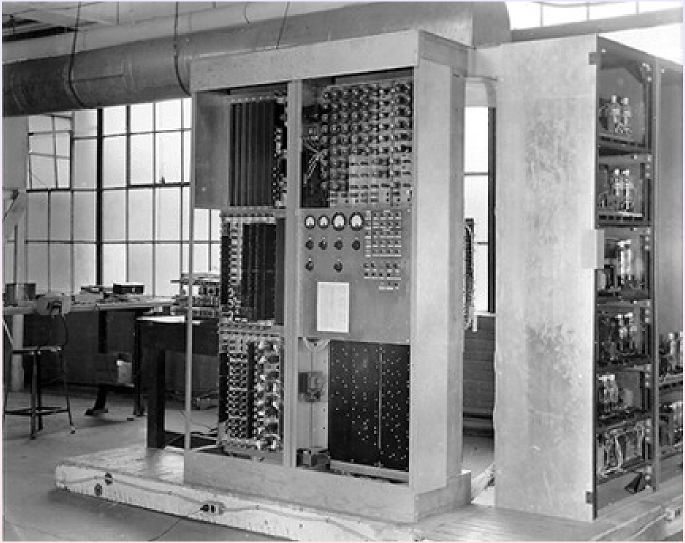
How inaccurate may become accurate?

Stachó László

10/02/2021

Computer $\sim 1940 - 50$



Simple task $\longrightarrow$ incorrect result

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\begin{array}{cccccc}
A_{1,1}x_1 & +A_{1,2}x_2 & +A_{1,3}x_3 & +\dots & +A_{1,36}x_{36} & =B_1 \\
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\end{array}
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WITH COMPUTER !!!

INCORRECT

WHAT IS THE REASON ?!

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ERROR IN SMALL SETTING

STUPID MASHINE
working **with 2 digits**

$$-x_1 + 97x_2 = 97 \qquad (1)$$

$$x_1 + x_2 = 2 \qquad (2)$$

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$$x_1 + x_2 = 2$$

SYMBOLIC VERSION:

$(2) + (1) \Longrightarrow$

$-x_1 + 97x_2 = 97$

$98x_2 = 99 \quad \Longrightarrow \quad x_2 = 99/98, \; x_1 = 97/98$

with 10 digits: $x_1 = 1.010204081 \approx 1, \quad x_2 = 0.9897959183 \approx 1$

Computation

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\begin{aligned}
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$x_2 = 99 : 98 = \mathbf{1.0}$

$(1) \Rightarrow \quad x_1 = 97x_2 - 97 = 97 - 97 = \mathbf{0.0} \quad$ **!!!**

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Are there sabotages or not?

Neumann János
1903 – 1957

Using arithmetics of 8 digits,

the solution of a linear system of equations with 40 variables
IS FALSE WITH A PROBABILITY OF ¿90% IF GAUSSIAN ELIMINATION IS USED

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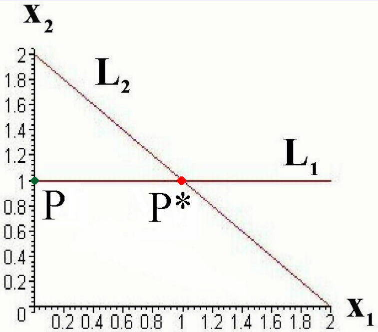
Correction with successive approximation

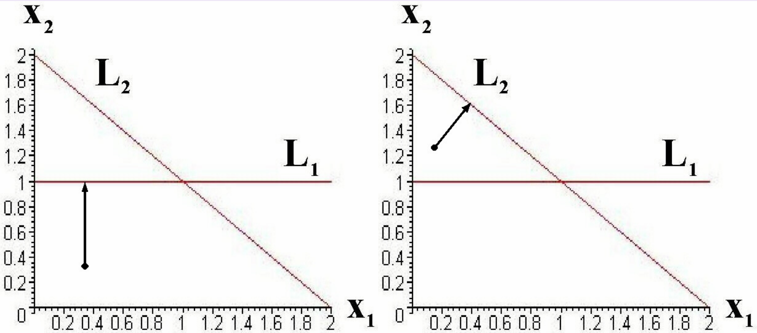


P^* =[True solution]
P = [point by Stupid Machine]

$$\begin{aligned} L_1 &= \{(x_1, x_2):\ -x_1 + 97x_2 = 97\} \\ &= \{(x_1, x_2):\ x_2 = 1 + x_1/97\} \\ &\approx \{(x_1, x_2):\ x_2 = 1\} \\ L_2 &= \{(x_1, x_2):\ x_1 + x_2 = 2\} \\ &= \{(x_1, x_2):\ x_2 = 2 - x_1\} \end{aligned}$$

Correction with successive approximation

$V_1 :=$ [Orthogonal projection to L_1^*-re]
$V_2 :=$ [Orthogonal projection to L_2^*-re]

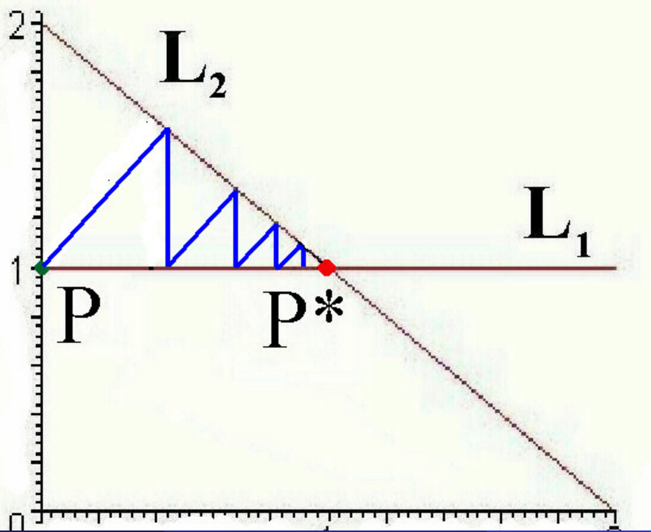


$V_1 : (x_1, x_2) \mapsto (x_1, 1)$

$V_2 : (x_1, x_2) \mapsto \big(1 + (x_1 - x_2)/2,\ 1 + (x_2 - x_1)/2\big)$

Correction with successive approximation

Apply the projections V_1, V_2 several times to the point P of Stupid Machine



Even with Stupid Machine

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$P_1 = V_2(P_1') = \left(1 + \frac{1-0}{2}, 1 + \frac{0-1}{2}\right) = (\mathbf{0.5}\ ,\ \mathbf{1.5})$

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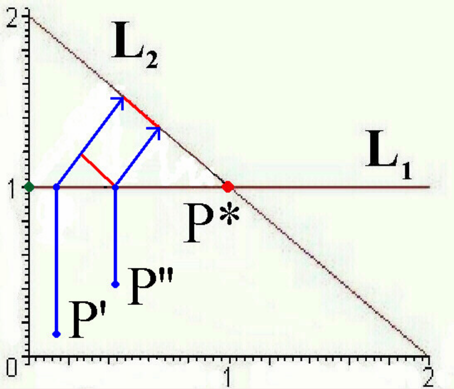
Why does this work?

$W = \left[V_1 \text{ and then } V_2 \text{ applied}\right] = V_2 \circ V_1$

$W : (x_1, x_2) \overset{V_1}{\mapsto} (x_1, 1) \overset{V_2}{\mapsto} \left(1 + \frac{x_1 - 1}{2}, 1 - \frac{x_1 - 1}{2}\right) = \left(\frac{1}{2} + \frac{x_1}{2}, \frac{3}{2} - \frac{x_1}{2}\right)$

W DECREASES THE DISTANCES VERY STRONGLY



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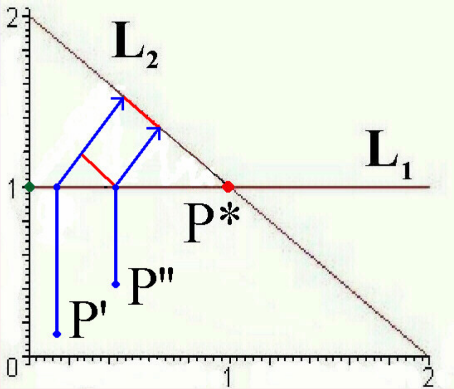

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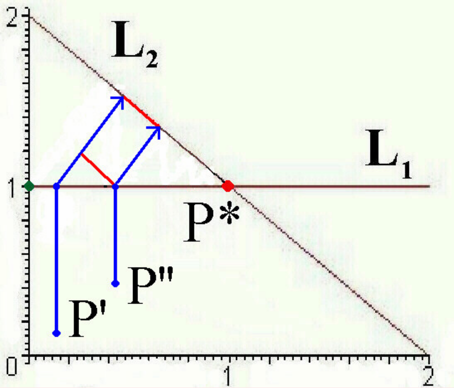

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FIXED POINT PRINCIPLE

$X = \left[\ \mathbb{R}\ \text{straight line or}\ \mathbb{R}^2\ \text{plane or}\ \textbf{closed}\ \text{half line}\ldots\right]$

$W : X \to X \qquad W = \alpha\left[\text{DISTANCE DECREASING}\right] \qquad \exists\ \alpha < 1$

- *Then* $\quad \exists!\ P^* \in X \quad W(P^*) = P^*$
- *Starting from arbitrary P*
  $W(P), W^2(P) = W\big(W(P)\big), W^3(P) = W\big(W^2(P)\big), \ldots \longrightarrow P^*$

Remark: (1)+(2) means that $W(P^*) = P^*$

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Square root

$x^2 = a$ no fixed point equation

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- E.g. $x = a/x$, $W(x) = a/x$

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Very effective method

Pl. $\sqrt{2}$ $\qquad x_{n+1} = \frac{1}{2}\left[x_n + 2/x_n\right], \;\; x_0 = 2$

1.5000000000000000000000000000000000000000000000

1.4166666666666666666666666666666666666666666666

1.41421568627450980392156862745098039215686274509803921

1.41421356237468991062629557889013491011655962211574400

1.414213562373095048801689623502530243614981925776197

1.41421356237309504880168872420969807856967187537772

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1.41421356237309504880168872420969807856967187537 69

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1.414213562374689910626295578890134910116559622115744 0

1.4142135623730950488016896235025302436149819257761 97

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GENERALIZATIONS

- Cubic root: $\sqrt[3]{a}$ Kepler $W(x) = \frac{2}{3}x + \frac{1}{3}a/x^2$
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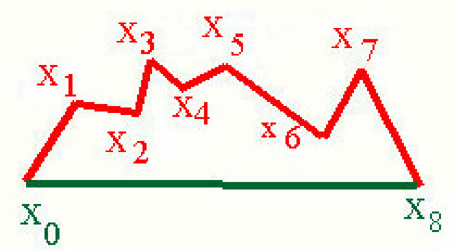
BANACH'S FIXED POINT THEOREM

$X = \{ \text{OBJECTS} \}$

d distance between the elements of X

- $d(x,y) = d(y,x) > 0$, if $x \neq y \in X$
- Bypass $\geq$ direct way:



$$d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \geq d(x_0, x_n)$$

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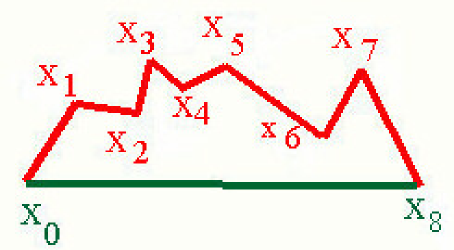

$$d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) \geq d(x_0, x_n)$$

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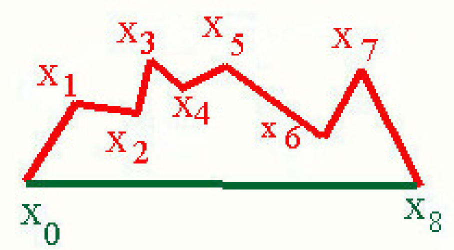

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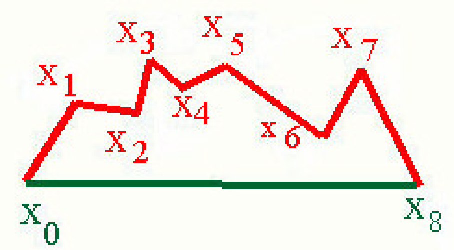

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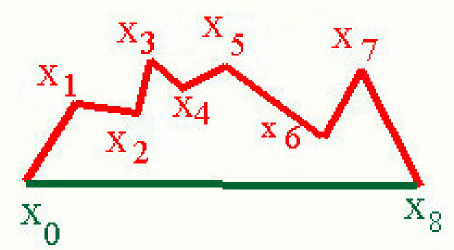

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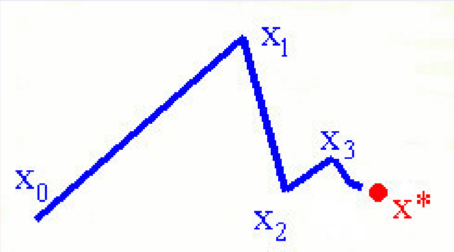
- **Completeness:** Finite paths (with infinite steps) end in X:



$d(x_0, x_1) \leq 1,\ d(x_1, x_2) \leq \frac{1}{2},\ d(x_2, x_3) \leq \frac{1}{4}, \ldots \implies$
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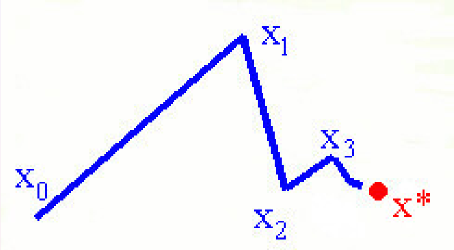

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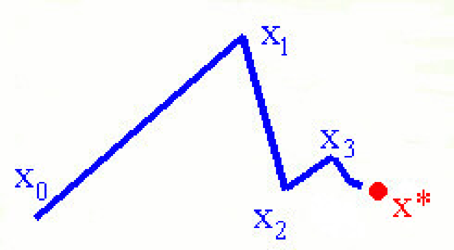

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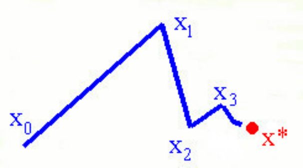
- **TÉTEL.** X, d *complete metric space*



- $W : X \to X$ mapping with α[d-DECREASING], $\alpha < 1$.

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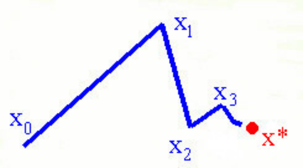

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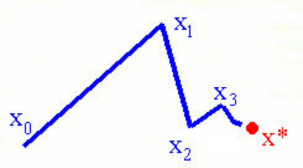

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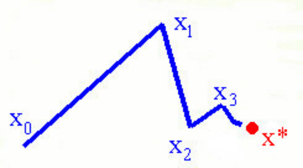

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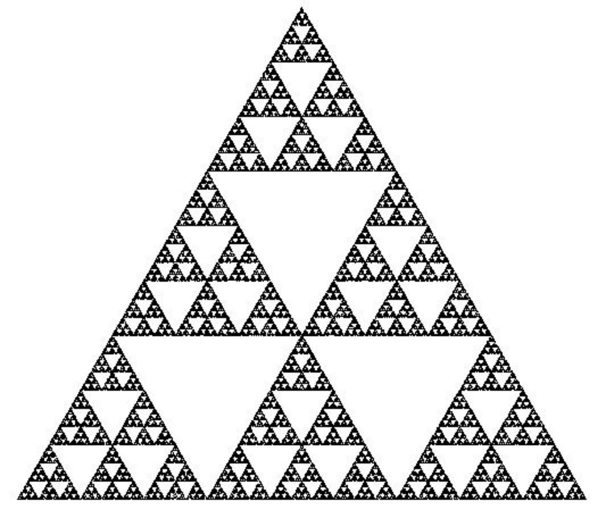

How can be deduced this from Banach's Fixed Point Thm?

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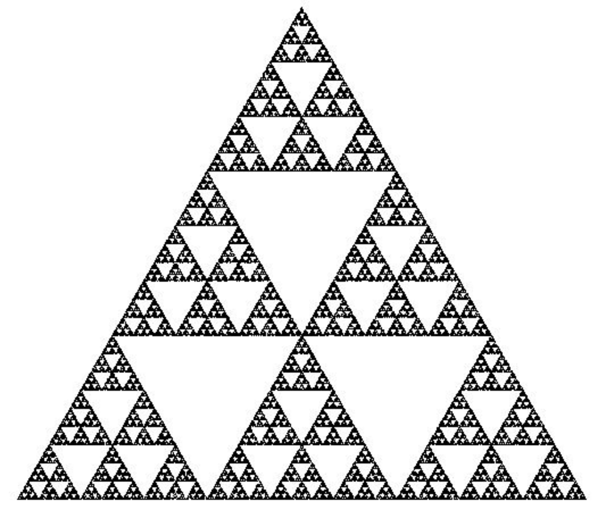

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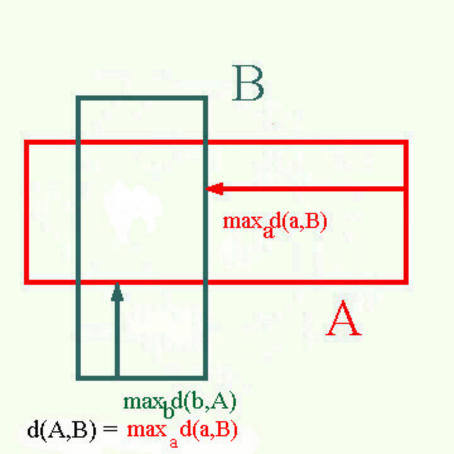

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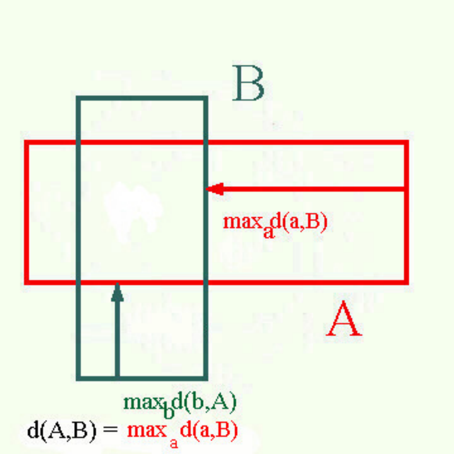

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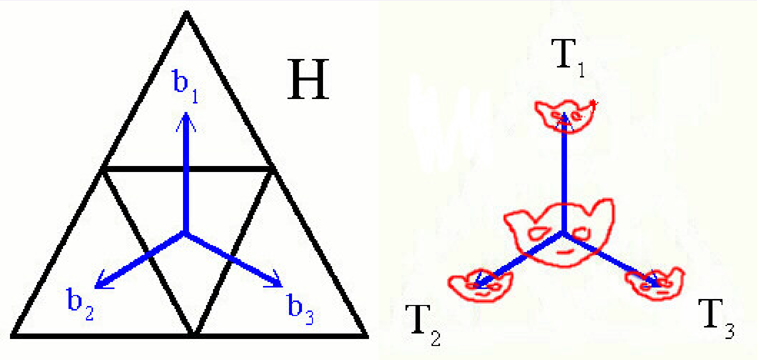

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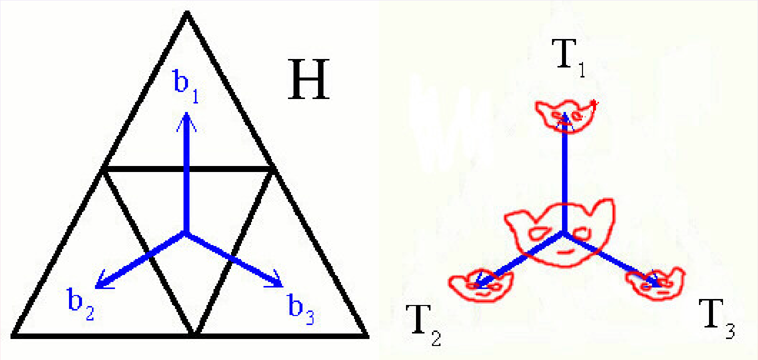

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THEREFORE $H,\ W(H),\ W^2(H), \ldots$ *CONVERGE to some figure !*

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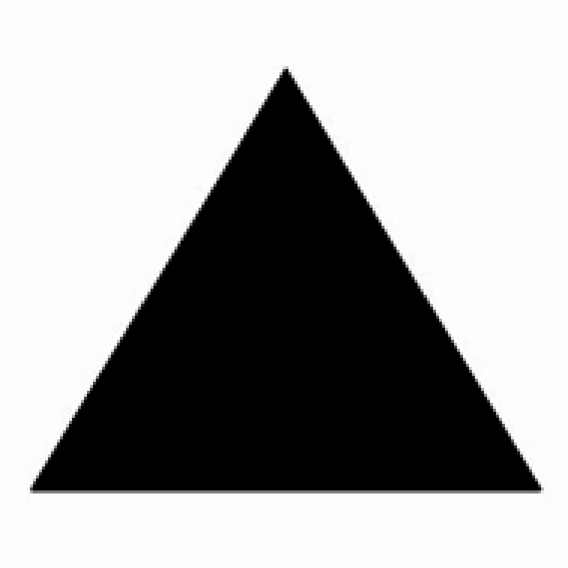
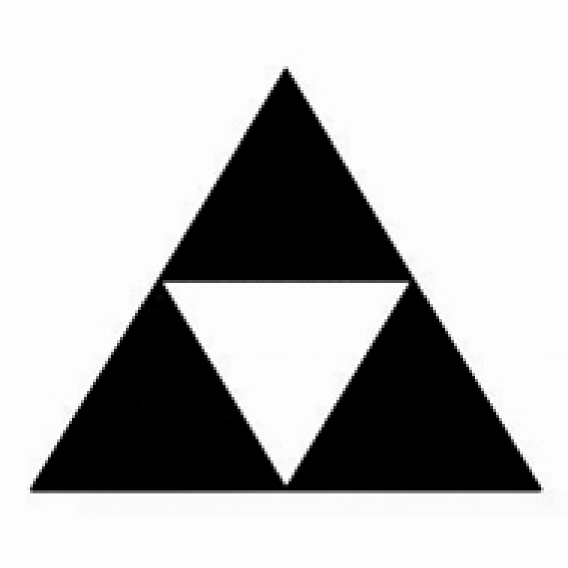
BACK TO THE TRIANGLE — H



$W(H)$



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TAJ MAHAL