SVD és főtengely-transzformáció

Emlékeztető. Ha $A \in \mathbb{R}^{N\times N}$ invertálható mátrix, akkor

$$\begin{aligned}
&A = F\Lambda E^{\mathrm{T}}, \quad \textit{ahol}\\
&F = \big[f_1, \dots, f_N\big] \in \mathrm{Ort}(N, \mathbb{R}),\\
&\Lambda = \mathrm{diag}\big(\lambda_1, \dots, \lambda_N\big), \quad \lambda_1 \geq \dots \geq \lambda_N \geq 0,\\
&E = \big[e_1, \dots, e_N\big] \in \mathrm{Ort}(N, \mathbb{R}).
\end{aligned}$$

Konstrukció:

$$\begin{aligned}
&e_1: \quad \|e_1\|^2 = \langle e_1|e_2\rangle = 1, \quad \|Ae_1\| \;\; MAX,\\
&\lambda_1 := \|Ae_1\|, \quad f_1 = Ae_1/\lambda_1.
\end{aligned}$$

Ha $(e_1, \lambda_1, f_1), \dots, (e_k, \lambda_k, f_k)$ adott,

$$\begin{aligned}
&e_{k+1}: \quad e_{k+1} \perp e_1, \dots, e_k, \quad \|e_{k+1}\| = 1, \quad \|Ae_{k+1}\| \;\; MAX,\\
&\lambda_{k+1} := \|Ae_{k+1}\|, \quad f_{k+1} = Ae_{k+1}/\lambda_{k+1}.
\end{aligned}$$

Mindegyik lépésnél

$$Ae_{k+1} \perp Af \quad \text{valahányszor } f \perp e_1, \dots e_k.$$

Szimmetrikus mátrix esete.

$A = A^{\mathrm{T}} \in \mathbb{R}^{N\times N}$, $\quad A = F\Lambda E^{\mathrm{T}}$, $\quad \Lambda = \mathrm{diag}\big(\lambda_1, \dots, \lambda_N\big)$ SVD-felbontás

$\mathrm{Sp}(A) = \big\{\mu_1, \dots, \mu_s\big\}$, $\quad \mu_1 > \dots > \mu_s > 0$, $\quad \mu_k$ multiplicitása $m_k(> 0)$:

$$\underbrace{\lambda_1 = \dots = \lambda_{m_1}}_{=\mu_1}, \; \underbrace{\lambda_{m_1+1} = \dots = \lambda_{m_1+m_2}}_{=\mu_2}, \dots, \underbrace{\lambda_{m_1+\dots+m_{s-1}+1} = \dots = \lambda_N}_{=\mu_s},$$

$$\begin{aligned}
A = F\Lambda E^{\mathrm{T}} &= F\,\mathrm{diag}(\lambda_1, \dots, \lambda_N)E^{\mathrm{T}} =\\
= A^{\mathrm{T}} = E\Lambda F^{\mathrm{T}} &= E\,\mathrm{diag}(\lambda_1, \dots, \lambda_N)F^{\mathrm{T}}.
\end{aligned}$$

Észrevétel: $E^{\mathrm{T}}\cdot, \cdot E$ szorzásokkal

$$Q\Lambda = \Lambda Q^{\mathrm{T}}, \quad \text{ahol } Q := E^{\mathrm{T}}F \in \mathrm{ORT}(N).$$

Innen

$$\begin{aligned}
\Lambda^{-1}Q\Lambda &= Q^{\mathrm{T}} = Q^{-1},\\
[\Lambda^{-1}Q\Lambda]^{\mathrm{T}} &= [\Lambda^{-1}Q\Lambda]^{-1} = Q\\
\Lambda Q^{\mathrm{T}}\Lambda^{-1} &= \Lambda^{-1}Q^{-1}\Lambda\\
\Lambda^2 Q^{\mathrm{T}} &= Q^{-1}\Lambda^2 = Q^{\mathrm{T}}\Lambda^2\\
\lambda_i^2[Q]_{ji} &= [Q]_{ji}\lambda_j^2.
\end{aligned}$$

Vagyis $[Q]_{ji} = 0$ valahányszor $\mu_{k_i} = \lambda_i \neq \lambda_j = \mu_{k_j}$, azaz ha az (i, j) indexpár nincs az $\bigcup_{k=1}^{s}\{m_{k-1}+1, \ldots, m_k\}^2$ átlóbeli négyzetekben. Ezért

$$Q = \operatorname{diag}(Q_1, \ldots, Q_s), \quad \text{ahol} \;\; Q_k \in \operatorname{ORT}(m_k).$$

Tehát

$$\begin{aligned} A &= F\Lambda E^{\mathrm{T}} = EQ \operatorname{diag}(\mu_1 I_{m_1}, \ldots, \mu_s I_{m_s})E^{\mathrm{T}} = \\ &= E \operatorname{diag}(\mu_1 Q_1, \ldots, \mu_s Q_s)E^{\mathrm{T}}. \end{aligned}$$

Mivel $A = A^{\mathrm{T}} = E \operatorname{diag}(\mu_1 Q_1^{\mathrm{T}}, \ldots, \mu_s Q_s^{\mathrm{T}})E^{\mathrm{T}}$ is áll,

$$Q_k = Q_k^{\mathrm{T}} = Q_k{-}1 \qquad (k = 1, \ldots, s),$$

azaz a Q_k mátrixok *szimmetrikus ortogonális* mátrixok, azaz *tükrözések*:

$$Q_k = G\operatorname{diag}(I_{\ell_k}, -I_{m_k-\ell_k})G^{\mathrm{T}}, \quad G_k \in \operatorname{ORT}(m_k)$$

alakúak. Ezekkel a főtengely alak

$$\begin{aligned} &A = \widetilde{E} \operatorname{diag}(\mu_1 I_{\ell_1}, -\mu_1 I_{m_1-\ell_1}, \ldots, \mu_s I_{\ell_s}, -\mu_s I_{m_s-\ell_s})\widetilde{E}^{\mathrm{T}}, \\ &\text{ahol} \quad \widetilde{E} = E \operatorname{diag}(G_1, \ldots, G_s). \end{aligned}$$