\mathrm{Clo}(\mathcal{W})$. Clearly, it suffices to establish this for the case $\bar{n} = n+1$.

In clone theoretic terms the construction of adding a 'fictitious variable' to an n-ary member $f \in C_n$ of a clone $\mathcal{C} = \langle C_1, C_2, \dots; \{\mathrm{comp}^m_k : m, k \geq 1\}, \{\pi^k_i : 1 \leq i \leq k\}\rangle$ to produce an $(n+1)$-ary member f° can be described by the following function:

$$^\circ\colon C_n \to C_{n+1}, \quad f \mapsto f^\circ := \mathrm{comp}^n_{n+1}(f, \pi^{n+1}_1, \dots, \pi^{n+1}_n).$$

-adding-fict-

Lemma 2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be clones of varieties or clones of algebras, let $n \geq 1$, and let $\mathbf{\Phi}\colon \mathcal{C} \to \mathcal{D}$ be a clone homomorphism.*

(1) *The map $^\circ\colon C_n \to C_{n+1}$ is injective.*
(2) *f° is independent of its last variable for every $f \in C_n$.*
(3) *$\Phi_{n+1}(f^\circ) = \big(\Phi_n(f)\big)^\circ$ holds for every $f \in C_n$.*

Proof. Let $f \in C_n$. The definition of f° and the clone axioms (namely the general associative law for the composition operations comp^n_k and the axioms for the projections π^n_k) yield that for any integer $k \geq 1$ and for any $g_1, \dots, g_{n+1} \in C_k$ we have

$$\begin{aligned}
&\mathrm{comp}^{n+1}_k(f^\circ, g_1, \dots, g_{n+1})\\
&\qquad = \mathrm{comp}^{n+1}_k(\mathrm{comp}^n_{n+1}(f, \pi^{n+1}_1, \dots, \pi^{n+1}_n), g_1, \dots, g_{n+1})\\
&\qquad = \mathrm{comp}^n_k(f, \mathrm{comp}^{n+1}_k(\pi^{n+1}_1, g_1, \dots, g_{n+1}), \dots, \mathrm{comp}^{n+1}_k(\pi^{n+1}_n, g_1, \dots, g_{n+1}))\\
&\qquad = \mathrm{comp}^n_k(f, g_1, \dots, g_n).
\end{aligned}$$

Applying this equality to $\langle g_1, \dots, g_{n+1}\rangle := \langle \pi^n_1, \dots, \pi^n_n, \pi^n_n\rangle$ we see that

$$\mathrm{comp}^{n+1}_k(f^\circ, \pi^n_1, \dots, \pi^n_n, \pi^n_n) = \mathrm{comp}^n_k(f, \pi^n_1, \dots, \pi^n_n) = f,$$

which proves statement (1). Similarly, the choices $\langle g_1, \dots, g_{n+1}\rangle := \langle \pi^{n+2}_1, \dots, \pi^{n+2}_n, \pi^{n+2}_{n+1}\rangle$ and $\langle g_1, \dots, g_{n+1}\rangle := \langle \pi^{n+2}_1, \dots, \pi^{n+2}_n, \pi^{n+2}_{n+2}\rangle$ prove statement (2).

Finally, since $\mathbf{\Phi}$ is a clone homomorphism, we get that

$$\begin{aligned}\Phi_{n+1}(f^\circ) &= \Phi_{n+1}(\mathrm{comp}^n_{n+1}(f, \pi_1^{n+1}, \dots, \pi_n^{n+1})) \\ &= \mathrm{comp}^n_{n+1}(\Phi_n(f), \Phi_{n+1}(\pi_1^{n+1}), \dots, \Phi_{n+1}(\pi_n^{n+1})) \\ &= \mathrm{comp}^n_{n+1}(\Phi_n(f), \pi_1^{n+1}, \dots, \pi_n^{n+1}) = (\Phi_n(f))^\circ,\end{aligned}$$

proving statement (3). □

3. The category of affine clones

sec-affine

We call an algebra $\mathbf{A}$ *affine* if it satisfies the following two conditions:

- $\mathbf{A}$ is *abelian*, that is, for every 2×2 matrix of elements of A of the form eq-matrix-DEFN

$$\begin{bmatrix} f(\mathbf{a},\mathbf{u}) & f(\mathbf{a},\mathbf{v}) \\ f(\mathbf{b},\mathbf{u}) & f(\mathbf{b},\mathbf{v}) \end{bmatrix} \in A^{2\times 2} \tag{3.1}$$

  where $f = f(\mathbf{x},\mathbf{y}) \in \mathrm{Clo}(\mathbf{A})$ and $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}$ are tuples of elements of A, we have that

$$f(\mathbf{a},\mathbf{u}) = f(\mathbf{a},\mathbf{v}) \quad \text{implies} \quad f(\mathbf{b},\mathbf{u}) = f(\mathbf{b},\mathbf{v}).$$

- $\mathbf{A}$ has a Maltsev term operation $d^{\mathbf{A}}$.

A variety is called *affine* (or *abelian*) if all of its algebras are.

We will also use the following characterizations of affine algebras from [1].

Theorem 3.1. [1, Chapter 5] *The following conditions on an algebra $\mathbf{A}$ are equivalent:* thm-char-affine

(a) $\mathbf{A}$ *is affine.*

(b) (i) $\mathbf{A}$ *has a unique Maltsev term operation $d^{\mathbf{A}}$, and*
 (ii) $d^{\mathbf{A}}$ *commutes with every (term) operation of $\mathbf{A}$, that is, $d^{\mathbf{A}}$ is a homomorphism $\mathbf{A}^3 \to \mathbf{A}$.*

(c) *For every choice $0 \in A$, there exists a unique (faithful) $\mathbf{R}$-module ${}_{\mathbf{R}}\underline{A} = (A; +, 0, -, R)$ on the base set A of $\mathbf{A}$ such that*
 (i) *the unique Maltsev term operation $d^{\mathbf{A}}(x,y,z)$ of $\mathbf{A}$ is the ternary abelian group operation $x - y + z$,*
 (ii) $\mathbf{R}$ *is a subring of the endomorphism ring of the abelian group $\underline{A} = (A; +, 0, -)$,*
 (iii) $\mathrm{Clo}(\mathbf{A})$ *is a subclone of* $\mathrm{Pol}({}_{\mathbf{R}}\underline{A})$*, that is, every term operation $f \in \mathrm{Clo}_n(\mathbf{A})$ $(n \geq 1)$ has the form*

$$f(x_1, \dots, x_n) = \sum_{i=1}^n r_i x_i + a$$

 for some $r_1, \dots, r_n \in R$ and $a \in A$, and

(iv) *every* $r \in R$ *occurs as a coefficient of some* $f \in \mathrm{Clo}_n(\mathbf{A})$, *that is,* $\mathbf{R}$ *is minimal for properties* (i)–(iii).

We note that in condition (c) of this theorem the choice of 0 is inessential in that changing $0 \in A$ to another element $0' \in A$ yields an isomorphic ring $\mathbf{R}'$ and an isomorphic $\mathbf{R}'$-module, ${}_{\mathbf{R}'}\underline{A}' = (A; +', 0', -', R')$, where the isomorphism between the modules ${}_{\mathbf{R}}\underline{A}$, and ${}_{\mathbf{R}'}\underline{A}'$ is the translation $\tau\colon x \mapsto x + 0'$ and the isomorphism between the rings $\mathbf{R}$ and $\mathbf{R}'$ is conjugation by τ.

For our goals in this section it will be useful to allow the ring $\mathbf{R}$ in part (c) of Theorem 3.1 to be replaced by an isomorphic copy. Therefore, we will relax condition (ii) in part (c) of Theorem 3.1 to

- $\mathbf{R}$ acts faithfully as a subring of the endomorphism ring of the abelian group $\underline{A} = (A; +, 0, -)$.

Notice that we will still insist that ${}_{\mathbf{R}}\underline{A}$ be a faithful $\mathbf{R}$-module. We incorporate this change into our next theorem from [8, Propositions 2.1, 2.6] (cf. [7, Lemma 4.3]), which extends part (c) of Theorem 3.1 to an explicit description for the clones of affine algebras.

ine-clones

Theorem 3.2 ([7, 8])**.** *Let* $\mathbf{A}$ *be an affine algebra, and let* ${}_{\mathbf{R}}\underline{A}$ *be a faithful* $\mathbf{R}$*-module on* A *such that conditions (i), (iii), (iv) in part (c) of Theorem 3.1 hold. There exists a unique* $\mathbf{R}$*-submodule* $\mathbf{M}$ *of* $\mathbf{R} \times {}_{\mathbf{R}}\underline{A}$ *such that the clone of* $\mathbf{A}$ *is the following subclone of* $\mathrm{Pol}({}_{\mathbf{R}}\underline{A})$*:*

$$\mathcal{C}({}_{\mathbf{R}}\underline{A}, \mathbf{M}) = \Big\{ r_1x_1 + r_2x_2 + \cdots + r_nx_n + a : \\ n \geq 1,\ r_1, \ldots, r_n \in R,\ a \in A,\ \Big(1 - \sum_{i=1}^{n} r_i, a\Big) \in M \Big\}.$$

For example, if $\mathbf{A}$ is itself a faithful $\mathbf{R}$-module ${}_{\mathbf{R}}\underline{A}$, then

$$\mathrm{Clo}({}_{\mathbf{R}}\underline{A}) = \mathcal{C}({}_{\mathbf{R}}\underline{A}, R \times \{0\}).$$

The largest clone for an affine algebra with associated module ${}_{\mathbf{R}}\underline{A}$ is the clone of the full constants expansion of ${}_{\mathbf{R}}\underline{A}$, which is

$$\mathrm{Clo}(({}_{\mathbf{R}}\underline{A}_A)) = \mathrm{Pol}({}_{\mathbf{R}}\underline{A}) = \mathcal{C}({}_{\mathbf{R}}\underline{A}, \mathbf{R} \times {}_{\mathbf{R}}\underline{A}).$$

Finally, the smallest clone for an affine algebra with associated (faithful) module ${}_{\mathbf{R}}\underline{A}$ is the clone of the full idempotent reduct of ${}_{\mathbf{R}}\underline{A}$ (that is, the clone of the affine module corresponding to ${}_{R}\underline{A}$), which is

$$\mathrm{Clo}(({}_{\mathbf{R}}\underline{A})_{\mathrm{id}}) = \mathcal{C}({}_{\mathbf{R}}\underline{A}, \{0\} \times \{0\}).$$

The equivalence of conditions (a) and (b) in Theorem 3.1 motivates the following definition of an (abstract) affine clone.

ne-clone

Definition 3.3. Let $\mathcal{C} = \langle C_1, C_2, \ldots; \{\mathrm{comp}_k^m : m, k \geq 1\}, \{\pi_i^k : 1 \leq i \leq k\}\rangle$ be a clone. We will say that $\mathcal{C}$ is *affine* if there exists $d \in C_3$ such that d is Maltsev (i.e., d satisfies (2.2)) and d commutes with every member of $\mathcal{C}$, i.e., the following holds for every $n \geq 1$ and $f \in C_n$:

$$\begin{aligned}\mathrm{comp}_{3n}^3\big(d, \mathrm{comp}_{3n}^n(f, \pi_1^{3n}, \ldots, \pi_n^{3n}), \mathrm{comp}_{3n}^n(f, \pi_{n+1}^{3n}, \ldots, \pi_{2n}^{3n}), \mathrm{comp}_{3n}^n(f, \pi_{2n+1}^{3n}, \ldots, \pi_{3n}^{3n})\big)\\ = \mathrm{comp}_{3n}^n\big(f, \mathrm{comp}_{3n}^3(d, \pi_1^{3n}, \pi_{n+1}^{3n}, \pi_{2n+1}^{3n}), \ldots, \mathrm{comp}_{3n}^3(d, \pi_n^{3n}, \pi_{2n}^{3n}, \pi_{3n}^{3n})\big).\end{aligned}$$

Clones of affine algebras, i.e., the clones $\mathcal{C}({}_{\mathbf{R}}\underline{A}, \mathbf{M})$ in Theorem 3.2 will also be referred to as *concrete affine clones.*

Note that by Lemma 2.2 and Theorem 3.1, for every affine clone $\mathcal{C} = \langle C_1, C_2, \ldots; \{\mathrm{comp}_k^m : m, k \geq 1\}, \{\pi_i^k : 1 \leq i \leq k\}\rangle$, the element $d \in C_3$ that is Mal'tsev is uniquely determined. Therefore, informally, we may write $\langle \mathcal{C}, d\rangle$ to indicate that $\mathcal{C}$ has d as its unique element that is Mal'tsev.

Corollary 3.4. *Let* $\mathbf{A}$ *be any algebra and* $\mathcal{V}$ *any variety.*

(1) $\mathbf{A}$ *is an affine algebra if and only if its clone* $\mathrm{Clo}(\mathbf{A})$ *is an affine clone.*
(2) $\mathcal{V}$ *is affine variety if and only if its clone* $\mathrm{Clo}(\mathcal{V})$ *is an affine clone.*

Proof. Statement (1) follows immediately from the equivalence of conditions (a) and (b) in Theorem 3.1 and from the definition of an affine clone.

For (2), we will use the following consequence of Definition 3.3: every clone that is a homomorphic image of an affine clone is affine. Since we know from Lemma 2.1 that the clone of every algebra $\mathbf{A} \in \mathcal{V}$ is a homomorphic image of $\mathrm{Clo}(\mathcal{V})$, moreover, $\mathrm{Clo}(\mathbf{A}) \cong \mathrm{Clo}(\mathcal{V})$ if $\mathbf{A}$ generates $\mathcal{V}$, the preceding observation implies that $\mathrm{Clo}(\mathcal{V})$ is affine if and only if $\mathrm{Clo}(\mathbf{A})$ is affine for every $\mathbf{A} \in \mathcal{V}$. By statement (1), $\mathrm{Clo}(\mathbf{A})$ is affine if and only if $\mathbf{A}$ is affine. Therefore, we get that $\mathrm{Clo}(\mathcal{V})$ is affine if and only if every algebra $\mathbf{A} \in \mathcal{V}$ is affine, which is the defining property for $\mathcal{V}$ to be affine. □

We will use the notation AClo for the category of (abstract) affine clones with all homomorphisms. By Corollary 3.4, the full subcategory of AClo where the objects are restricted to clones of algebras is the category of clones of affine algebras (i.e., the category of concrete affine clones). Similarly, the full subcategory of AClo where the objects are restricted to clones of varieties is the category of clones of affine varieties. We will denote these two subcategories of AClo by $\mathsf{AClo}_{\mathrm{alg}}$ and $\mathsf{AClo}_{\mathrm{var}}$, respectively.

Lemma 2.2 immediately implies the following.

Corollary 3.5. AClo*,* $\mathsf{AClo}_{\mathrm{alg}}$*, and* $\mathsf{AClo}_{\mathrm{var}}$ *are equivalent categories.*

By Theorem 3.2 we have a fairly good understanding of concrete affine clones, the objects of $\mathsf{AClo}_{\mathrm{alg}}$, and therefore of the objects of the other two categories as well, up to isomorphism. Our main goal in this section is to describe all homomorphisms between affine clones, by finding a fourth, much simpler category that is equivalent to AClo, $\mathsf{AClo}_{\mathrm{alg}}$, and $\mathsf{AClo}_{\mathrm{var}}$.

Definition 3.6. The *objects of the category* RM are 2-sorted algebras, denoted $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$, where $\mathbf{R}$ is a ring, ${}_{\mathbf{R}}\mathbf{M}$ is a faithful left $\mathbf{R}$-module, and † is an $\mathbf{R}$-module homomorphism ${}_{\mathbf{R}}\mathbf{M} \to {}_{\mathbf{R}}\mathbf{R}$. So, the operations of $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$ are

- the ring operations $+$, $-$, 0, $\cdot$, 1 on the first sort, R, the underlying set of the ring $\mathbf{R}$,
- the abelian group operations $+$, $-$, 0 on the second sort, M, the underlaying set of the abelian group $\mathbf{M}$,
- the two-sorted binary operation $\cdot : R \times M \to M$ describing the action of $\mathbf{R}$ on $\mathbf{M}$ that makes $\mathbf{M}$ into a left $\mathbf{R}$-module, and
- the $\mathbf{R}$-module homomorphism ${}^{\dagger}\colon {}_{\mathbf{R}}\mathbf{M} \to {}_{\mathbf{R}}\mathbf{R}$, viewed as a two-sorted unary operation.

Note that ${}_{\mathbf{R}}\mathbf{R}$ (i.e., $\mathbf{R}$ as a left $\mathbf{R}$-module) is given by the action of the ring $\mathbf{R}$ on the additive group of $\mathbf{R}$ that coincides with the ring multiplication of $\mathbf{R}$. Hence, ${}_{\mathbf{R}}\mathbf{R}$ as a 2-sorted structure is the same as the ring $\mathbf{R}$. Therefore, from now on, we will use the notation $\mathbf{R}$ for both of these structures.

The *morphisms of the category* RM are defined to be all homomorphisms between its objects. In more detail, for arbitrary objects $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$ and $(\mathbf{S}, {}_{\mathbf{S}}\mathbf{N}; {}^{\dagger})$ of RM, a homomorphism $\varphi\colon (\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger}) \to (\mathbf{S}, {}_{\mathbf{S}}\mathbf{N}; {}^{\dagger})$ is a pair $\varphi = (\varphi_1, \varphi_2)$ of maps where

- φ_1 is a ring homomorphism $\mathbf{R} \to \mathbf{S}$,
- φ_2 is an abelian group homomorphism $\mathbf{M} \to \mathbf{N}$,
- $\varphi = (\varphi_1, \varphi_2)$ commutes with the ring action of the first sort on the second sort; i.e., $\varphi_2(r \cdot m) = \varphi_1(r) \cdot \varphi_2(m)$ holds for all $r \in R$ and $m \in M$, and
- $\varphi = (\varphi_1, \varphi_2)$ commutes with the unary operation mapping the second sort into the first sort; i.e., $\varphi_1(m^{\dagger}) = (\varphi_2(m))^{\dagger}$ holds for all $m \in M$.

If ${}_{\mathbf{R}}\mathbf{M}$ is an $\mathbf{R}$-module and ${}_{\mathbf{S}}\mathbf{N}$ is an $\mathbf{S}$-module for some rings $\mathbf{R}$ and $\mathbf{S}$, then a pair $\varphi = (\varphi_1, \varphi_2)$ of maps satisfying the first three conditions of the last list above will be referred to as a *two-sorted module homomorphism* ${}_{\mathbf{R}}\mathbf{M} \to {}_{\mathbf{S}}\mathbf{N}$.

To define the object function of a functor $\mathscr{F}$ from the category of affine clones to the category RM, we need to assign to each affine clone $\mathcal{C}$ the associated 2-sorted module $\mathscr{F}(\mathcal{C})$ with additional operation †. In the next definition we describe the sorts and the operations of $\mathscr{F}(\mathcal{C})$, and in Lemma 3.8 we will prove that $\mathscr{F}(\mathcal{C})$ is indeed an object of RM.

g-module-

Definition 3.7. Let $\mathcal{C} = \langle C_1, C_2, \dots; \{\mathrm{comp}^m_k : m, k \geq 1\}, \{\pi^k_i : 1 \leq i \leq k\}\rangle$ be an affine clone, and let $d \in C_3$ be its unique element that is Mal'tsev. Let $C_{2,\mathrm{idem}}$ denote the set consisting of all $f \in C_2$ that are idempotent (i.e., satisfy (2.1) for $n = 2$). We define the operations of a 2-sorted structure with sorts $C_{2,\mathrm{idem}}$ and C_1 as follows:

- Operations $+$, $\cdot$ (binary), $-$ (unary), and 1, 0 (nullary) on the first sort, $C_{2,idem}$: for any $f, g \in C_{2,\text{idem}}$,

$$\begin{aligned} f + g &:= \text{comp}_2^3(d, f, \pi_2^2, g), \\ -f &:= \text{comp}_2^3(d, \pi_2^2, f, \pi_2^2), \\ 0 &:= \pi_0^2, \\ 1 &:= \pi_1^2, \\ fg &:= \text{comp}_2^2(f, g, \pi_2^2). \end{aligned}$$

- Operations $+$ (binary), $-$ (unary), and 0 (nullary) on the second sort, C_1: for any $u, v \in C_1$,

$$\begin{aligned} u + v &:= \text{comp}_1^3(d, u, \pi_1^1, v), \\ -u &:= \text{comp}_1^3(d, \pi_1^1, u, \pi_1^1), \\ 0 &:= \pi_1^1. \end{aligned}$$

- Multisorted operation $\cdot$ (binary), mapping $C_{2,\text{idem}} \times C_1 \to C_1$: for any $f \in C_{2,\text{idem}}$, and $u \in C_1$,

$$fu := \text{comp}_1^2(f, u, \pi_1^1).$$

- Multisorted operation † (unary) mapping $C_1 \to C_{2,\text{idem}}$: for any $u \in C_1$,

$$u^\dagger := \text{comp}_2^3(d, \text{comp}_2^1(u, \pi_1^2), \text{comp}_2^1(u, \pi_2^2), \pi_2^2).$$

lm-F(C)-in-LM

Lemma 3.8. *Let $\mathcal{C} = \langle C_1, C_2, \ldots; \{\text{comp}_k^m : m, k \geq 1\}, \{\pi_i^k : 1 \leq i \leq k\}\rangle$ be an affine clone, and let $d \in C_3$ be its unique element that is Mal'tsev.*

(1) *If $\mathcal{C}$ is a concrete affine clone, i.e., $\mathcal{C} = \mathcal{C}({}_{\mathbf{R}}\underline{A}, \mathbf{M})$ for some faithful $\mathbf{R}$-module ${}_{\mathbf{R}}\underline{A}$ and some submodule $\mathbf{M}$ of $\mathbf{R} \times {}_{\mathbf{R}}\underline{A}$, then for the maps*

$$\begin{aligned} &\rho \colon R \to C_{2,\text{idem}},\ r \mapsto rx_1 + (1-r)x_2 \quad \textit{and} \\ &\mu \colon M \to C_1,\ (r, a) \mapsto (1-r)x_1 + a, \end{aligned}$$

(ρ, μ) is an isomorphism between the 2-sorted module $(\mathbf{R}, \mathbf{M}; {}^\dagger)$ with additional operation † and the 2-sorted structure $\mathscr{F}(\mathcal{C})$ described in Definition 3.7. Hence, $\mathscr{F}(\mathcal{C})$ is an object of RM and (ρ, μ) is an isomorphism in RM.

(2) *$\mathscr{F}(\mathcal{C})$ is an object of RM for every (abstract) affine clone.*

thm-functorF

Theorem 3.9. *The following functions define a functor $\mathscr{F} \colon \mathsf{AClo} \to \mathsf{RM}$:*

- *$\mathscr{F}$ assigns to every affine clone $\mathcal{C}$ the 2-sorted module $\mathscr{F}(\mathcal{C})$ with additional operation † described in Definition 3.7, and*
- *$\mathscr{F}$ assigns to every clone homomorphism $\mathbf{\Phi} \colon \mathcal{C} \to \mathcal{D}$ between affine clones the morphism $\mathscr{F}(\mathbf{\Phi}) := (\Phi_2|_{\mathcal{C}_{2,\text{idem}}}, \Phi_1)$ in RM.*

df-Mtilde

Definition 3.10. For any object $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$ of RM let $\widetilde{\mathbf{M}}$ be the submodule of $\mathbf{R} \times {}_{\mathbf{R}}(\mathbf{R} \times \mathbf{M})$ with underlying set

$$\widetilde{M} := \{(m^{\dagger}, (m^{\dagger}, m)) : m \in M\},$$

and let ${}^{\dagger}\colon \widetilde{\mathbf{M}} \to \mathbf{R}$ be the $\mathbf{R}$-module homomorphism projecting $\widetilde{\mathbf{M}}$ onto its first coordinate.

tilde-
ts

Lemma 3.11. *For any object* $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$ *of* RM*,* $(\mathbf{R}, {}_{\mathbf{R}}\widetilde{\mathbf{M}}; {}^{\dagger})$ *of* RM *isomorphic to* $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$*; namely, the pair* $(id, {}^{\sim})$ *where the second function is defined by*

$${}^{\sim}\colon {}_{\mathbf{R}}\mathbf{M} \to {}_{\mathbf{R}}\widetilde{\mathbf{M}}, \quad m \mapsto \widetilde{m} := (m^{\dagger}, (m^{\dagger}, m))$$

is an isomorphism in RM*.*

tilde-
hisms

Lemma 3.12. *For any morphism*

$$(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger}) \to (\mathbf{S}, {}_{\mathbf{S}}\mathbf{N}; {}^{\dagger})$$

in RM*, there is a unique clone homomorphism*

$$\mathbf{\Phi}\colon \mathcal{C}({}_{\mathbf{R}}(\mathbf{R} \times {}_{\mathbf{R}}\mathbf{M}), \widetilde{\mathbf{M}}) \to \mathcal{C}({}_{\mathbf{S}}(\mathbf{S} \times {}_{\mathbf{S}}\mathbf{N}), \widetilde{\mathbf{N}})$$

such that the diagram

$$\begin{array}{ccc}
(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger}) & \xrightarrow{(\varphi_1,\varphi_2)} & (\mathbf{S}, {}_{\mathbf{S}}\mathbf{N}; {}^{\dagger}) \\
\big\downarrow{\scriptstyle (id,{}^{\sim})} & & \big\downarrow{\scriptstyle (id,{}^{\sim})} \\
(\mathbf{R}, {}_{\mathbf{R}}\widetilde{\mathbf{M}}; {}^{\dagger}) & \longrightarrow & (\mathbf{S}, {}_{\mathbf{S}}\widetilde{\mathbf{N}}; {}^{\dagger}) \\
\big\downarrow{\scriptstyle (\rho,\mu)} & & \big\downarrow{\scriptstyle (\rho,\mu)} \\
\mathscr{F}(\mathcal{C}({}_{\mathbf{R}}(\mathbf{R} \times {}_{\mathbf{R}}\mathbf{M}), \widetilde{\mathbf{M}})) & \xrightarrow{\mathscr{F}(\mathbf{\Phi})} & \mathscr{F}(\mathcal{C}({}_{\mathbf{S}}(\mathbf{S} \times {}_{\mathbf{S}}\mathbf{N}), \widetilde{\mathbf{N}}))
\end{array}$$

commutes; i.e., $\mathscr{F}(\mathbf{\Phi}) = (\Phi_{2,\mathrm{idem}}, \Phi_1)$ *coincides with* (φ_1, φ_2) *modulo the isomorphism* $(\rho, \mu) \circ (id, {}^{\sim})$ *in* RM*.*

unctorG

Theorem 3.13. *The following functions define a functor* $\mathscr{G}\colon \mathsf{RM} \to \mathsf{AClo}$*:*

- $\mathscr{G}$ *assigns to every object* $(\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger})$ *of* RM *the affine clone* $\mathcal{C}({}_{\mathbf{R}}(\mathbf{R} \times \mathbf{M}), \widetilde{\mathbf{M}})$*, and*
- $\mathscr{G}$ *assigns to every morphism* $(\varphi_1, \varphi_2)\colon (\mathbf{R}, {}_{\mathbf{R}}\mathbf{M}; {}^{\dagger}) \to (\mathbf{S}, {}_{\mathbf{S}}\mathbf{N}; {}^{\dagger})$ *the unique clone homomorphism* $\mathbf{\Phi}\colon \mathcal{C}({}_{\mathbf{R}}(\mathbf{R} \times {}_{\mathbf{R}}\mathbf{M}), \widetilde{\mathbf{M}}) \to \mathcal{C}({}_{\mathbf{S}}(\mathbf{S} \times {}_{\mathbf{S}}\mathbf{N}), \widetilde{\mathbf{N}})$ *found in Lemma 3.12.*

FGcateq

Theorem 3.14. *The functors* $\mathscr{F}\colon \mathsf{AClo} \to \mathsf{RM}$ *and* $\mathscr{G}\colon \mathsf{RM} \to \mathsf{AClo}$ *establish a categorical equivalence between the category* AClo *of (abstract) affine clones and the category* RM *of 2-sorted modules with additinal operation* †*.*

4. Minimal affine varieties

inaffine

Definition 4.1. Let R be a ring and let L be a left ideal of R. For the R-module ${}_RR$ and R-submodule $L \times \{0\}$ of $R \times {}_RR$ we will denote the clone $\mathcal{C}({}_RR, L \times \{0\})$ by $\mathcal{C}(R, L)$; i.e., $\mathcal{C}(R, L)$ is the following clone of linear functions on R:

$$\mathcal{C}(R, L) = \Big\{ r_1x_1 + r_2x_2 + \cdots + r_nx_n : n \geq 1,\ r_1, \ldots, r_n \in R,\ 1 - \sum_{i=1}^{n} r_i \in L \Big\}.$$

By an (R, L)-*clone* we mean any clone isomorphic to $\mathcal{C}(R, L)$. A variety whose clone is an (R, L)-clone will be called an (R, L)-*variety*.

Our interest in (R, L)-varieties stems from the fact (see Corollary 4.6) that every minimal affine variety is an (R, L)-variety. However, not every affine variety is an (R, L)-variety. For example, the variety generated by the full constant expansion of a non-trivial module is not an (R, L)-variety. This follows from our characterization of (R, L)-varieties in Theorem 4.4 below.

Before proving Theorem 4.4 we establish some basic facts about (R, L)-varieties.

lm-RL

Lemma 4.2. *Let R be a ring and L a left ideal of R. Every (R, L)-clone is an affine clone with ring isomorphic to R, and every (R, L)-variety is an affine variety with ring isomorphic to R.*

Proof. To prove the first statement, let $\mathcal{C}$ be a clone of operations a set A, and assume that $\mathcal{C}$ is an (R, L)-clone. By Definition 4.1, we have that $\mathcal{C} \cong \mathcal{C}(R, L)$. Hence, $\mathcal{C}$ contains a Maltsev operation which commutes with all operations in $\mathcal{C}$. The latter condition easily implies that the algebra $(A; \mathcal{C})$ is abelian, so $\mathcal{C}$ is an affine clone. The definition of the ring of an affine clone shows also that isomorphic affine clones have isomorphic rings. Therefore, the ring of $\mathcal{C}$ is isomorphic to R. This proves the first statement of the lemma.

The second statement is a consequence of the first, since the clone of a variety is isomorphic to the clone of its countably generated free algebra. $\square$

An easy consequence of Theorem **??** is that the pair (R, L) describing an (R, L)-variety is uniquely determined, up to isomorphism.

cor-iso-RL

Corollary 4.3. *For arbitrary rings R, R' and left ideals L of R and L' of R', we have $\mathcal{C}(R, L) \cong \mathcal{C}(R', L')$ if and only if there exists a ring isomorphism $'\colon R \to R'$ that maps L onto L'.*

Proof. Using Theorem **??** we see that $\mathcal{C}(R, L) \cong \mathcal{C}(R', L')$ if and only if there exist a ring isomorphism $'\colon R \to R'$, $r \mapsto r'$ (making R' and R-module), and an R-module isomorphism $\varphi\colon L \times \{0\} \to L' \times \{0\}$, $(\ell, 0) \mapsto (\ell', 0)$. Given $'$, such a φ will exist if and only if $'$ maps L onto L'. This completes the proof. $\square$

Now we are ready to discuss our characterization theorem for (R, L)-clones. First we introduce some terminology.

For any faithful module ${}_R\underline{A}$, the functions of the form $x + a$ $(a \in A)$ will be called *translations*. The translations in an affine clone $\mathcal{C}({}_R\underline{A}, M)$ are exactly the functions $x + a$ with constant terms $a \in A$ satisfying $(0, a) \in M$. An affine algebra $\mathbf{A}$, or its clone $\mathcal{C}({}_R\underline{A}, M)$, will be called *translation-free* if the unary projection function, $x = x + 0$, is the only translation in $\mathcal{C}({}_R\underline{A}, M)$.

r-free

Theorem 4.4. *Let R be a ring, ${}_R\underline{A}$ a faithful R-module, M an R-submodule of $R \times {}_R\underline{A}$, and let L be the left ideal of R obtained by projecting M onto its first coordinate. The following conditions on the clone $\mathcal{C}({}_R\underline{A}, M)$ are equivalent:*

(a) *$\mathcal{C}({}_R\underline{A}, M)$ is an (R', L')-clone for some ring R' and left ideal L' of R'.*

(b) *$\mathcal{C}({}_R\underline{A}, M)$ is translation-free.*

(c) *There exists an R-module homomorphism $\psi\colon L \to {}_R\underline{A}$ such that M is the graph $\{(\ell, \psi(\ell)) : \ell \in L\}$ of ψ.*

(d) *$\mathcal{C}({}_R\underline{A}, M)$ is an (R, L)-clone.*

Proof. (a) $\Rightarrow$ (b): Assume first that condition (a) holds, and let us fix a clone isomorphism $\Phi\colon \mathcal{C}({}_R\underline{A}, M) \to \mathcal{C}(R', L') = \mathcal{C}({}_{R'}R', L' \times \{0\})$. By Theorem **??** (1) there exist a ring isomorphism $R \to R'$, $r \mapsto r'$, and an R-module isomorphism $M \to L' \times \{0\}$ such that Φ sends each function $\sum_{i=1}^n r_i x_i + a \in \mathcal{C}({}_R\underline{A}, M)$ to the function $\sum_{i=1}^n r_i' x_i \in \mathcal{C}(R', L')$. In particular, Φ sends every translation $x + a \in \mathcal{C}({}_R\underline{A}, M)$ to $x = x + 0 \in \mathcal{C}(R', L')$. Since Φ is one-to-one, it follows that $\mathcal{C}({}_R\underline{A}, M)$ is translation-free, proving (b).

(b) $\Rightarrow$ (c): Assume now that condition (b) holds, that is, $\mathcal{C}({}_R\underline{A}, M)$ is translation-free. Then a pair $(0, a) \in R \times {}_R\underline{A}$ belongs to M if and only if $a = 0$. This means that the R-module homomorphism $M \to R$ sending each pair in M to its first coordinate is one-to-one. Since this map has range L, it is a bijection $M \to L$. Hence, there is a function $\psi\colon L \to {}_R\underline{A}$ such that M is the graph of ψ. As M is an R-submodule of $R \times {}_R\underline{A}$ — in fact, an R-submodule of $L \times {}_R\underline{A}$ —, we get that ψ is an R-module homomorphism $L \to {}_R\underline{A}$. This proves (c).

(c) $\Rightarrow$ (d): Assuming (c) holds we see that the map $\varphi\colon M \to L \times \{0\}$, $(\ell, \psi(\ell)) \mapsto (\ell, 0)$ is an R-module isomorphism. Moreover, the identity isomorphism $R \to R$ and φ satisfy conditions (a)–(b) in Theorem **??**. Hence, Theorem **??** (2) implies that the clone $\mathcal{C}({}_R\underline{A}, M)$ is isomorphic to $\mathcal{C}(R, L \times \{0\}) = \mathcal{C}(R, L)$. This completes the proof of (d).

(d) $\Rightarrow$ (a): This implication is trivial. □

A characterization of minimal affine varieties is a consequence of the following more general theorem on subvarieties of affine varieties.

ffine-RL

Theorem 4.5. *Let $\mathcal{V}$ be an affine variety with ring R.*

(1) *$\mathcal{V}$ has a subvariety that is an (R, L)-variety for some left ideal L of R.*

(2) *$\mathcal{V}$ is an (R, L)-variety for some left ideal L of R if and only if* $\mathrm{Clo}(\mathcal{V})$ *is translation-free.*
(3) *If $\mathcal{V}$ is an (R, L)-variety, then the subvarieties of $\mathcal{V}$ are in one-to-one correspondence with the ideals of R.*

Proof. Let $\mathbf{A}$ be any algebra generating $\mathcal{V}$. Thus, $\mathbf{A}$ is an affine algebra with $\mathrm{Clo}(\mathbf{A}) \cong \mathrm{Clo}(\mathcal{V})$ and hence with ring isomorphic to R. By Theorem 3.2, $\mathrm{Clo}(\mathbf{A}) = \mathcal{C}({}_R\underline{A}, M)$ for some faithful R-module ${}_R\underline{A}$ and some submodule M of $R \times {}_R\underline{A}$. Let L be the projection of M onto its first coordinate. Thus, L is an R-submodule, and hence a left ideal, of R. eq-linhom

Statement (1) of the theorem follows from the fact that the map

$$\Phi\colon \mathcal{C}({}_R\underline{A}, M) \to \mathcal{C}(R, L), \quad \sum_{i=1}^{n} r_i x_i + a \mapsto \sum_{i=1}^{n} r_i x_i \tag{4.1}$$

is an onto clone homomorphism. (In the notation of Theorem **??**, Φ is the clone homomorphism that corresponds to the pair $({}', \varphi)$ where ${}'\colon R \to R$ is the identity isomorphism and $\varphi\colon M \to L \times \{0\}$ is the R-module homomorphism $(\ell, a) \mapsto (\ell, 0)$.) By Lemma 2.4 (2), the kernel of Φ yields a subvariety, $\mathcal{U}$, of $\mathcal{V}$ whose clone is isomorphic to $\mathcal{C}(R, L)$. Hence, $\mathcal{U}$ is an (R, L)-variety.

For (2), recall that the equivalence of conditions (b) and (d) in Theorem 4.4 show that

($*$) $\mathrm{Clo}(\mathbf{A}) = \mathcal{C}({}_R\underline{A}, M)$ is an (R, L)-clone for some left ideal L of R if and only if $\mathrm{Clo}(\mathbf{A})$ is translation-free.

Isomorphisms between affine clones preserve the property of being an (R, L)-clone (by Definition 4.1), as well as the property of being translation-free (by Theorem **??**). Therefore, ($*$) remains true if $\mathrm{Clo}(\mathbf{A})$ is replaced by the isomorphic clone $\mathrm{Clo}(\mathcal{V})$.

To prove (3), assume that $\mathcal{V}$ is an (R, L)-variety. Thus, $\mathrm{Clo}(\mathcal{V}) \cong \mathcal{C}(R, L)$. Hence, Lemma 2.4 (2) implies that the subvarieties of $\mathcal{V}$ are in one-to-one correspondence with the congruences of the clone $\mathcal{C}(R, L)$. The congruences of $\mathcal{C}(R, L)$ are the kernels of homomorphism with domain clone $\mathcal{C}(R, L)$, therefore Theorem **??** implies that the congruences of $\mathcal{C}(R, L)$ are in one-to-one correspondence with the congruences of R, or equivalently, with the ideals of R. □

cor-affine-main

Corollary 4.6. *An affine variety is minimal if and only if it is an (R, L)-variety for some simple ring R and left ideal L of R.*

Proof. As we noted at the beginning of Section 3, the ring R of an affine variety $\mathcal{V}$ has the binary terms x and y (up to the identities of $\mathcal{V}$) as their identity element and zero element, respectively. Hence $\mathcal{V}$ is nontrivial if and only if $|R| > 1$. By Theorem 4.5(1), every nontrivial affine variety with ring R has a subvariety that is an (R, L)-variety. By Lemma 4.2, this subvariety has ring isomorphic to R, and hence is also nontrivial. It follows that a minimal affine variety $\mathcal{V}$ is necessarily an (R, L)-variety, where R is the ring of $\mathcal{V}$ and L is a left ideal of R.

By Theorem 4.5 (3), an (R, L)-variety is minimal if and only if its ring R is simple. This completes the proof. $\square$

Note that if $\mathcal{V}$ is an affine variety with ring R and $\mathbf{A}$ is an algebra generating $\mathcal{V}$ as in the proof of Theorem 4.5 (1), then the (R, L)-subvariety, $\mathcal{U}$, of $\mathcal{V}$ constructed in that proof is generated by the linearization, $\mathbf{A}^{\Delta} := (\mathbf{A} \times \mathbf{A})/\Delta$ of $\mathbf{A}$ where Δ is the congruence of $\mathbf{A} \times \mathbf{A}$ defined by

$$(a_1, a_2)\,\Delta\,(a_3, a_4) \quad \Leftrightarrow \quad a_1 - a_2 = a_3 - a_4.$$

Indeed, the same assignment as in (4.1) yields a clone homomorphism $\mathrm{Clo}(\mathbf{A}) = \mathcal{C}({}_R\underline{A}, M) \to \mathrm{Clo}(\mathbf{A}^{\Delta})$, and we have $\mathrm{Clo}(\mathbf{A}^{\Delta}) \cong \mathcal{C}(R, L)$ via the map that assigns to each operation in $\mathrm{Clo}(\mathbf{A}^{\Delta})$ with linear expression $\sum_{i=1}^{n} r_i x_i$ the operation in $\mathcal{C}(R, L)$ with the same linear expression.

Thus, the (R, L)-subvariety of $\mathcal{V}$ constructed in the proof of Theorem 4.5 (1) is generated by an algebra which has a singleton subalgebra. In particular, this implies (by the proof of Corollary 4.6) that every minimal affine variety $\mathcal{M}$ is generated by an algebra with a singleton subalgebra. If $\mathcal{M}$ is locally finite, more is known to be true: every member of $\mathcal{M}$ has a singleton subalgebra. Surprisingly, without assuming local finiteness, this property fails in some minimal affine varieties.

To see that this is the case, we first establish a necessary and sufficient condition for an (R, L)-variety to have the property that every member has a singleton subalgebra.

ffine-trivial

Theorem 4.7. *The following are equivalent for an (R, L)-variety, $\mathcal{V}$.*

(1) *Every member of $\mathcal{V}$ has a singleton subalgebra.*
(2) *L is generated by an idempotent (i.e., $L = Re$ for some $e \in R$ satisfying $e^2 = e$).*

Proof. For the implication (1) $\Rightarrow$ (2) assume that every member of $\mathcal{V}$ has a singleton subalgebra. Since $\mathcal{V}$ is an (R, L)-variety, there exists an isomorphism $\Xi\colon \mathrm{Clo}(\mathcal{V}) \to \mathcal{C}(R, L)$. Thus, we obtain an algebra $\mathbf{R}$ in $\mathcal{V}$ with universe R by defining the operations of $\mathbf{R}$ to be the Ξ-images of the terms (modulo the identities of $\mathcal{V}$) in $\mathrm{Clo}(\mathcal{V})$, which represent the operation symbols of $\mathcal{V}$. Clearly, $\mathrm{Clo}(\mathbf{R}) = \mathcal{C}(R, L)$, so the unary term operations of $\mathbf{R}$ are the functions $(1 - \ell)x$ with $\ell \in L$.

It follows that the subalgebra of $\mathbf{R}$ generated by the element $1 \in R$ is $\langle 1 \rangle = \{1 - \ell : \ell \in L\}$. By our assumption (1), this algebra has a singleton subalgebra, say $\{1 - e\}$ where $e \in L$. Hence, $(1 - \ell)(1 - e) = 1 - e$ holds for all $\ell \in L$. Equivalently, $\ell e = \ell$ holds for all $\ell \in L$. This implies that $e^2 = e$ and $Re \subseteq L = Le \subseteq Re$, which proves (2).

For the converse, (2) $\Rightarrow$ (1), assume that $L = Re$ for some $e \in R$ satisfying $e^2 = e$. Since every member of $\mathcal{V}$ is a homomorphic image of a generating algebra in $\mathcal{V}$ (e.g., a free algebra with sufficiently many free generators), the desired conclusion (1) will follow if we argue that every generating algebra in $\mathcal{V}$ has a singleton subalgebra.

So, let $\mathbf{A}$ be an arbitrary algebra generating $\mathcal{V}$. Then, as we saw in the proof of Theorem 4.5, $\mathbf{A}$ is an affine algebra with ring R and clone $\mathrm{Clo}(\mathbf{A})$ isomorphic to $\mathrm{Clo}(\mathcal{V})$, which has the form $\mathrm{Clo}(\mathbf{A}) = \mathcal{C}({}_R\underline{A}, M)$ for some R-submodule M of $R \times {}_R\underline{A}$. Moreover, since $\mathcal{V}$ is an (R, L)-variety, and hence $\mathcal{C}({}_R\underline{A}, M)$ is an (R, L)-clone for some left ideal L of R, Corollary 4.3 and Theorem 4.4 imply that L is the projection of M onto its first coordinate, and M is the graph of an R-module homomorphism $\psi\colon L \to {}_R\underline{A}$.

Let $c := \psi(e)\,(\in A)$. Then, in ${}_R\underline{A}$, we have that $ec = e\psi(e) = \psi(e^2) = \psi(e) = c$. Our goal is to show that $\{c\}$ is a subalgebra of $\mathbf{A}$. It suffices to show that c is fixed by every unary term operation of $\mathbf{A}$, that is, by every unary operation in $\mathcal{C}({}_R\underline{A}, M)$. By Theorem 3.2, every such operation f has the form $(1-\ell)x + a$ with $(\ell, a) \in M$. Since, $L = Re$ by assumption, we get that $\ell = re$ for some $r \in R$. Moreover, as M is the graph of the R-homomorphism ψ, it must be the case that $a = \psi(\ell) = \psi(re) = r\psi(e) = rc$. Thus, f has the form $(1-re)x + rc$. Hence, using $ec = c$, we get that $f(c) = (1-re)c + rc = c - rec + rc = c$, completing the proof. □

Recall that a ring R is called *left artinian* if R satisfies the descending chain condition on left ideals. By the Wedderburn–Artin Theorem every left artinian simple ring is isomorphic to a matrix ring $M_n(\mathbb{D})$ $(n \geq 1)$ where $\mathbb{D}$ is a division ring.In such a ring, it is known that all left ideals are generated by idempotents. Therefore, Theorem 4.7 yields the following generalization of the well-known property of locally finite minimal affine varieties mentioned earlier.

Corollary 4.8. *If $\mathcal{V}$ is a minimal affine variety whose ring is left artinian, then every member of $\mathcal{V}$ has a singleton subalgebra.*

The next example shows that there exist minimal affine varieties in which some members have no singleton subalgebras.

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Example 4.9. Let $A_1 = k[x, y]/(xy - yx - 1)$ be the first Weyl algebra over a field k of characteristic zero. Here x and y are noncommuting indeterminates. It is known that A_1 is a simple ring with no nonzero zero divisors, and that A_1 is not a division ring. The fact that A_1 has no nonzero zero divisors implies that it has no idempotents other than 0 and 1 (since if $e^2 = e$, then $e(e-1) = 0$, so $e = 0$ or $e - 1 = 0$). Therefore the only left ideals of A_1 generated by idempotents are $L = A_1 \cdot 0 = (0)$ and $L = A_1 \cdot 1 = A_1$. But there are other left ideals, since A_1 is not a division ring. Let $L \lhd A_1$ be a left ideal other than (0) or A_1. Then, by Corollary 4.6 and Theorem 4.7, any (A_1, L)-variety will be a minimal affine variety with the property that not all members have singleton subalgebras.

Finally, we mention another application of the characterization of minimal affine varieties in Corollary 4.6. Recall from Lemma 2.4 (1) that two varieties are term equivalent if and only if they have isomorphic clones. Therefore Corollary 4.6, combined with Corollary 4.3, yields the following characterization of minimal affine varieties, up to term equivalence.

Corollary 4.10. *Given a simple ring R, the minimal affine varieties with ring R, up to term equivalence, are in one-to-one correspondence with the orbits of the left ideals of R under the automorphism group of R.*

For example, this implies that if R is a simple artinian ring, i.e., R is isomorphic to a matrix ring $M_n(\mathbb{D})$ over a division ring $\mathbb{D}$, then the number of minimal affine varieties with ring R, up to term equivalence, is $n+1$, one variety for each d with $0 \leq d \leq n$ where d is the dimension of the corresponding left ideal over $\mathbb{D}$.

In particular, for the case $n = 1$ when R itself is a division ring, we get that, up to term equivalence, there are exactly two minimal affine varieties with ring R. They correspond to the left ideals $L = R$ and $L = \{0\}$, respectively. So, by the description of the corresponding (R, L)-clones in Definition 4.1 we see that, up to term equivalence, the minimal affine varieties whose ring R is a division ring are the the variety of vector spaces over $\mathbb{D}$ and the variety of affine spaces over $\mathbb{D}$.

Recall that the clone $\mathrm{Clo}(\mathcal{V})$ of a variety $\mathcal{V}$ is said to be *commutative* if for any operation symbols f, g in the language of $\mathcal{V}$ (and hence for any terms f, g),

$$\mathcal{V} \models f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})) = g(f(x_{11}, \ldots, x_{m1}), \ldots, f(x_{1n}, \ldots, x_{mn})),$$

where m, n are the arities of f, g, respectively. It is straightforward to check that if $\mathcal{V}$ is an affine variety with a commutative clone, then the ring of $\mathcal{V}$ must be commutative. Since a commutative simple ring is a field, we get the following.

Corollary 4.11. *A minimal affine variety with a commutative clone is term equivalent either to a variety of vector spaces over a field, or to a variety of affine spaces over a field.*

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