Non-continuous T-norms

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FUZZY LOGICS (Motivation)

Classical (*crisp*) set A $\equiv$ its indicator function

$1_A : X \ni x \mapsto [1 \text{ if } x \in A, 0 \text{ else}]$ [X set for "UNIVERSE"]

$F : X \to [0,1]$ *fuzzy set*

Interpretation: $F(x) = [$ sureness for x to belong to $F]$

For crisp $A, B (: X \to \{0,1\})$,
$\overline{A} = 1 - A$, $A \cap B = \min\{A, B\}$, $A \cup B = \max\{A, B\}$

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ZADEH, L.A. 1965: $F(x)$ *logical value*

$\overline{A}(x) = c\big(A(x)\big), \quad A \cap B(x) = t\big(A(x), B(x)\big),$
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Operations on logical values — de Morgan identities

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De Morgan: $\overline{\overline{A}} = A, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}, \quad \overline{A \cup B} = \overline{A} \cap \overline{B}$

Algebra: $\cap, \cup$ ASSOCIATIVE, COMMUTATIVE

Monotonicity: $A \subset B \iff A \leq B$ (as functions)
$\bar{\cdot}$ decreasing, $\cap, \cup$ increasing

Extreme cases: $\emptyset = 0, \quad X = 1, \quad \overline{\emptyset} = X, \quad A \cap \emptyset = \emptyset, \quad A \cup \emptyset = A$

WHAT DOES IT MEAN FOR
$\overline{\lambda} = c(\lambda), \ \lambda \wedge \mu = t(\lambda, \mu), \ \lambda \vee \mu = s(\lambda, \mu)$?

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$\bar{\cdot}, \wedge, \vee$

Algebraic and monotonicity properties with
$0, 1, \bar{\cdot}, \wedge, \vee$ in place of $\emptyset, 1, \bar{\cdot}, \cap, \cup$

Full range: $\{\bar{\lambda} : \ 0, \leq \lambda\} = [0,1]$, $\{\lambda \wedge \mu : \ 0 \leq \mu \leq 1\} = [0, \lambda]$
$\Longrightarrow$ CONTINUITY
[Φ monotone, continuous $\Longleftrightarrow$ Φ(INTERVAL)=INTERVAL]

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NEGATION, T-NORM

$\lambda \mapsto \overline{\lambda}$ any strictly decreasing continuous self-inverse function $[0,1] \leftrightarrow [0,1]$

In any case: $\lambda \vee \mu = \overline{\overline{\lambda} \wedge \overline{\mu}}$, $s = c \circ t(c, c)$

If $\phi : [0,1] \nearrow [0,1]$ onto and c, t, s de Morgan $\Rightarrow$
$C := \phi^{-1} \circ c \circ \phi$, $T := \phi^{-1} \circ t(\phi, \phi)$, $S := \phi^{-1} \circ s(\phi, \phi)$
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T-NORM AXIOMS

$T : [0,1]^2 \to [0,1]$

$T(x,y) = T(y,x)$ commutative
$T(T(x,y),z) = T(x,T(y,z))$ associative
$y \leq z \Rightarrow T(x,y) \leq T(x,z)$ increasing

$T(1,x) = x, \quad T(0,x) = 0$ marginal conditions

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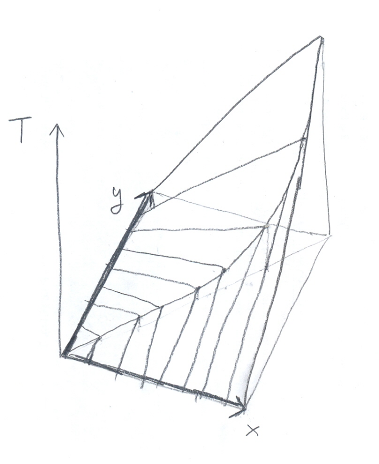
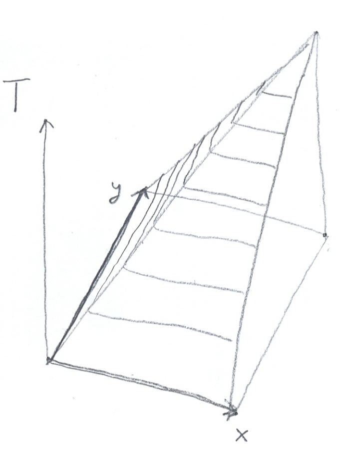
EXAMPLES: $T = xy$, $T = \min\{x, y\}$





Multiplicative presentation

$\big([0,1], T\big)$ ordered commutative topological semigroup, unit 1, sink 0

$x \cdot y$ instead of $T(x,y)$

$xy = yx, \quad (xy)z = x(yz), \quad 1x = x, \quad 0x = 0$

$x_1 \le x_2,\ y_1 \le y_2 \Longrightarrow x_1 y_1 \le x_2 y_2$

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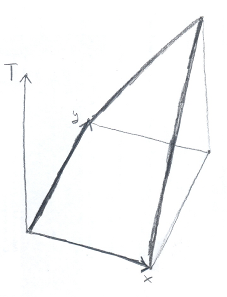

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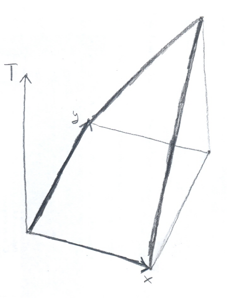

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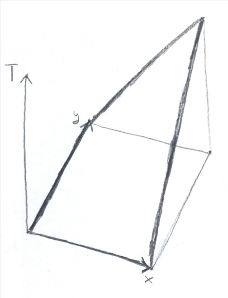

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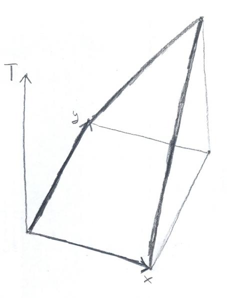

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MAXIMAL PAIRS

$$M := \{(x, y) : \ xy = \min\{x, y\}\}$$

Assume: $(x, y) \in M$, e.g. $xy = x \le y$

Consider $z \ge y$

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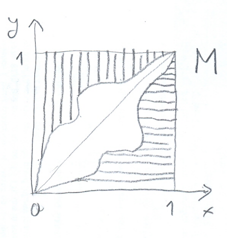

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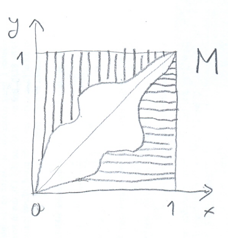

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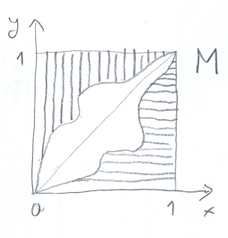

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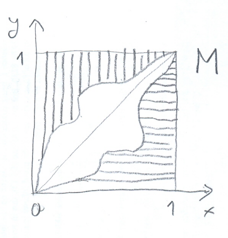

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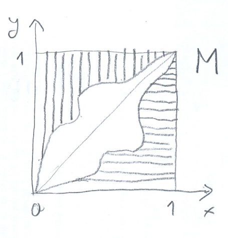

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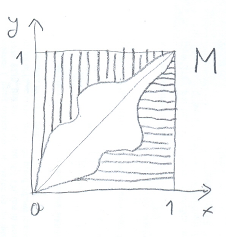

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IDEMPOTENTS

$$E := \{e \in [0,1] : \ e^2\big(= T(e,e)\big) = e\}$$

Lemma. E *is LEFT-CLOSED*
$E \ni e_n \nearrow e \Rightarrow e = \sup_n e_n = \sup_n e_n^2 \le e^2 \le e \Rightarrow e \in E$
$[0,1] \setminus E = \bigcup \big[$*disjoint left-open intervals*$\big]$

$P_e : x \mapsto ex$ projection $P_e^2 x = eex = ex = P_e x$

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(starting point z)

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Lemma. E *is LEFT-CLOSED*

$E \ni e_n \nearrow e \Rightarrow e = \sup_n e_n = \sup_n e_n^2 \leq e^2 \leq e \Rightarrow e \in E$

$[0,1] \setminus E = \bigcup \big[\textit{disjoint left-open intervals}\big]$

$P_e : x \mapsto ex$ projection $P_e^2 x = eex = ex = P_e x$

$x \in \mathrm{ran}(P_e) \quad x = P_e y = ey \quad P_e x = P_e^2 y = P_e y = x$

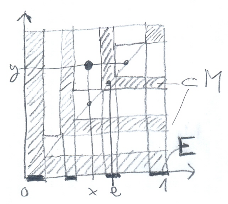
$x_1, x_2 \in \mathrm{ran}(P_e) \ \ P_e x_1 = P_e x_2 = z$

$x_1 < x < x_2 \implies P_e x \in [P_e x_1, P_e x_2] = \{z\}$

$P_e^{-1}\{z\} = \{y : \ P_e y = x\}$ LEFT-CLOSED INTERVAL

(starting point z)

OUTSIDE THE SQUARES



On the stripes above the squares: $x \leq e \leq y \quad \exists\, e = e^2 \in E$

Observation: if $x = ze, \; e = e^2 \leq y \Rightarrow \quad ey = \min\{e, y\} = e$

$$xy = zey = ze = x \Rightarrow xy = x = \min\{x, y\}$$

Let $[0,1] \setminus E = \bigcup_n I_n \quad (I_1 = (e_1, f_1\rangle_?, I_2 = (e_2, f_2\rangle_?, \ldots$ disjoint intervals)

- $$\bigcup_{e \in E} \left[\mathrm{ran}(P_e) \times [e,1] \,\cup\, [e,1] \times \mathrm{ran}(P_e)\right] \subset M$$

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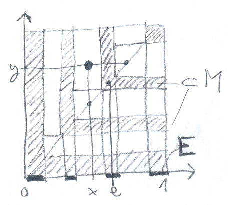

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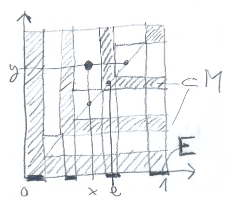

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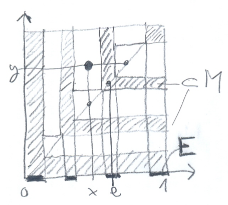

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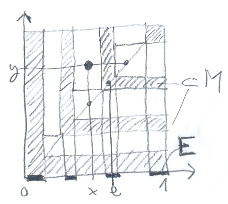

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THE CLASSICAL CONTINUOUS CASE

$[0,1] \setminus E = \bigcup_n^{\bullet} (e_n, f_n)$

QUESTION: $[e_n, f_n] \cdot f_n \overset{?}{=} [e_n, f_n]$ resp. $f_n \cdot [e_n, f_n] \overset{?}{=} [e_n, f_n]$

YES

$\phi_n : x \mapsto x f_n$ INCREASING, CONTINUOUS

$\phi_n(e_n) = \min\{e_n, f_n\} = e_n, \quad \phi_n(f_n) = f_n^2 = f_n$

In general: *An increasing function* $\psi : [a,b] \to \mathbb{R}$ *is continuous*

$\Leftrightarrow \quad \psi[a,b] = \big[\psi(a), \psi(b)\big]$

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THEOREM 1

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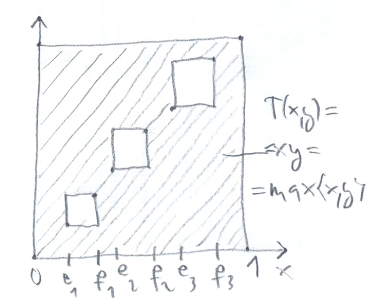

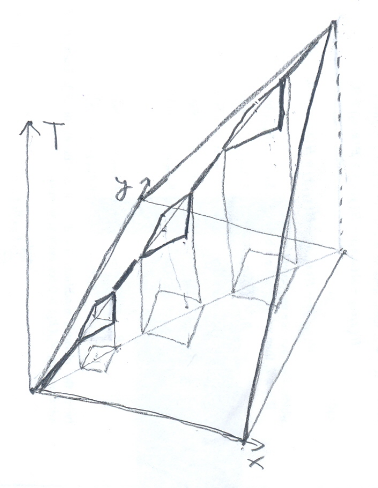

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FIX $I_n = (e_n, f_n)$

We know: 1) $x^2 = T(x,x) < x$ for $x \in I_n$
2) $xy = T(x,y) = \min\{x,y\}$ for $x,y \in \{e_n, f_n\}$

$\big([e_n, f_n], T|[e_n, f_n]\big)$ ordered top. semigroup with unit f_n, sink e_n

TRIVIALLY: If $\Phi : [0,1] \nearrow [e_n, f_n]$ increasing continuous onto $\Rightarrow$
$\big([0,1], \Phi^{-1}T(\Phi,\Phi)\big) \longleftrightarrow \big([e_n, f_n], T|[e_n, f_n]\big)$
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$\Phi^{-1}T(\Phi(x),\Phi(y)))$ T-norm (on $[0,1]$)
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ARCHIMEDIAN T-NORMS

DEFINITION. A function $T : [a, b] \to [a, b]$ is a *T-norm on* $[a, b]$ if

- increasing
- $T(x, y) = T(y, x)$
- $T(T(x, y), z) = T(x, T(y, z))$
- $T(x, a) = a$ and $T(x, b) = x$

DEFINITION. T is *Archimedean* if $T(x, x) < x$ for $x \neq a, b$.

FOR STRUCTURE $[a, b] = [0, 1]$ without loss of generality

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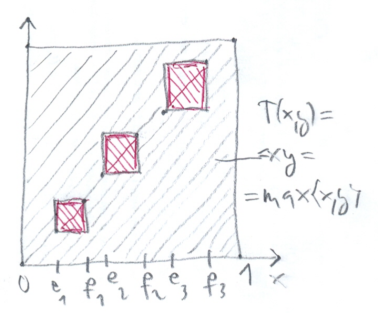
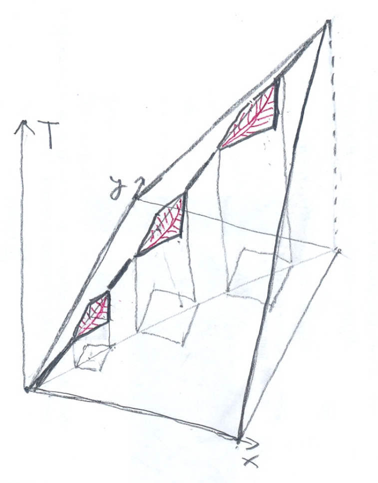
INSERTION INTO THE HOLES





THEOREM 2

THEOREM 2. $I_1 = (e_1, f_1), I_2 = (e_2, f_2), \ldots$ *disjoint* $\subset [0,1]$.
$T_k : [e_k, f_k]^2 \longrightarrow [e_k, f_k]$ Archimedean T-norms, $\Longrightarrow$

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STRUCTURE OF CONTINUOUS ARCHIMEDEAN T-NORMS

DEF. Let $f : [a, b] \to [0, \infty]$ strictly decreasing continuous, $f(b) = 0$

Write: $$f^{-1}(y) := \begin{cases} [x : \; f(x) = y] & \text{for } \; 0 \le y < f(b) \\ b & \text{for } \; f(a) \le y \le \infty \end{cases}$$

$$T^f(x, y) := f^{-1}\big(f(x) + f(y)\big) \quad x, y \in [a, b]$$

THEOREM 3. *Each* T^f *is a continuous Archimedean T-norm, each Archimedean T-norm has the form* T^f*,*
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THEOREM 3. *Each T^f is a continuous Archimedean T-norm, each Archimedean T-norm has the form T^f,* $x_1 \overset{T^f}{\bullet} \cdots \overset{T^f}{\bullet} x_n = f^{-1}\big(f(x_1) + \cdots + f(x_n)\big)$.

EX. 1) $T^{-\log x}(x,y) = xy$; 2) $T^{1-x}(x,y) = [(x+y) - 1]_+$

Equivalence classes

$a \geq a^2 \geq \cdots \geq 0 \qquad \underline{a^{(\infty)} := \lim_n a_n} = \inf_n a_n$

$a \geq b \implies a^{(\infty)} \geq b^{(\infty)}$

DEF. $\sim b : \quad a^{(\infty)} = b^{(\infty)}$

$a < x < b \;\; a \sim b \quad \implies x^{(\infty)} = a^{(\infty)} = b^{(\infty)}$

$a^{\sim} := \{x : \; x \sim a\}$ EQUIV CLASSES ARE INTERVALS

LEMMA. $a^n \sim b \implies a \sim b \quad a^n < b \Rightarrow a^{(\infty)} \leq b^{(\infty)}$

NON-COMMUTATIVE T-NORMS

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ORDERING ON EQUIVALENCE CLASSES

$a^{\sim} b \;:\Leftrightarrow\; a^{(\infty)} < b^{(\infty)}$ [Rem: $a^{(\infty)} = \inf a^{\sim}$]

Notation. $A := \{a^{(\infty)} : \; a \in [0,1]\}$
$\{I_\alpha : \; \alpha \in A\} = \{\text{EQUIV CLASSES OF } \sim\}$

$I_\alpha := \{x : \; x^{(\infty)} = \alpha\}, \quad \alpha < \beta \Rightarrow \sup I_\alpha \le \inf I_\beta \quad I_\alpha < I_\beta$

LEMMA. $e := \sup I_\alpha \quad e \notin I_\alpha \quad \Rightarrow e \in E$ (idempotent)
$\alpha_1 < \alpha_2 < \cdots \quad e := \sup \bigcup_n I_{\alpha_n} \quad \Rightarrow e \in E.$

Proof.

1) $e > I_\alpha \quad e^{(\infty)} > \inf I_\alpha = \alpha \quad x \in I_\alpha \Rightarrow e^2 \ge x^2 \in I_\alpha$
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ORDERING ON EQUIVALENCE CLASSES

THEOREM. $e \in E$, (e, f) is a component interval of $[0,1] \setminus E$
$\Rightarrow \ \forall\, f' \in (e, f) \quad A \cap (e, f')$ is well-ordered wrt. $<$
(there is no sequence $e < \alpha_1 < \alpha_2 < \cdots < f'$ in A)

EITHER $\quad f \in E$ and $\quad A \cap (e, f) = [\text{well-ord}] \cup \{\alpha_1, \alpha_2, \ldots\}$
with $\alpha_1 < \alpha_2 < \cdots \nearrow f$

OR $\quad A \cap (e, f)$ is well-ordered.

T-LOGARITHM ON I_α

$I := I_\alpha$ fixed equivalence class (of $\sim$)

DEF. $L : I \to (0, \infty)$ <u>T-logarithm</u>:

$$L(\underbrace{xy}_{T(x,y)}) = L(x) + L(y)$$

L T-log $\Rightarrow$ λL T-log $(\lambda > 0)$
$L(a^k) = kL(a) \quad (k = 1, 2, \ldots)$

$a^N = a^{(\infty)} \in E$ idempotent $\Rightarrow$
$N\,L(a) = (N+1)L(a) = \cdots \Rightarrow L(a) = 0$ contrd.

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RESTRICTED T-LOG

DEF. $L : I \setminus \inf I \to (0, \inf)$ RESTRICTED T-log :

$$L(xy) = L(x) + L(y) \quad \text{if} \quad xy > \inf I (= \alpha)$$

1) $a > a^2 > a^3 > \cdots \searrow a^{(\infty)} = \alpha$
If $L(a) = 1, \quad b \in I \quad \ell \in \{1, 2, \ldots\}$
$b^\ell \in I \qquad \exists\, k_\ell \quad a^{k_\ell} \geq b^\ell \geq a^{k_\ell + 1} \qquad L(b) \in [k_\ell/\ell, (k_\ell + 1)/\ell]$
At most one value for $L(b)$ [L arb. large]

LEMMA. $a > a^2 > a^3 > \cdots$
$\Rightarrow \sup_\ell \{k/\ell : \ a^k \geq b^\ell\} \leq \inf_{\overline{\ell}} \{\overline{k}/\overline{\ell} : \ a^{\overline{k}} \leq b^{\overline{\ell}}\}$

Proof. Assume $\quad a^k \geq b^\ell \quad a^{\overline{k}} \leq b^{\overline{\ell}}$
$a^{k\overline{\ell}} \geq b^{\ell\overline{k}} \qquad b^{\ell\overline{\ell}} \geq a^{\overline{k}\ell}$
$a > a^2 \cdots \Rightarrow k\overline{\ell} \leq \overline{k}\ell \Rightarrow k/\ell \leq \overline{k}/\overline{\ell}$

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RESTRICTED T-LOG

2) $\forall\ N \quad \exists\ x \in I \qquad x^N < a \neq a^{(\infty)}$

$x^i \geq a \geq x^{i+1} \qquad x^j \geq b \geq x^{j+1}$

$1 = L(a) \in \left[iL(x), (i+1)L(x)\right] \qquad L(b) \in \left[jL(x), (j+1)L(x)\right]$

$$L(x) \in \left[\imath i + 1, \imath i\right] \quad L(b) \in \left[\jmath i + 1, \jmath + \imath i\right]$$

LEMMA. $\forall\ b \in I \setminus \{\alpha\} \quad \forall\ N \quad \exists\ x \in I \qquad x^N < b \implies$

$$\sup_{x,i,j} \left\{\jmath i + 1 :\ a \geq x^{i+1},\ x^j \geq b\right\} \leq \inf_{x,i,j} \left\{\overline{\jmath i} + 1 :\ x^{\bar{i}+1} \geq a,\ b \geq x^{\bar{j}}\right\}$$

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Proof. $x > x^2 > x^3 \cdots \searrow x^{(\infty)} = a^{(\infty)} = \inf I = \alpha$

Assume $a \geq x^{i+1}, \ \ x^j \geq b, \ \ \overline{x}^{\bar{i}} \geq a, \ \ b \geq \overline{x}^{\bar{j}}$

$\exists\ z \in I \setminus \{\alpha\} \quad \exists\ n, \overline{n} \qquad z^n \geq x \geq z^{n+1} \quad z^{\overline{n}} \geq \overline{x} \geq z^{\overline{n}+1}$

$a \geq x^{i+1} \geq z^{(i+1)(n+1)} \qquad z^{jn} \geq x^j \geq b$

$z^{\bar{i}\overline{n}} \geq \overline{x}^{\bar{i}} \geq a \qquad b \geq \overline{x}^{\bar{j}+1} \geq z^{(\bar{j}+1)(\overline{n}+1)}$

$z^{\bar{i}\overline{n}} \geq z^{(i+1)(n+1)} \qquad z^{jn} \geq z^{(\bar{j}+1)(\bar{i}+1)}$

$\bar{i}\overline{n} \leq (i+1)(n+1) \qquad jn \leq (\bar{j}+1)(\overline{n}+1)$

$$\bar{i}jjn \leq (i+1)(n+1)(\bar{j}+1)(\overline{n}+1)$$

$$\frac{j}{i+1}\underbrace{\frac{n}{n+1}}_{\downarrow 1} \leq \frac{\bar{j}+1}{\bar{i}}\underbrace{\frac{\overline{n}+1}{\overline{n}}}_{\downarrow 1} \quad (n, \overline{n} \to \infty)$$

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Proof. $x > x^2 > x^3 \cdots \searrow x^{(\infty)} = a^{(\infty)} = \inf I = \alpha$

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RESTRICTED T-LOG

Proof. $x > x^2 > x^3 \cdots \searrow x^{(\infty)} = a^{(\infty)} = \inf I = \alpha$

Assume $a \ge x^{i+1}, \;\; x^j \ge b, \;\; \overline{x}^{\bar{i}} \ge a, \;\; b \ge \overline{x}^{\bar{j}}$

$\exists\, z \in I \setminus \{\alpha\} \quad \exists\, n, \overline{n} \qquad z^n \ge x \ge z^{n+1} \quad z^{\overline{n}} \ge \overline{x} \ge z^{\overline{n}+1}$

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GENERALIZED ARCHIMEDEAN PROPERTY

DEF. $T|I_\alpha$ gen. Archimedean:

$$\forall\ N \quad \exists\ x \in I_\alpha \qquad x^N > \alpha$$

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INTERPLAY BETWEEN I_α AND I_β

Let $\alpha, \beta \in A$, $\alpha < \beta$

LEMMA. $x \in I_\alpha,\ y \in I_\beta \ \Rightarrow\ xy \in I_\alpha = I_{\min\{\alpha,\beta\}}$

Proof. $[xy]^{(\infty)} = \lim_n (xy)^n \le \min\{\lim_n x^n, \lim_n y^n\} \le \min\{x^{(\infty)}, y^{(\infty)}\} = \min\{\alpha, \beta\}$
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LEMMA. $e \in E \quad e > a \in I_\alpha \quad a^3 > \alpha \ \Rightarrow\ L_\alpha(ea) = L_\alpha(a)$

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$a^2 e \le a \ \Rightarrow\ L_\alpha(a^2 e)$ well-def

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T-LOG BETWEEN ARCHIMEDEAN EQUIV. CLASSES

THEOREM. $T|I_\alpha$ Archimedean, $a \in I_\alpha,\ x \in I_\beta,\ \beta > \alpha)$,

$$\Longrightarrow \qquad L_\alpha(xa) = L_\alpha(a)$$

Proof.

1) $\forall\ N \quad \exists\ z \in I_\alpha \qquad z^N < \alpha, \ \Rightarrow \inf_{z \in I_\alpha} L_\alpha(z) = 0$

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STRICT T-NORMS

DEF. T strict: $xy_1 < xy_2 \qquad (0 < x,\ y_1 < y_2)$

Let T be strict

$E = \{0, 1\}$ because $ex = e$ for $e \leq x \leq 1$

For $z < 1$, $A \cap (0, z)$ $>$-well ordered

$\nexists\ \alpha_1 < \alpha_2 < \cdots < z$ sequence in A

LEMMA. $I_\alpha = (\alpha, \beta]$

Proof. Indirect:
$\alpha \in I_\alpha \Rightarrow \alpha^2 \in I_\alpha \quad \alpha^2 \leq \alpha = \inf I_\alpha \quad \alpha^2 = \alpha \quad \alpha \in E$ contrd.

$\beta := \sup I_\alpha \quad \beta \notin I_\alpha \quad \Rightarrow$

$$\beta \in I_\gamma \quad \exists\ \gamma > \alpha \quad (\text{indeed } \beta = \gamma)$$
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LOGARITHMS ON A STRICT T-NORM

Fix $\alpha \in A$ arbitrarily

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L_α $\searrow$ decreasing, $L_\alpha^{-1}\{\xi\}$ INTV. or POINT

LEMMA. $L_\alpha^{-1}\{\xi\}$ INTV. $\eta \in \mathrm{range}(L_\alpha) \Rightarrow L_\alpha^{-1}\{\xi + \eta\}$ INTV.

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COR. $\{\xi \in \mathrm{range}(L_\alpha) : \ L_\alpha^{-1}\{\xi\} \text{ INTV}\}$ countable ideal in $[\mathrm{range}(L_\alpha), +]$ subsemigroup in $[[1,\infty), +]$.

LOGARITHMS ON A STRICT T-NORM

Fix $\alpha \in A$ arbitrarily

DEF. $\underline{a_\alpha := \sup I_\alpha}$ ($= \max I_\alpha$ by the lemma)

$a_\alpha > a_\alpha^2 > a_\alpha^3 > \cdots \searrow \alpha$

$L_\alpha(a_\alpha) = 1$ may be assumed

L_α $\searrow$ decreasing, $L_\alpha^{-1}\{\xi\}$ INTV. or POINT

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STRUCTURE THEOREM

Recall: $E := \{e: \ e^2 = e\}$, $A := \{a^{(\infty)}: \ a \in [0,1]\}$,

$I_\alpha = \big[\sim$-equiv. cl. whose infimum$=\alpha\big]$, $a_\alpha :=$ sup I_α

THEOREM. T strict $\Longrightarrow$ $E = \{0,1\}$, $I_0 = [0, a_0]$;

EITHER A does not contain infinite increasing sequence

and $I_{\max(A\setminus\{1\})}$ =[open intv.],

$I_\alpha = (\alpha, a_\alpha]$ for $0 \neq \alpha \in A < \max(A \setminus \{1\}$,

OR $A = A_0 \cup \{\alpha_1, \alpha_2, \ldots\} \cup \{1\}$, $A_0 < \alpha_1 < \alpha_2 < \cdots \nearrow 1$,

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EITHER $\quad L_\alpha$ is strictly increasing

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$G \subset \mathbb{R}^2$, $f : G \to \mathbb{R}$ left semicont.:

$$f(x_n, y_n) \to f(x, y) \qquad (x_n \nearrow x,\ y_n \nearrow y)$$

(right sc.: with $x_n \searrow x,\ y_n \searrow y$)

LEMMA. f increasing $\Rightarrow$

$$f \text{ is left [right] sc. iff piecewise left [right] sc.}$$

that is $f(., y),\ f(x, .)$ left [right] sc. $\forall\, x, y$

$\varphi : I$ intv. $\to \mathbb{R}$ increasing

φ is left [right] sc. $\iff$ $\mathrm{range}(\varphi)$ is right [left] closed

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THEOREM. Assume $T|I_\alpha$ gen. Archimedean. Then

T left [right] sc. $\Longrightarrow$ L_α left [right] sc.

Proof. Four cases with similar arguments
Approximate L_α uniformly with left [right] step functions
1) $a > a^2 > a^3 > \cdots$ in I_α, T left sc.
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STRUCTURE OF STRICT LEFT-CONTINUOUS T-NORMS

Let T be strict and left-continuous

THEOREM. All the $\sim$-equivalence classes < 1 are Archimedean (in classical sense).

EITHER the family of all $\sim$-equivalence classes is well-ordered and the first $\sim$-class (neighboring with 1) is continuous

OR there is an increasing sequence of $\sim$-classes converging to 1 and, given any $x < 1$, the family of all $\sim$-equivalence classes $< x$ is well-ordered.

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