NEWTON-Raphson ITERÁCIÓ

Ókor: $\sqrt{2}$

2   1.5   1.41666   1.414215

$2,\ \frac{1}{2}\left[2+\frac{2}{2}\right],\ \frac{1}{2}\left[1.5+\frac{2}{1.5}\right],\ \frac{1}{2}\left[1.41\dot{6}+\frac{2}{1.41\dot{6}}\right]$

$x^2=2 \qquad x=\frac{2}{x} \qquad \left.\begin{array}{l} x=\frac{2}{x} \quad \cdot 1/2 \\ x=x \quad \cdot 1/2 \end{array}\right\} + \qquad x=\frac{1}{2}\left(x+\frac{2}{x}\right)$

$\sqrt{9}$-re is $x^2=9$ $\left.\begin{array}{l} x=\frac{9}{x} \\ x=x \end{array}\right\}$ $x=\frac{1}{2}\left[x+\frac{9}{x}\right]$

<u>Kérdés</u> 1) Miért konvergál

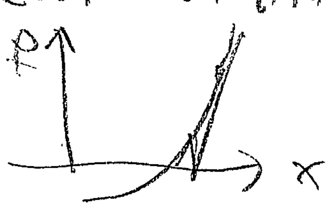
2) Hogy kell módosítani pl $\sqrt[3]{9}=?$

2)-höz: KEPLER $\left.\begin{array}{l} x^3=9 \quad \cdot\frac{1}{3} \\ x=x \quad \cdot\frac{2}{3} \end{array}\right\}$ $x=\frac{2}{3}x+\frac{1}{3}\frac{9}{x^2}$

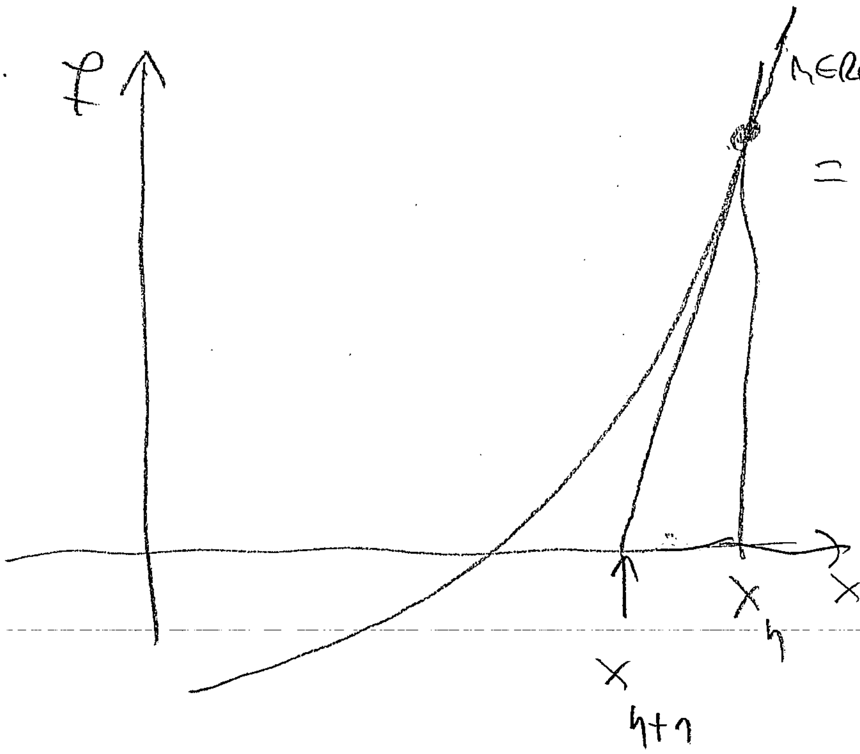
Sejtés $\sqrt[N]{9}$-hoz $x_0:=9$

$$x_{n+1}:=\left(1-\frac{1}{N}\right)x_n+\frac{1}{N}\frac{9}{x_n^{N-1}}$$

1) NEWTON ötlete $f(x)=0$ [Pl $x^2-2=0$]



érintő-módszer



meredekség =

$= f'(x_n)$

$$(x_{n+1} - x_n) \cdot f'(x_n) = f(x_n)$$

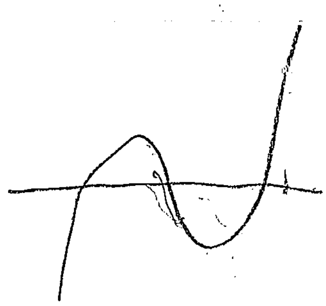
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Pl. $\sqrt[N]{a}$ $\quad x^N - a = 0 \quad f(x) := x^N - a \quad f'(x) = N x^{N-1}$

$$x_{n+1} = x_n - \frac{x_n^N - a}{N x_n^{N-1}} = \left(1 - \frac{1}{N}\right) x_n + \frac{1}{N} \frac{a}{x_n^{N-1}}$$

1) miért konvergál?

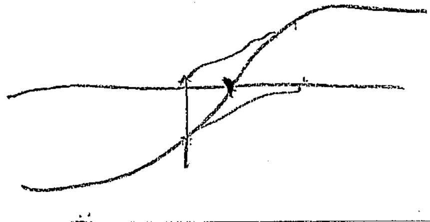
„nyakló nélkül" nem megy



a) $f(x) := x^2 - x$

„0 eltalálása"

b) $f(x) = \operatorname{arctg} x$



c) $\operatorname{arctg}(x^3 - x)$  kaotikus.

TAYLOR FORMULA + LAGRANGE MARADÉKTAG

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \dots + \frac{1}{N!}f^{(N-1)}(x)h^{N-1} +$$
$$+ \frac{1}{N!}f^{(N)}(t_{x,h})h^N$$

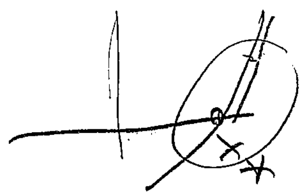
HA f N-SZER FOLYT DIFF

$N = 2$

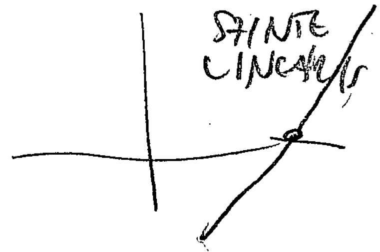
$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(t_{x,h})h^2$$

NEWTON [illegible]

$x_n \approx x_* \qquad f(x_*) = 0$



$$f(x_n + h) \approx f(x_n) + f'(x_n)h$$



$$0 = f(x_*) \approx f(x_n) + f'(x_n)\cdot(x_* - x_n)$$

$$x_* \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

BECSLÉS

$$0 = f(x_*) = f(x_n) + f'(x_n)(x_* - x_n) + \frac{1}{2}f''(t_{x_n,x_*})(x_* - x_n)^2$$

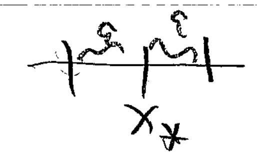
$$0 = \text{közelítés}(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

$(2) - (1) \Rightarrow$

$$0 = f'(x_n)(x_{n+1}-x_*) - \frac{1}{2} f''(t_{x_n,x_*})(x_n-x_*)^2$$

$$|x_{n+1}-x_*| = \underbrace{\frac{1}{2}\,\frac{|f''(t_{x_n,x_*})|}{|f'(x_n)|}}_{\le M \text{ ha } x_n\in(x_*-\varepsilon,\, x_*+\varepsilon)}\, |x_n-x_*|^2$$

$\exists \varepsilon > 0$



$e_n := |x_n - x_*|$ $\quad$ x_n TÁVOLSÁGA A MEGOLDÁSTÓL

$$\boxed{e_{n+1} \le M e_n^2}$$

DE M LEHET NAGYON NAGY

$e_1 \le M e_0^2$ — 2

$e_2 \le M e_1^2 \le M(M e_0^2) = M^3 e_0^4$ — 4

$e_3 \le M e_2^2 \le M(M^3 e_0^4) = M^7 e_0^8$ — 8

$\vdots$ — 16

$$e_{n+1} \le M^{2^n-1} e_0^{2^n} = \frac{1}{M}[M e_0]^{2^n}$$

HA MÁR $M e_0 < 1$ (AZAZ x_0 „NAGYON KÖZEL" VAN x_*-HOZ)

AKKOR $x_n \to x_*$ SŐT $|x_n - x_*| \le \text{const}\cdot(1-\delta)^{2^n}$

TÖBBVÁLTOZÓS NEWTON-RAPHSON ITERÁCIÓ

$\underline{f} : \mathbb{R}^k \to \mathbb{R}^k$ $\quad x_i := \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix} \mapsto \xi_i$ KOORDINÁTÁK $\underline{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}$

$\nabla f_i(\underline{x}) = \left[\frac{\partial f_i}{\partial x_1} \; \frac{\partial f_i}{\partial x_2} \cdots \frac{\partial f_i}{\partial x_k} \right]$ GRADIENSEK (SORVEKTOR)

$\nabla^2 f_i(\underline{x}) := \left[\frac{\partial^2 f_i}{\partial x_h \partial x_\ell} \right]_{h,\ell=1}^{k}$ HESSE-MÁTRIXOK

$$\boxed{f_i(\underline{x}+\underline{h}) = f_i(\underline{x}) + \langle \nabla f_i(\underline{x}), \underline{h} \rangle + \frac{1}{2} \underline{h}^* \nabla^2 f_i(\xi_{x,h}) \underline{h}}$$

Tegyük fel: x_n ADOTT [n. KÖZELÍTÉSE x_*-NAK]

$f(x_*) = 0$

$f_i \simeq [f_i]_n(\underline{x}) := f_i(x_n) + \langle \nabla f_i(x_n) \,|\, x - x_n \rangle$

$[f_i]_n(\underline{x}) = 0 \quad (i = 1, 2, \ldots, k)$ EGYENLETRENDSZER

Pl $k = 2$ $\quad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \underline{f} = \begin{bmatrix} f \\ g \end{bmatrix}$

$$\left.\begin{aligned} f\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{\partial f}{\partial x}(x_{n+1} - x_n) + \frac{\partial f}{\partial y}(y_{n+1} - y_n) &= 0 \\ g\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{\partial g}{\partial x}(x_{n+1} - x_n) + \frac{\partial g}{\partial y}(y_{n+1} - y_n) &= 0 \end{aligned}\right\}$$

$$\underline{\underline{f}}(\underline{\underline{x}}_n) + \underbrace{\begin{bmatrix} \nabla f(x_n) \\ \nabla g(x_n) \end{bmatrix}}_{} (\underline{\underline{x}}_{n+1} - \underline{\underline{x}}_n) = 0$$

2×2-es mátrix $\quad \underline{\underline{f}}$ JACOBI MÁTRIXA

Általában: (TETSZ. $k \geq 1$ DIM-BAN)

$$\boxed{\underline{\underline{f}}(\underline{\underline{x}}_n) + \underbrace{\begin{bmatrix} \nabla f_1(\underline{\underline{x}}_n) \\ \vdots \\ \nabla f_k(\underline{\underline{x}}_n) \end{bmatrix}}_{k \times k\text{-s MÁTRIX}} (\underline{\underline{x}}_{n+1} - \underline{\underline{x}}_n) = 0}$$

MEGOLDANDÓ $\underline{\underline{x}}_{n+1}$-RE

$$\boxed{\underline{\underline{x}}_{n+1} := \underline{\underline{x}}_n - \underbrace{\begin{bmatrix} \nabla f_1(\underline{\underline{x}}_n) \\ \vdots \\ \nabla f_n(\underline{\underline{x}}_n) \end{bmatrix}}_{\underline{\underline{f}}'(\underline{\underline{x}}_n)}^{-1} \underline{\underline{f}}(\underline{\underline{x}}_n)}$$

Becslés

$$\underline{\underline{x}}_{n+1} - \underline{\underline{x}}_* = \frac{1}{2} \underline{\underline{f}}'(\underline{\underline{x}}_n)^{-1} \begin{bmatrix} (\underline{\underline{x}}_n - \underline{\underline{x}}_*)^* \nabla^2 f_1(\underline{\underline{t}}) (\underline{\underline{x}}_n - \underline{\underline{x}}_*) \\ \vdots \\ (\underline{\underline{x}}_n - \underline{\underline{x}}_*)^* \nabla^2 f_k(\underline{\underline{t}}) (\underline{\underline{x}}_n - \underline{\underline{x}}_*) \end{bmatrix}$$

$$\boxed{\|x_{n+1} - x_*\| \leq M \|x_n - x_*\|^2}$$