

LU felbontás a Gauss-elimináció alapján

$$\pi_2 = \{0, 1\} \quad a \square^2 b = \text{mod}_2 a \square b$$

$$1 + 1 = 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = LU$$

Algoritmus: GE  $A$ -ra  $\rightarrow U$

$$L = ?$$

$$A = A_0 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{Szűkítve} \quad A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Szűkítve

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & x & x \end{bmatrix}$$

$\rightarrow$  szűkítve

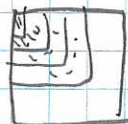
$$\rightarrow (1. \text{ sor}) - x(2. \text{ sor})$$

$\rightarrow$  kész

Mikor működik a Gauss-elimináció?

GE működik  $\Leftrightarrow$  a főátlós elemek mindegyike nem 0

$$\text{itt: } \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$$





$$A = A_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & x & x \\ 0 & x & x \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A_1$$

$$\begin{cases} A_1 \cdot 1.m \\ A_1 \cdot 2.m - A_1 \cdot 1.m \end{cases}$$

$$A_1 \cdot 2.m$$

$$U = A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{cases} -A_1 \cdot 1.m \\ -A_1 \cdot 2.m \\ -A_1 \cdot 3.m - A_1 \cdot 2.m \end{cases}$$

$A_2$  - ober triangularis

$$A_2 = U$$

$L$ : Matrix geben, so dass  $U$  ist, oder ist einstellbar

Matrix-Matrix

$$L = [l_{ij}]_{i,j=1}^m$$

$L$  hat die Form

$$L \cdot A = \begin{bmatrix} l_{11} a_1 + l_{12} a_2 + \dots + l_{1m} a_m \\ l_{21} a_1 + l_{22} a_2 + \dots + l_{2m} a_m \\ \vdots \\ l_{m1} a_1 + l_{m2} a_2 + \dots + l_{mm} a_m \end{bmatrix}$$

$$A = [a_{ij}]_{i,j=1}^n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$L$  ist ein  $m \times m$  Matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ A \cdot 2.m \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Orthonormal

$$A = [a_{ij}]_{i,j=1}^m$$

$$R = [r_{ij}]_{i,j=1}^m$$

$$A = [a_1' \ a_2' \ a_3']$$

Orthonormal

$$AR = \begin{bmatrix} r_{11} a_1' + r_{12} a_2' + r_{13} a_3' & r_{11} a_1' + r_{12} a_2' + r_{13} a_3' \\ r_{21} a_1' + r_{22} a_2' + r_{23} a_3' & r_{21} a_1' + r_{22} a_2' + r_{23} a_3' \\ r_{31} a_1' + r_{32} a_2' + r_{33} a_3' & r_{31} a_1' + r_{32} a_2' + r_{33} a_3' \end{bmatrix}$$

R ist ein  $m \times m$  Matrix



$$A_0 = A$$

$$A_1 = L_1 A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A_0$$

row manipulation:  $A_0$ -Zeile: (1. Zeile) und

$$(2. \text{ Zeile}) - (1. \text{ Zeile})$$

$$(3. \text{ Zeile}) \text{ nicht}$$

$$\begin{array}{l} 1.1w + 0.2w + 0.3w \\ -1.1w + 1.2w + 0.3w \\ 0.1w + 0.2w + 1.3w \end{array}$$

$$A_2 = L_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot A_1$$

$$A_0 = A$$

$$A_1 = L_1 A_0$$

$$A_2 = L_2 A_1 = L_2 (L_1 A) = (L_2 L_1) A = U$$

$$L_2 L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = (L_2 L_1)^{-1} U = L \cdot U$$

$$L = (L_2 L_1)^{-1} = L_1^{-1} \cdot L_2^{-1}$$

und  $L_2$ -beim

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L_1$$

$$L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = L_2$$

alt:  $\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

||

$$\tilde{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = L U$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# QR - felbontás

$$A = QR$$

$$A = QR$$

$$x = A^{-1} b$$

$$x = A^{-1} b$$

$$A^{-1} = R^{-1} Q^T$$

Q ortogonális

R háromszögletes (primitív)

Vektorok stabil, de a normák instabil

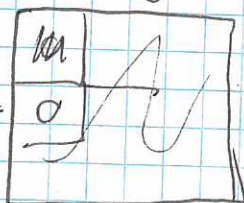
Csak valószínűleg komplex tartomány

Givens - forgatás elv.

Elimináció

$$A_0 = A$$

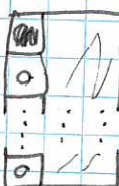
$$A^{(1)} = Q^{(1)} A_0$$



$$A^{(2)} = Q^{(2)} A^{(1)}$$



$$A^{(3)} = Q^{(3)} A^{(2)}$$



$$A^{(4)} = Q^{(4)} A^{(3)}$$



$$A^{(5)} = Q^{(5)} A^{(4)}$$



$$A^{(m)} = Q^{(m)} A^{(m-1)}$$



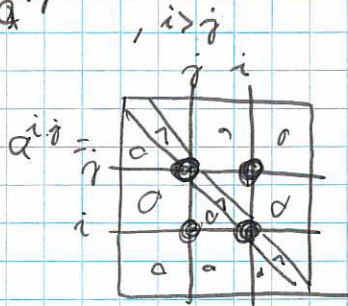
R → háromszögletes



$$Q^{m,m-1} \dots Q^{2,1} A = R$$

$$A = (Q^{2,1})^T \dots (Q^{m,m-1})^T \cdot R$$

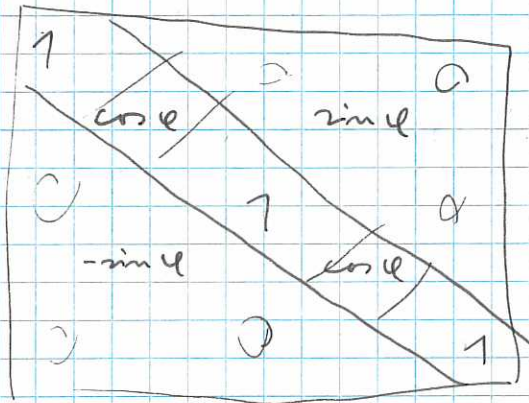
$Q^{ij}$



Givens rotation

→ orthogonal

Givens rotation:



→ orthogonal  
symmetric  
unitary

algebraic:  $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \left. \vphantom{\begin{pmatrix} x & y \\ -y & x \end{pmatrix}} \right\} x^2 + y^2 = 1 \rightarrow \text{on a circle}$

Given test: Find the prime number sequence of  $n$ , in  $0 \leq n \leq 10^6$ .

may be



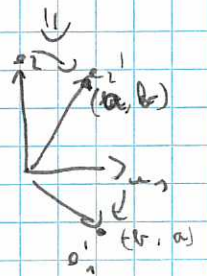
Givens -yötel pe'lela

$$A = \begin{bmatrix} 36 & 192 & -517 \\ -48 & 354 & 581 \\ 25 & 480 & 170 \end{bmatrix} = A_0$$

$$A_1 = \begin{bmatrix} 0 \\ 25 & 480 & 170 \end{bmatrix}$$

Sat =  $\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(lengthi 1'ine,  
 $a^2 + b^2 = 1$ )



$$Q_1 \cdot A = A_1$$

$$\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 36 & 192 \\ -48 & 354 \end{bmatrix} = \begin{bmatrix} -36b + 48a \\ -48a + 354b \end{bmatrix}$$

$$-36a - 48b = 0 \rightarrow b = -\frac{4}{3}a$$

$$a^2 + b^2 = 1$$

$$Q_1 = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-36a - 48b = -36a - 48(-\frac{4}{3}a) = -36a + 64a = 28a$$

$$a^2 + (-\frac{4}{3}a)^2 = 1$$

$$a^2 + \frac{16}{9}a^2 = 1$$

$$\frac{25}{9}a^2 = 1$$

$$(\frac{5}{3}a)^2 = 1$$

$$\frac{5}{3}a = \pm 1$$

$$a = \pm \frac{3}{5}$$

valutini belitizig

$$\Rightarrow a = \frac{3}{5}, b = -\frac{4}{5}$$

$$Q_1 \cdot A = A_1 = \begin{bmatrix} 60 & -200 & -175 \\ 0 & 30 & -65 \\ 25 & 480 & 170 \end{bmatrix}$$



$$Q_2 \cdot A_1 = A_2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 65 & 0 & -65 \\ 0 & -650 & 520 \\ 0 & 750 & 455 \end{bmatrix} = \begin{bmatrix} 65 & 0 & -650 \\ 0 & 750 & -65 \\ 0 & 520 & 455 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{bmatrix}$$

$$\begin{cases} 0 = -60b + 25a \\ 1 = a^2 + b^2 \end{cases} \Rightarrow \begin{cases} a = 12/13 \\ b = 5/16 \end{cases}$$

$$A_3 = Q_3 A_2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}$$

$$\begin{cases} -520b + 455a = 0 \\ a^2 + b^2 = 1 \end{cases} \Rightarrow \begin{cases} a = 3/5 \\ b = 4/5 \end{cases}$$

$$A_3 = \begin{bmatrix} 65 & 0 & -650 \\ 0 & 650 & 725 \\ 0 & 0 & 725 \end{bmatrix}$$

$$u = A_3 = Q_3 \cdot Q_2 \cdot Q_1 \cdot A$$

$$A = \underbrace{Q_1^{-1} Q_2^{-1} Q_3^{-1}}_Q u = Q^T u$$

$$Q = \begin{bmatrix} 16/65 & 96/125 & -257/125 \\ -48/65 & 137/125 & -96/125 \\ 5/17 & 48/65 & 16/65 \end{bmatrix} = \frac{1}{125} \begin{bmatrix} 256 & 96 & -257 \\ -96 & 137 & -96 \\ 85 & 48 & 16 \end{bmatrix}$$

QR faktorisasi berdasarkan:

$$Ax = b$$

$$A = QR$$

$$QRx = b$$

$$Rx = Q^T b$$

→ lebih sederhana  
dibaca

$$r_{11}x_1 + r_{12}x_2 + \dots + r_{1n}x_n = b_1$$

$$\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

→ lebih mudah  
dibaca



$$A = LU$$

$$Ax = b$$

$$LUx = b$$

$$Ux = L^{-1}b \rightarrow \text{nr} \text{ számok hiába (e's nem kell eigen)} \\ \uparrow \\ \text{indefiniten közelíti}$$

Végső lépés feltétlenül analitikus pontosság.

~~Transformáció~~ (SVD elvérté)  $\rightarrow$  transzformáció

SVD felbontás

Singular Value Decomposition (Releigh-Prin)

$$Ax = b$$

$$A = Q \Lambda R^T$$

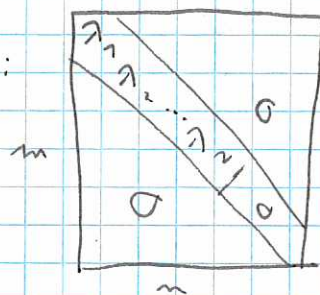
$A$ :  $n \times n$  mátrix

$Q$ :  $n \times n$  ortogonális mátrix

$R$ :  $n \times n$  ortogonális mátrix

$A$

$\Lambda$ :



$r \leq \min(n, m)$

$\text{rank}(A)$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$$

$$A \neq 0$$



$$Q = \begin{bmatrix} q_1 & q_2 & q_3 & \dots & q_m \end{bmatrix}$$

$$R = \begin{bmatrix} r_1 & \dots & r_m \end{bmatrix}$$



$q_i$ : Spaltenvektor

$\mathbb{R}^m$ -ve



$r_i$ : Spaltenvektor

$\mathbb{R}^m$ -ve

Permutation - Koordinatensystem auf  $\mathbb{R}^{m-1}$ ,  $\mathbb{R}^{m-1}$

$$[P \text{ und}] = \begin{bmatrix} (p|r_1) \\ \vdots \\ (p|r_m) \end{bmatrix}$$

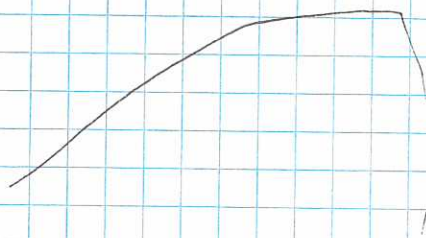
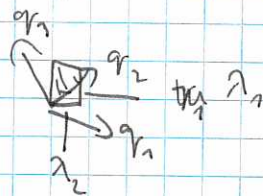
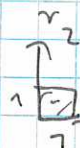
$$[P' \text{ und}] = \begin{bmatrix} (p'|r_1) \\ \vdots \\ (p'|r_m) \end{bmatrix}$$

$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear

$n=2, m=3$



1. Teil ist nun wichtig  
~ ist gegeben, dass  $n$  Teil ist  
Teil ist nun wichtig





Railigh-Ritz Lemma: "von SVD"

Lineare Algebra: (Youtuber, MIT, 2x2)

$$A = Q \Lambda R^T$$

$$A^T A = (Q \Lambda R^T)^T Q \Lambda R^T = R^T \Lambda^T Q^T Q \Lambda R^T = R \Lambda^T \Lambda R^T =$$

$$I = Q^T Q$$

$$R^T R = I$$

$$\Lambda^T \Lambda = \begin{matrix} m & & n \\ \begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{matrix} & \begin{matrix} m \\ \begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{matrix} \end{matrix} = \begin{matrix} m & & n \\ \begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{matrix} \end{matrix}$$

$$= R \begin{matrix} m & & n \\ \begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{matrix} \end{matrix} R^T$$



$$v_i \mapsto \lambda_i^2 v_i$$

$\Lambda^T \Lambda = A^T A$   $v_i$ : rechteckige B-matrix,  $\lambda_i^2$  rechteckige B-matrix

B ist positiv definit, also  $\lambda_1^2 \dots \lambda_n^2$  positiv, also  $\lambda_1 \dots \lambda_n$  positiv.

Es ist B positiv definit, also rechteckig



# átlag transzformáció:

$$B = \begin{bmatrix} 77 & -76 \\ -76 & 52 \end{bmatrix}$$

Így  $\lambda_p(B) = \langle B \text{ sajátértékei} \rangle =$   
 $= \langle B \text{ karakterisztikus polinomjának gyökei} \rangle$

$$\text{CharPoly}(B, \lambda) = \det(\lambda I - B) = \det$$

$$\det \begin{pmatrix} \lambda - 77 & 76 \\ 76 & \lambda - 52 \end{pmatrix} = (\lambda - 77)(\lambda - 52) - 76^2 =$$

$$= \lambda^2 + ((-77) + (-52))\lambda + (-77) \cdot (-52) - 76^2 =$$

$$= \lambda^2 - 125\lambda + 2500 =$$

$$\lambda_{1,2} = \frac{125 \pm \sqrt{125^2 - 4 \cdot 2500}}{2}$$

$$\lambda_{1,2} = \frac{125 \pm 75}{2} = \begin{cases} 100 \\ 25 \end{cases}$$

$$s_p(n) = \langle 100, 25 \rangle$$

sajátvektorok:  $v_1 = \begin{bmatrix} 4/5 \\ -7/5 \end{bmatrix}$   $v_2 = \begin{bmatrix} 7/5 \\ 4/5 \end{bmatrix}$

SVD

$$A = \frac{1}{125} \begin{bmatrix} 1728 & -656 \\ -1728 & 2228 \\ 1728 & 210 \end{bmatrix} = \underbrace{Q}_{\text{ortogonális}} \underbrace{\Lambda}_{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underbrace{R^T}_{\text{ortogonális}}$$

Trükk:  $A^T A = R \underbrace{\Lambda^T \Lambda}_{\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}} R^T = B$





$$\begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

$R \quad \Lambda^T \quad R^+$

Think:  $A v_1 = \lambda_1 R^T v_1$

$$R^T R = \begin{bmatrix} R^T v_1 & R^T v_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T v_1 & R^T v_2 \end{bmatrix}$$

Let:  $R^T v_i = e_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}_i$

$$A v_1 = \lambda_1^2 e_1$$

$$A v_2 = \lambda_2^2 e_2$$

$$\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\uparrow$   
100

$$= \lambda_2^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\uparrow$   
25

$$A v_1 = \lambda_1^2 e_1 = \lambda_1^2 \cdot Q e_1 = \lambda_1^2 \cdot v_1$$

$$Q = [q_1 \ q_2 \ \dots \ q_n]$$

$$A v_2 = \lambda_2^2 e_2 = \lambda_2^2 \cdot Q e_2 = \lambda_2^2 v_2$$

$$q_2 = ?$$



$$x_{1/1}^2 = \frac{1}{125} \begin{bmatrix} 1728 & -696 \\ -729 & 2228 \\ 7728 & 210 \end{bmatrix} \cdot \begin{bmatrix} 4/5 \\ -7/5 \end{bmatrix} = \begin{bmatrix} -7605/65 \\ -48/65 \\ 25/65 \end{bmatrix}$$

$$25q_c = \frac{1}{225} \begin{bmatrix} 1728 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 96/225 \\ 128/225 \\ 140/225 \end{bmatrix}$$

$\gamma_3$ : Liegen in  $\langle \gamma_1, \gamma_2 \rangle$ -t. Jedes orthonormale endliche  $\mathbb{R}^3$ -l.u.



$$r_3 = \frac{1}{\sqrt{2}} r_1 \times r_2$$

$$A = Q \cap R^T = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix}$$

Platix alata (Moore-Perrone 1988) immature

$A_K = b$       ballone

line'let/  
e'ne's  
ballini'el  
ad'itil

ballone ballini  
line'let  
red'ge

in'ad'lon, l'orrett

$$Ax = b \quad \left. \begin{array}{l} 2x + 4y - 5z = 3 \\ 2x + 4y - 0z = 2,5 \\ 2x + 4y - 5z = 5 \end{array} \right\} \text{elementarzeile}$$

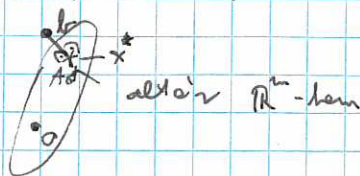
Is it a legally negotiable / compromisable?





Kipróbáljuk az írás lehetőséget leállításkor.

$$\{Ax: x \in \mathbb{R}^n\} - \text{range}(A)$$



legjobb leállítási: ahol  $Ax$  legközelebb van  $b$ -hez



$$Ax = [b \text{ a legközelebbi pontja } \text{range}(A) \text{-nek}]$$

Probléma:  $x_4$  a létező hogy egyértelmű

$$A(x_4 + u) = Ax_4 \text{ ha } Au = 0$$

$$\text{pl: } A = \begin{bmatrix} 2 & 4 & 5 \\ 2 & 4 & 5 \\ 2 & 4 & 5 \end{bmatrix} \quad u = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 10 \end{bmatrix} \text{ stb}$$

$\{x_4 + u: A(x_4 + u) = Ax_4\} - \text{Affin altér}$   
(hipersík)

Ez az ideális a legjobb közelítő leállítási.

EGYÉRTÉKŰ  
(b alapján)

Megoldás a legjobb?

$A$  legközelebbi az origóhoz.

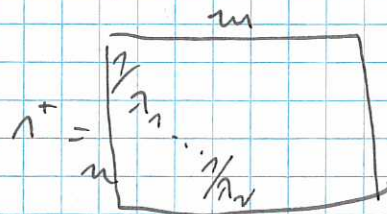
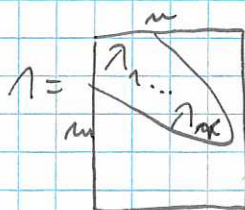
( $A$  normája az origóhoz a legközelebbi)



$$v \mapsto A^+(v) = \begin{bmatrix} x_1(b_1 \dots b_m) \\ \vdots \\ x_m(b_1 \dots b_m) \end{bmatrix}$$

Teil 1:  $v \mapsto A^+(v)$  linearisierbar, a

$\Rightarrow A^+ = R \Lambda^+ Q^T$  da  $A = Q \Lambda R^T$  SVD-Zerlegung



Mit: Ohne Koordinaten mit, nehme  $m$  Vektoren  $\mathbb{R}^m$ -ten

$v_1 \dots v_m$  a Orthonormal,  $\mathbb{R}^m$ -ten  $v_1 \dots v_m$  Orthonormal.

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_m \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{bmatrix} \Rightarrow$$

$$\tilde{x}_1 = \frac{1}{\lambda_1} \tilde{b}_1$$

$$\tilde{x}_2 = \frac{1}{\lambda_2} \tilde{b}_2$$

$$\tilde{x}_m = \frac{1}{\lambda_m} \tilde{b}_m$$

$$\tilde{x}_{m+1} = \text{etw.}$$

$\Downarrow$

gleich 0-l

Beispiel: 1, da  $A$  invertierbar, aber  $A^+ = A^{-1}$

2,  $A \rightarrow SVD \rightarrow A^+$  (einfach)

$$A = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

$r = \text{rang}(A) \leq m$  ( $r$  linear unabhängige rows)

$$A = \begin{bmatrix} \times \\ \times \\ \times \end{bmatrix}$$

$r = \text{rang}(A) \leq n$  ( $r$  linear unabhängige columns)

$$a) A^+ = A^T (A A^T)^{-1}$$

$$b) A^+ = (A^T A)^{-1} A^T$$



$$a) \quad AA^T = \boxed{r \times r} \quad (AA^T)^{-1} \text{ nur}$$

1VV  
2m. RT

$$A^+ = A^T (AA^T)^{-1}$$

$$b) \quad A^T A^+ = \boxed{r \times r} \quad (A^T A)^{-1} A^T = A^+$$

$$\text{hier: } r = \text{rank}(A) \leq n$$

$$A = Q \cancel{R}^T$$

$$AA^T = Q \cancel{R}^T \widetilde{R} \cancel{R}^T Q^T = Q \cancel{R}^T Q^T$$

$$(AA^T)^{-1} = Q \cancel{R}^T Q^T$$

$$\cancel{(AA^T)^{-1} A^T} = Q \cancel{R}^T Q^T Q \cancel{R}^T \widetilde{R} \cancel{R}^T Q^T = Q \cancel{R}^T Q^T$$

$$A^T (AA^T)^{-1} = R^* \cancel{R}^T Q^T Q \cancel{R}^T Q^T = \cancel{R}^T Q^T Q^T$$

$$= R A^+ Q^T$$

$$4) \quad \text{Ableiten zeigen, also } (AB)^+ = \overset{B^+ A^+}{\cancel{A^+ B^+}}, \text{ richtig } (A B)^{-1} = B^{-1} A^{-1}$$

$$5) \quad (\tilde{r}_1 \tilde{r}_2)^+ = \tilde{r}_2^+ \tilde{r}_1^+ \quad \text{da } \tilde{r}_1 = \begin{bmatrix} r \\ 0 \end{bmatrix} \quad \text{rank}(\tilde{r}_1) = r \quad \text{so } \tilde{r}_2 = \begin{bmatrix} m \\ 0 \end{bmatrix} \quad \text{rank}(\tilde{r}_2) = r$$



Te'fel:  $A$ :  $n \times n$ -es mátrix  $\rightarrow$  ártípusú mátrix  $\mathbb{K}$ -n  $\rightarrow$   $\mathbb{K}$ -n felbontás (pl.  $\mathbb{Z}_p$ -n felbontás)

Frobenius  
felbontás

$$r := \text{rank}(A)$$

$$\exists F_1, F_2, \text{ invertálható, hogy } A = F_1 F_2$$

példa Frobenius felbontásra

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

$\Delta$  mátrix  $r = \text{rank}(A)$  lineáris független sor.

$\downarrow$

(Gauss-jordan eljárással)

$$A_1 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- 1. sor marad
- 2. sor  $- 2 \cdot 1$  sor
- 3. sor  $- 1 \cdot 1$  sor
- 4. sor  $- 1 \cdot 1$  sor

$$A_2 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- 1. sor
- 2. sor  $- 2 \cdot 1$  sor
- 3. sor  $- 2 \cdot 1$  sor
- 4. sor

$0-2$

lineáris független

$$F_2 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}$$

$$F_1 F_2 = A$$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

$$F_1 F_2 = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}$$

$$F_1 F_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 5 \end{bmatrix} = A$$

$$F_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A^+ = F_1^+ F_2^+$$



# Altalános invers

szinguláris szétbontás  $\rightarrow$  általános invers

$$A = \begin{matrix} m \\ \square \end{matrix} \quad A^+ = \begin{matrix} n \\ \square \end{matrix}$$

Augmentáció

$$A = Q \Lambda R^T$$

$\begin{matrix} m \\ \square \end{matrix} \quad \begin{matrix} n \\ \square \end{matrix} \quad \begin{matrix} m \\ \square \end{matrix}$

$\begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{matrix}$

$$A^+ = R \Lambda^+ Q^T$$

$\Lambda^+ = \begin{matrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{matrix}$

$$A \xrightarrow{\text{SVD}} Q \Lambda R^T \rightarrow R \Lambda^+ Q^T = A^+$$

hülye felbontás

$\mathbb{R}^N$ -en

$\mathbb{C}^N$ -en

(T leggyakrabban \*  
↑  
transponált  
szimmetrikus)

ALG -ül  $\mathbb{R}^N$ -en

$A^+$  inverz tulajdonságai  $\mathbb{R}^n \sim \mathbb{C}^n$ -en

$$(A \text{ inv} \Leftrightarrow AA^+ = A^+A = I)$$

$\mathbb{D}$

$$A \text{ } n \times n, \det A \neq 0$$

$$A^{-1} A A^{-1} = A^{-1}$$

$$A A^{-1} A = A$$

} Jordan normális

$$A^+ A A^+ = A^+$$

$$A A^+ A = A$$

legjobb koordináták választása:

$$X A Y^{-1} \rightarrow \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \sigma \end{bmatrix} = \tilde{A}$$

$$A^+ = Y \tilde{A}^+ X^{-1}$$

$$X = \begin{bmatrix} x_1 & \dots & x_n \\ \vdots & & \vdots \end{bmatrix} \quad Y = \begin{bmatrix} y_1 & \dots & y_m \\ \vdots & & \vdots \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sigma} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma} \end{bmatrix} \rightarrow \text{mátrix}$$



# Frobenius - algebra

$$A = F_1 F_2$$

$$A^+ = F_2^+ F_1^+$$

$$A = \begin{matrix} m \\ \boxed{\phantom{0}} \end{matrix}$$

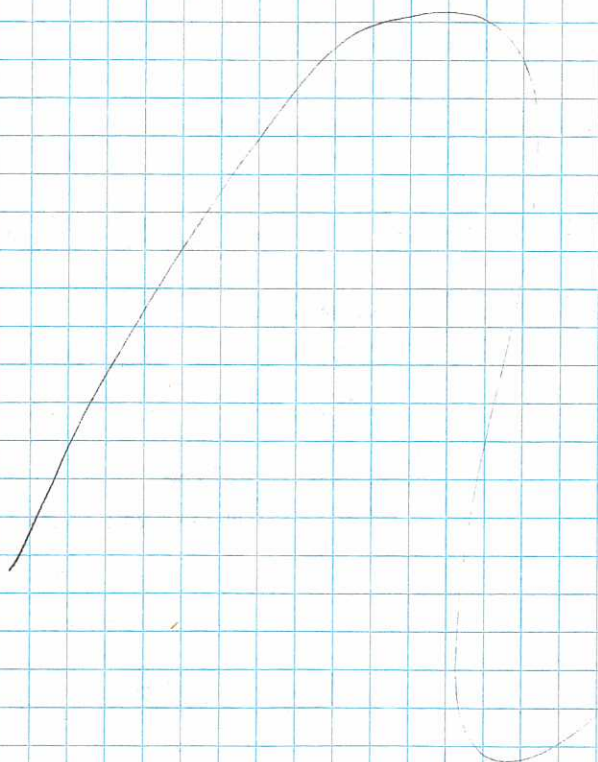
$$F_1 = \begin{matrix} n \\ \boxed{\phantom{0}} \end{matrix} \begin{matrix} m \\ \phantom{0} \end{matrix}$$

$$F_2 = \begin{matrix} m \\ \boxed{\phantom{0}} \end{matrix} \begin{matrix} r \\ \phantom{0} \end{matrix}$$

$$F_1^+ = (F_1^T F_1)^{-1} F_1^T$$

$$F_2^+ = F_2^T (F_1 F_2^T)^{-1}$$

Q: Wie sieht man sich  $A^+$ . (Rationalisierbarkeit ist nicht gegeben)





## Vektoranalysis

### Feldformeln

Alte Formeln:  $\mathbb{R}^N$ -de  $k$ -dim  $C^1$  lokal ( $k \leq N$ )

Integrale's  $C^1$  lokal

"Anquadrat"

Def:  $Q: [a_1, b_1] \times \dots \times [a_k, b_k] \rightarrow \mathbb{R}^N$

Skalar

reizes interval

$$Q(\tau_1, \tau_2, \dots, \tau_k) = \begin{bmatrix} Q_1(\tau_1, \dots, \tau_k) \\ \vdots \\ Q_k(\tau_1, \dots, \tau_k) \end{bmatrix} \quad \begin{matrix} a_1 \leq \tau_1 \leq b_1 \\ \vdots \\ a_k \leq \tau_k \leq b_k \end{matrix}$$

$$Q_1, \dots, Q_k \xrightarrow{\text{1st differentiable}} (C^1\text{-loc}) \quad (C^1\text{-loc})$$

pl:

$$k=2, N=3 \quad (2D \rightarrow 3D \text{ Felder})$$

$$Q_1(\tau_1, \tau_2) = \tau_1^2 \quad Q_2(\tau_1, \tau_2) = \tau_2^2 \quad Q_3(\tau_1, \tau_2) = \tau_1 + \tau_2$$

$$(a_1, b_1) = (a_2, b_2) = [0, 1]$$

Parameter

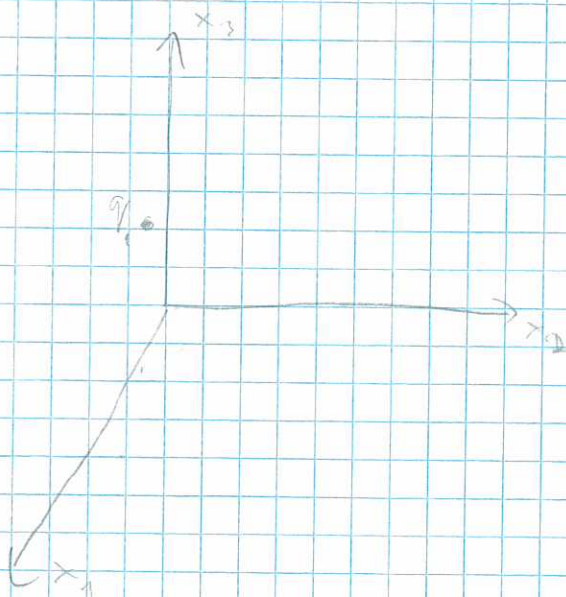
$$\tau_1 = 0.0, 0.1, 0.2, \dots, 0.9, 1.0$$

$$\tau_2 = 0.0, 0.1, 0.2, \dots, 0.9, 1.0$$

11x11 = 121 unter part

$$q_{ij} = Q(\tau_i, \tau_j)$$

$$q_{2,7} = Q(0.1, 0.2) = \begin{bmatrix} 0.01 \\ 0.04 \\ 0.3 \end{bmatrix}$$





$$x(q_{ij}) \quad (q_{ij} \times \text{Bewertung})$$

$$x(q_{ij}) = 0,01 \quad \tau_i = \sqrt{0,01} = 0,1 = \sqrt{x(q_{ij})}$$

$$y(q_{ij}) = 0,04 \quad \tau_j = \sqrt{0,04} = 0,2 = \sqrt{y(q_{ij})}$$

$$z(q_{ij}) = 0,1 = 0,1 + 0,2 = \tau_i + \tau_j = \sqrt{x(q_{ij})} + \sqrt{y(q_{ij})}$$

$$z(q) = \sqrt{x(q)} + \sqrt{y(q)}$$

$$z = \sqrt{x} + \sqrt{y}$$

H7: linearisierbar

Def:  $Q$  injektiv, da  $(\tau_1, \dots, \tau_n) \neq (\tau'_1, \dots, \tau'_n)$ , aber  $Q(\tau_1, \dots, \tau_n) = Q(\tau'_1, \dots, \tau'_n)$

Algebraisch

$$\mathbb{R}^V\text{-en: } x_1, x_2, \dots, x_n \mid x_i \in \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \rightarrow \xi_i \quad \text{pl: } x_2 \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} \rightarrow 2$$

$$x_4 \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} \rightarrow 6$$

Techniken also,  $\mathbb{R}^k$  fiktiv

$t_1, \dots, t_n$

$$t_j: \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \mapsto \tau_j$$

$$\text{pl: } A_2 \text{ also } Q(\tau_1, \tau_2) = \begin{bmatrix} \tau_1^2 \\ \tau_2^2 \\ \tau_1 + \tau_2 \end{bmatrix}$$

$$Q = \begin{bmatrix} t_1^2 \\ t_2^2 \\ t_1 + t_2 \end{bmatrix}$$

$$x_2(Q) = t_2^2$$

$$x_1 \circ Q = t_1^2$$

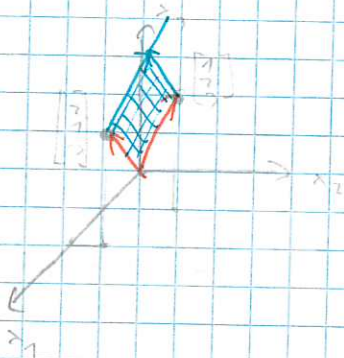


## Algorithmen

Parallelepipedum:  $[p \text{ als Punkt}, v_1, \dots, v_n \text{ als Vektoren}] =$   
 $\mathbb{R}^N$ -ten  $kD$ -en

$$= \{ p + \tau_1 v_1 + \tau_2 v_2 + \dots + \tau_n v_n : 0 \leq \tau_1, \tau_2, \dots, \tau_n \leq 1 \}$$

Bz:  $N=3, k=2, p = [\text{origi}] \quad v: \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$



Algorithmen:  $k$ -dim.  $\mathbb{R}^N$ -ten

Häufig parallelepipedale Vektoren  $v_1, \dots, v_n$  linear, es  
 kann sein, dass sie linear unabhängig sind: (Vektoren)  $v_1, \dots, v_n$   
 müssen nicht linear unabhängig sein

Matrizenproblem:

günstig darstellen:  $\forall A \in \mathbb{R}^{n \times n}, \exists V = V(A)$  long  $V$   $\mathbb{R}^n$ -ten  
 orthogonal

NEUMANN's:  $G = G_1 \cup G_2 \cup G_3$

$\uparrow$   
 gibt  $\mathbb{R}^n$ -ten

$G_1, G_2, G_3$  orthogonal

alternativ:  $\exists G_1, G_2, G_3$  orthogonal  $G_1, G_2, G_3$ -ten

$$G_1 \cup G_2 \cup G_3 = [\text{negativ} \text{ gibt}]$$

günstig darstellen es gibt,  
 auch für Vektoren  
 im  $\mathbb{R}^n$ -ten



~~Handout~~

Handout: Teilmenge eines  $n$ -dim

$$A \subset \mathbb{R}^n \quad \text{mit } \bigcup_{i=1}^{\infty} A_i \quad \delta > 0 \text{ (festw.)} / \delta = 0,1; 0,01; 0,001; 0,0001 \dots$$

$A$  ist bedeckt durch  $\delta$  aus  $\mathbb{R}^n$  beliebig kleinen Lebstücken

$$\text{wobei } B = \{B_1, \dots, B_N, \dots\} \text{ bedeckt } \text{Vol}_{n,\delta}(A) = \sum_{i=1}^{\infty} (\text{diam } B_i)^n$$

$$\text{Vol}_{n,\delta}(A) = \inf_{\delta} \text{Vol}_{n,\delta}(A)$$

$$\overline{\text{Vol}}_n(A) = \lim_{\delta \rightarrow 0} \text{Vol}_{n,\delta}(A) \cdot \mu_n \quad (\text{Längen Vorzeichen})$$

$$\mu_n \text{ definition: } \mu_n \overline{\text{Vol}}_n([0,1]^n \times \mathbb{R}^{n-k}) = 1$$

$$A \subset \mathbb{R}^n \quad k \in \mathbb{R} \quad (\text{edlig } k \leq n \text{ gegen } n(A))$$

$$\text{Vol}_{n,\delta}(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } B_i)^k : B_1 \cup B_2 \cup \dots \supset A \right\}$$

$$\overline{\text{Vol}}_k(A) = \lim_{\delta \rightarrow 0} \text{Vol}_{n,\delta}(A)$$

Wichtig!

$$1, A' \text{ spiegelt } A \text{ an } (\exists T: A \rightarrow A' \quad d(T(p), T(q)) = d(p, q) \quad \forall p, q \in \mathbb{R}^n)$$

$$\overline{\text{Vol}}_k(A') = \overline{\text{Vol}}_k(A)$$

$$2, \lambda A = \{ \lambda a : a \in A \}$$

$$\overline{\text{Vol}}_k(\lambda A) = \lambda^k \overline{\text{Vol}}_k(A)$$

3, Abbildung total

$$T: A \rightarrow \mathbb{R}^n$$

$$[d(TA', T\emptyset) \leq A(h, \forall a, b \in A)]$$

$$\Rightarrow \overline{\text{Vol}}_k(TA) \leq \overline{\text{Vol}}_k(A)$$



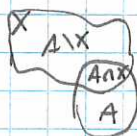
Kérdés: Melyik  $A \subset \mathbb{R}^N$  reális értéket adhat?

Kelgondar:

$$\left( A \text{ reális} \Rightarrow \begin{array}{l} \text{Vál}_k(0) = \text{Vál}_k(A) \\ A \text{ definiál } A \in \end{array} \right)$$

$A \subset \mathbb{R}^N$  reális, ha idegen egy  $x \in \mathbb{R}^N$  -re

$$\text{Vál}_k(x) = \text{Vál}_k(A \cap x) + \text{Vál}_k(x \setminus A)$$



Tétel: 1)  $A$  zárt halmaz  $\Rightarrow \text{Vál}_k$  reális értékű függvény

$$2) A = \bigcup_{n=1}^{\infty} A_n \quad A_1, A_2, \dots \quad \text{Vál}_k \text{ reális} \Rightarrow A \text{ reális}$$

$$\text{és } \text{Vál}_k(A) = \sum_{n=1}^{\infty} \text{Vál}_k(A_n)$$

Kelgondar: 1)  $A$  zárt halmaz  
reális, egyértelmű

2) Ha  $A$  és  $B$  reálisak  $\Rightarrow A \cup B$  is reális

egyszerűsítés

$$k=N \quad \text{Vál}_k([0,1]^N)$$

Ha  $N=2$



$$\text{Terület } \frac{4\pi}{4} = \pi$$

$A$  halmazok reálisak:  $\pi, \frac{1}{2}$  stb. reálisak

$$\text{Terület } \frac{4\pi}{4} = \pi$$

$$\sum_{n=1}^{\infty} \left( \frac{4\pi}{4} \right)^2 = 1 \quad \text{és a másodikkal}$$

$$\text{Vál}_k([0,1]^2) = \sum_{n=1}^{\infty} \left( \frac{4\pi}{4} \right)^2 = \frac{8}{4\pi}$$



$$\overline{\text{Vol}}_1 [0,1] = \sum_{n=1}^{\infty} \delta_n^1 = \frac{1}{\frac{\sqrt{12}}{2} \cdot 8} \sum_{n=1}^{\infty} \frac{4n \delta_n^1}{8} = \frac{8}{\sqrt{12}}$$

Alternativ is Volumen:  $\overline{\text{Vol}}_V [0,1]^V = \frac{2^V}{\omega_V}$   $\omega_V \leftarrow$  Volumen des  $V$ -Einheitswürfels

pl:  $V=2$   $\overline{\text{Vol}}_2 [0,1]^2 = \frac{2^2}{\omega_2} = \frac{4}{\pi}$

$\omega_2 =$  Kreisfläche

$$\omega_V = \pi^{V/2} \cdot \Gamma\left(1 + \frac{V}{2}\right)$$

$$\omega_2 = \pi^{2/2} \cdot \Gamma\left(1 + \frac{2}{2}\right) = \pi \cdot 1!$$

$$\Gamma(n) = (n-1)! \\ \uparrow \\ \text{Faktoriel}$$

Def: Normiert  $k$ -dimensional Hausdorff-Maß:  $\overline{\text{Vol}}_k(A)$

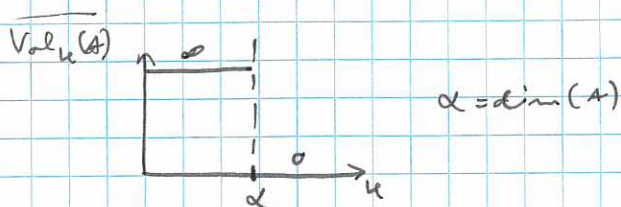
$$\text{Vol}_k(A) = \frac{2^k}{\omega_k} \cdot \overline{\text{Vol}}_k(A)$$

$$\text{Vol}_k [0,1]^k = 1, \text{ da } k\text{-Einheitswürfel}$$

$$\omega_k = \pi^{k/2} \Gamma\left(1 + \frac{k}{2}\right)$$

### Abhängigkeit

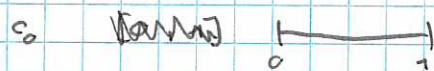
$$\left. \begin{array}{ll} \text{I} & \text{Ist } \text{Vol}_k(A) = 0 \text{ oder } \text{Vol}_k(A) = \infty, \lambda > k \text{ existiert} \\ \text{II} & \text{Ist } \text{Vol}_k(A) < \infty \text{ oder } \text{Vol}_k(A) = \infty, \lambda < k \text{ existiert} \\ \text{III} & \text{Ist } \text{Vol}_k(A) = 0 \text{ oder } \text{Vol}_k(A) = \infty, \lambda = k \text{ existiert} \end{array} \right\} \overline{\text{Vol}}_k(A) \text{ ist endlich}$$





$C_n$  sein dimensionale:

Cantor halmen (Ösperale)



$C_{n+1} = C_n$  intervalle mit halber Länge entfernt

$$C = \bigcap_{n=1}^{\infty} C_n \quad (\neq \emptyset)$$

$$\dim C = \lim_{n \rightarrow \infty} \frac{\ln 2}{\ln 3}$$

Zu folgend:

- 1/ LU faktorisieren,  $3 \times 3$ ,  $\mathbb{Z}_5$  faktorisieren
- 2/ QR faktorisieren,  $3 \times 3$
- 3/ Faltungstransformation,  $2 \times 2$  Matrix,  $\mathbb{Z}_5$
- 4/ SVD  $3 \times 2$  faktorisieren
- 5/ Determinante berechnen,  $3 \times 3$  ( $\det=0$ )

Kugeln in einem Raum (euklidisch)

Kugeln in einem Raum (euklidisch)



Uz sag

Paralelepipedon înfuzata

$$P = \alpha \left( p + [0,1]x_1 + [0,1]x_2 + [0,1]x_k \right) \quad p, x_1, \dots, x_k \in \mathbb{R}^N \quad k \leq N$$

$$\text{Val}_v(P) = ?$$

1,  $Val_N(p) = Val_N([0,1]r_1 + \dots + [0,1]r_m)$  ( $p = \text{avg}$ , also eine Verteilung)

$$V = \begin{bmatrix} r_1 & r_2 & \dots & r_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{array}{c} N \\ K \end{array}$$

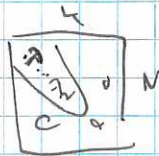
liquid

Orthogonales Transferlernen = alternat. Transferarten.

*Onychomys leucogaster* is a terrologist.

$$\{V\} \Rightarrow V = Q \Lambda R^T$$

originally  
is not



$$G \quad n \times n$$

$R^{L \times K}$

$$\text{Vol}_K(Q^T V R) = \text{Vol}_K(V)$$

$$\text{Val}_K(A) = \lambda_1 \cdot \dots \cdot \lambda_k$$

Ozan kartını alabilir  
eğer, her onun ego  
teğlatatır ve tınızın ve  
alabilir mi?

Valapji- $\lambda$ -l line'itara?

Cn



$$V^T V = R^T Q^T Q R = R^T I R = R \begin{matrix} \nearrow \text{diag}(\lambda_1, \dots, \lambda_n) \\ \boxed{\begin{smallmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{smallmatrix}} R^T$$

$$\det(V^T V) = \det R \boxed{\begin{smallmatrix} \lambda_1^2 \\ \vdots \\ \lambda_n^2 \end{smallmatrix}} R^T = \det(R) \cdot \det(\text{diag}(\lambda_1^2, \dots, \lambda_n^2)) \cdot \det(R^T) =$$

$$= (\pm 1) \lambda_1^2 \cdot \dots \cdot \lambda_n^2 \cdot (\pm 1) = \lambda_1^2 \cdot \lambda_2^2 \cdot \dots \cdot \lambda_n^2$$

(R-orthogonal)

$$\Downarrow$$

$$\text{Vol}_k(p) = \sqrt{\det(V^T V)}$$

pl:  $\text{Vol}_2\left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ parallelogram} \right) = \left[ \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \right]^{1/2} = \left[ \det \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 7 \\ 3 & 7 & 18 \end{pmatrix} \right]^{1/2} = \sqrt{16 - 7 + 32} = \sqrt{54}$

— — — — —

$$r_1 \times r_2 = \text{Vol}_2[r_1, r_2 \text{ parallelogram}] \cdot e$$

$\uparrow$   $\text{grünlich}$   
 $\downarrow$   $\text{grünlich}$

$$\text{Vol}_2(p) = \|r_1 \times r_2\| = \left[ \det \begin{pmatrix} y(r_1) & y(r_2) \\ z(r_1) & z(r_2) \end{pmatrix}^2 + \det \begin{pmatrix} x(r_1) & x(r_2) \\ z(r_1) & z(r_2) \end{pmatrix}^2 + \det \begin{pmatrix} x(r_1) & y(r_1) \\ z(r_1) & z(r_2) \end{pmatrix}^2 \right]^{1/2}$$

$\uparrow$   
 $\times \text{ang}$

$$= \left[ \det \begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix}^2 + \det \begin{pmatrix} 1 & 6 \\ 1 & 4 \end{pmatrix}^2 + \det \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix}^2 \right]^{1/2} = \sqrt{9 + 36 + 9} = \sqrt{54}$$





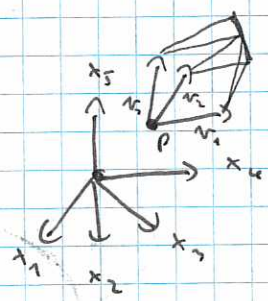
$$P = p + [0, 1] \begin{matrix} \text{Intervallum} \\ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{n_1} + [0, 1] \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}}_{n_2} + [0, 1] \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{n_3}$$

Vol, P = ?

$P \subset \mathbb{R}$

reelles  $\mathbb{R}^5$ -LSE

P 3dim parallelepipedum  $\mathbb{R}^5$ -LSE



$$|Vol_n P| = \begin{cases} 0 & \text{in } K^2 \\ \infty & \text{in } K^3 \end{cases}$$

...

$$\sqrt{\det(V^T V)}$$

$$V = \begin{bmatrix} \vdots & \vdots & \vdots \\ n_1 & n_2 & n_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix} E = \begin{bmatrix} [1, 1] \\ [1, 2] \\ [2, 3] \end{bmatrix} \cdot \begin{bmatrix} n_1 & n_2 & n_3 \\ \vdots & \vdots & \vdots \end{bmatrix} =$$

Intervallum

$$\rightarrow \underbrace{V^T V}_{3 \times 3} = \begin{bmatrix} \langle n_1 | n_1 \rangle & \langle n_1 | n_2 \rangle & \langle n_1 | n_3 \rangle \\ \langle n_2 | n_1 \rangle & \langle n_2 | n_2 \rangle & \langle n_2 | n_3 \rangle \\ \langle n_3 | n_1 \rangle & \langle n_3 | n_2 \rangle & \langle n_3 | n_3 \rangle \end{bmatrix} = \begin{bmatrix} 11 & 10 & 3 \\ 10 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\underbrace{\det(V^T V)}_{\geq 0} = \det \begin{bmatrix} 1 & -5 & 0 \\ 10 & 5 & 1 \\ -17 & -5 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -5 & 0 \\ 0 & 0 & 1 \\ -17 & -5 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 \\ 10 & 5 & 1 \\ -17 & -15 & 0 \end{bmatrix} \\ = \det \begin{bmatrix} 1 & -5 & 0 \\ 0 & 55 & 1 \\ -17 & -5 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & -5 & 0 \\ 0 & 55 & 1 \\ 0 & -94 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 55 \\ 0 & 0 & -94 \end{bmatrix} = 94$$

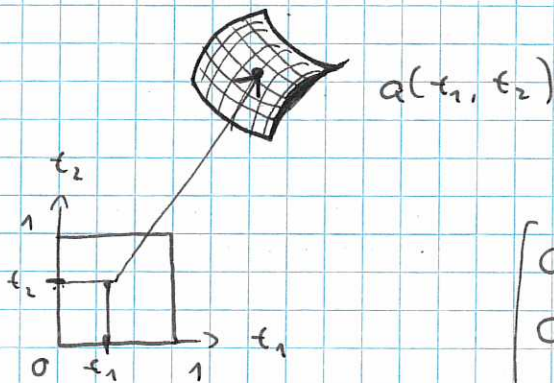
Gauss-Jordan: spart 2 Zeilen, 2 Spalten



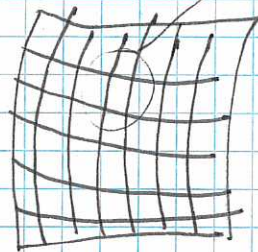
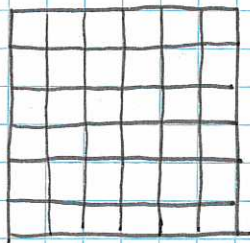
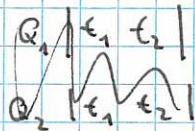
Felvin formula

$C^1$ -surface [2-dim  $C^1$ -surface  $\mathbb{R}^n$ -dim]

$$W=3 \quad k=2$$



$$\begin{bmatrix} Q_1 | t_1, t_2 \\ Q_2 | t_1, t_2 \\ Q_3 | t_1, t_2 \end{bmatrix} = Q(t_1, t_2)$$



$Q[0,1]^2 \rightarrow 2$  dimensions surface over  $\mathbb{R}^3$ -dim

$Q[0,1]^2 \rightarrow \mathbb{R}^3$   $C^1$ -surface,  $Q_1, Q_2, Q_3$  fglg. diff

injektiv  $Q(\tau_1, \tau_2) \neq Q(\sigma_1, \sigma_2) \Leftrightarrow$

$Q(\tau_1, \tau_2) \neq Q(\sigma_1, \sigma_2)$

Vol<sub>k</sub>

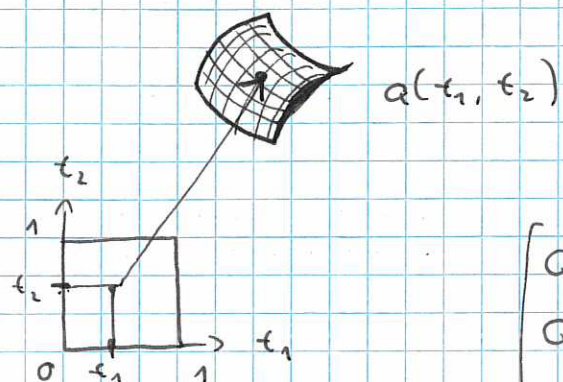
$Vol_k(Q[0,1]^k)$



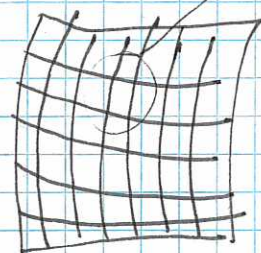
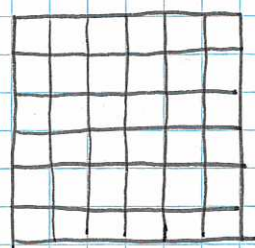
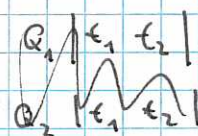
# Felvin formula

$C^1$ -locha [2-dim  $C^1$ -locha  $\mathbb{R}^n$ -lin]

$W=3$   $k=2$



$$\begin{bmatrix} Q_1 | t_1, t_2 \\ Q_2 | t_1, t_2 \\ Q_3 | t_1, t_2 \end{bmatrix} = Q(t_1, t_2)$$



$Q[0,1]^2 \rightarrow 2$  dimensijs felület domén  $\mathbb{R}^3$ -ben

$Q[0,1]^2 \rightarrow \mathbb{R}^3$   $C^1$ -reim,  $Q_1, Q_2, Q_3$  folyt. diff

injektiv

$$Q(t_1, t_2) \neq Q(s_1, s_2) \Leftrightarrow$$

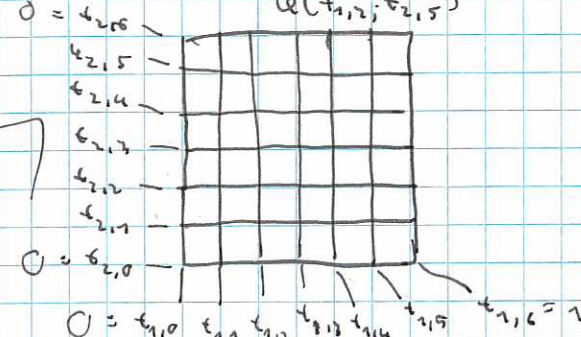
$$Q(t_1, t_2) \neq Q(s_1, s_2)$$

$Vol_k$

$$Vol_k(Q[0,1]^k)$$

Euler-pikkfogel  $Q(t_{1,1}, t_{2,1})$  görbe, végül labette mellek alakot

$$P_{2,4} = Q(t_{1,2}, t_{2,4}) + [0,1] \quad [diff. rövidítés] \quad Q(t_{1,4}, t_{2,4}) - Q(t_{1,2}, t_{2,4}) - ne$$



+ (diff. rövidítés)

Taylor-rov 1. tag

$$[0,1] Q(t_{1,1}, t_{2,1}) - Q(t_{1,1}, t_{2,1})$$

ed, id legk

Alt.

$$P_{i_1, i_2} = Q(t_{1, i_1}, \dots, t_{n, i_n}) + \sum_{d=1}^k [0,1] \cdot [diff. Q(t_{1, i_1}, \dots, t_{n, i_n}) - Q(t_{1, i_1}, \dots, t_{n, i_n})]$$

$$[diff. Q(t_{1, i_1}, \dots, t_{n, i_n}) - Q(t_{1, i_1}, \dots, t_{n, i_n})] =$$

$$= \frac{\partial Q}{\partial t_d} (t_{1, i_1}, \dots, t_{n, i_n}) \cdot (t_{d, i_d+1}, \dots, t_{d, i_d}) =$$

$$= \frac{\partial Q}{\partial t_d} (t_{1, i_1}, \dots, t_{n, i_n}) \Delta_{i_d} t_d \quad (rövidítés)$$

$$Vol_k(P_{i_1, i_2}) = \sum_{i_1, \dots, i_n} Vol_k \left( \frac{\partial Q}{\partial t_1} \cdot \Delta_{i_1} t_1, \dots, \frac{\partial Q}{\partial t_n} \cdot \Delta_{i_n} t_n \right) \text{ helyi paralella pipak}$$

$$Vol_k(P_{i_1, i_2}) = \frac{\partial Q}{\partial t_1} \Big|_{t_{1, i_1}} \Delta_{i_1} t_1, \frac{\partial Q}{\partial t_2} \Big|_{t_{2, i_2}} \Delta_{i_2} t_2, \dots, \frac{\partial Q}{\partial t_n} \Big|_{t_{n, i_n}} \Delta_{i_n} t_n$$

$\uparrow$   
N. alapvet



$$\text{Vol}_k(i_1, \dots, i_k)\text{-differential} = \sqrt{\det(V_{i_1, \dots, i_k}^T V_{i_1, \dots, i_k})} =$$

$$= \sqrt{\det \left( \frac{\partial \mathbf{q}}{\partial t_1}, \dots, \frac{\partial \mathbf{q}}{\partial t_k} \right)^T \left( \frac{\partial \mathbf{q}}{\partial t_1}, \dots, \frac{\partial \mathbf{q}}{\partial t_k} \right)} \Bigg|_{t_{i_1}, \dots, t_{i_k} \text{ fest}} \Delta t_{i_1} \dots \Delta t_{i_k}$$

→ *simult. d.h.*

$$\int_{t_1=0}^1 \int_{t_2=0}^1 \dots \int_{t_k=0}^1 \sqrt{\det \left( \frac{\partial \mathbf{q}}{\partial t} \right)^T \left( \frac{\partial \mathbf{q}}{\partial t} \right)} dt_1 \dots dt_k$$

$Q' - Q$  deriviert matrix

Textel (eigen formula)

$$\text{Vol}_k(Q[0,1]^k) = \int_{t_1=0}^1 \dots \int_{t_k=0}^1 \sqrt{\det(Q'^T \cdot Q')} dt_1 \dots dt_k$$

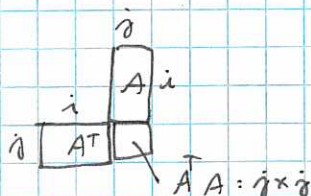
$$\text{RE } Q(t_1, t_2) = \begin{bmatrix} t_1 \cos(2\pi t_2) \\ t_1 \sin(2\pi t_2) \\ t_1^2 \end{bmatrix} \quad (t_1, t_2) \in [0,1]^2$$

$$Q' = \frac{dQ}{dt} = \begin{bmatrix} \cos(2\pi t_2) & -2\pi t_1 \sin(2\pi t_2) \\ \sin(2\pi t_2) & 2\pi t_1 \cos(2\pi t_2) \\ 2t_1 & 0 \end{bmatrix}$$

$$A \in \mathbb{R}^{i \times j} \quad i \times j$$

$$A^T A = \left[ \langle \text{row } l \mid \text{row } l \rangle \right]_{l,l=1 \dots j}$$

$$A A^T = \left[ \langle \text{col } l \mid \text{col } l \rangle \right]_{l,l=1 \dots i}$$



*Ähnlichkeit*





$$[0, 1]^2$$

$$[t_1, t_2] \mapsto \begin{bmatrix} t_1 \cos 2\pi t_2 \\ t_1 \sin 2\pi t_2 \\ t_1^2 \end{bmatrix} = Q(t_1, t_2) = \begin{bmatrix} Q_1(t_1, t_2) \\ Q_2(t_1, t_2) \\ Q_3(t_1, t_2) \end{bmatrix}$$

$$\text{Vol}_2 Q [0, 1]^2 = \int_{t \in [0, 1]^2} [\det Q'^T Q']^{1/2} dt = *$$

$t \in [0, 1]^2$   
 $\uparrow$   
 $(t_1, t_2)$

$$Q' = \begin{matrix} Q_1 \\ Q_2 \\ Q_3 \end{matrix} \begin{bmatrix} \frac{dQ_1}{dt_1} & \frac{\partial Q_1}{\partial t_2} \\ \frac{dQ_2}{dt_1} & \frac{\partial Q_2}{\partial t_2} \\ \frac{dQ_3}{dt_1} & \frac{\partial Q_3}{\partial t_2} \end{bmatrix} = \frac{\partial Q}{\partial T} = \begin{bmatrix} \cos 2\pi t_2 & -2\pi t_1 \sin 2\pi t_2 \\ \sin 2\pi t_2 & 2\pi t_1 \cos 2\pi t_2 \\ 2t_1 & 0 \end{bmatrix}$$

$\uparrow$   $\uparrow$   
Vektor-igliche  $\partial/\partial T$   $\partial/\partial t_2$

$$Q'^T Q'$$

\*

$$\int_{t_1=0}^1 \int_{t_2=0}^1 [\det \text{diag}(1+t_1^2, 4\pi^2 t_1^2)]^{1/2} dt_2 dt_1 = *$$

$$Q'^T Q' = \begin{bmatrix} 1+t_1^2 & 0 \\ 0 & 4\pi^2 t_1^2 \end{bmatrix}$$

M

$$* \int_{t_1=0}^1 \int_{t_2=0}^1 \sqrt{(1+t_1^2)(4\pi^2 t_1^2)} dt_2 dt_1 = 2\pi \int_{t_1=0}^1 \int_{t_2=0}^1 \sqrt{1+t_1^2} dt_2 dt_1 =$$

$$= 2\pi \int_{t_1=0}^1 \sqrt{1+t_1^2} dt_1 =$$





Wegintegralen in der Ebene:

$$\int x^a dx = \frac{x^{a+1}}{a+1}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\begin{aligned} u &= 1 + 4t_1^2 \\ \frac{du}{dt_1} &= 8t_1 \end{aligned} \quad \left\{ \begin{aligned} du &= 8t_1 dt_1 \end{aligned} \right.$$

$$\begin{aligned} 2\pi \int_{t_1=0}^1 \sqrt{1+4t_1^2} t_1 dt_1 &= 2\pi \int_{u=1}^5 \frac{1}{8} du = \frac{\pi}{4} \int_1^5 u^{1/2} du = \frac{\pi}{4} \left[ \frac{2}{3} u^{3/2} \right]_1^5 \\ &= \frac{\pi}{6} \left( \frac{2}{3} \cdot 5^{3/2} - \frac{2}{3} \right) = \frac{\pi}{6} (5^{3/2} - 1) = \end{aligned}$$

$$\text{Vol}_2 Q[0,1]^2 = \frac{\pi}{6} (5^{3/2} - 1)$$



Vonwertintegral über

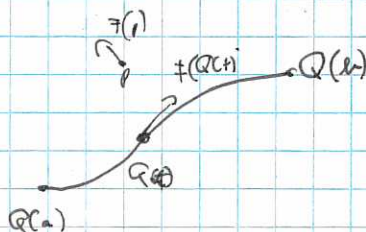
ist: so man kann

ist: ist  $t \rightarrow$  Teil der

$$\begin{aligned} & \begin{array}{c} t=b \\ \nearrow \\ t \text{ ist } \\ \nwarrow \\ t=a \end{array} \\ Q(t) &= \begin{bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{bmatrix} \in \mathbb{R}^3 \end{aligned}$$

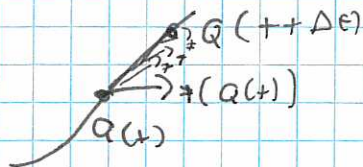
$$Q: [a, b] \rightarrow \mathbb{R}^3$$

$$F: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad \left( \text{für } p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \text{ setzen} \right)$$





Formula:



Derivata della funzione  $Q(t)$  rispetto al tempo  $t$  è  $\frac{dQ(t)}{dt}$  che è uguale a  $\frac{dQ(t)}{dt}$  all'istante  $t$

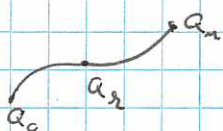
$$\Delta W = \langle \psi(Q(t)) | \frac{\partial Q}{\partial t} \Delta t \rangle \approx \langle \psi(Q(t)) | Q(t+\Delta t) - Q(t) \rangle$$

$$\psi(Q(t)) = \begin{cases} \psi_1(Q(t)) \rightarrow x_0 \psi \\ \psi_2(Q(t)) \rightarrow y_0 \psi \\ \psi_3(Q(t)) \rightarrow z_0 \psi \end{cases}$$

$$\langle \psi(Q(t)) | Q(t+\Delta t) - Q(t) \rangle = \psi_1(Q(t)) \cdot [Q_1(t+\Delta t) - Q_1(t)] + \psi_2(Q(t)) \cdot [Q_2(t+\Delta t) - Q_2(t)] + \psi_3(Q(t)) \cdot [Q_3(t+\Delta t) - Q_3(t)]$$

(x, y, z) in  $\Delta t$

$$W \approx \sum_{i=1}^n x_0 \psi(q_i) \underbrace{(z_i - z_{i-1})}_{\Delta_i z} + \sum_{i=1}^n y_0 \psi(q_i) \underbrace{(y_i - y_{i-1})}_{\Delta_i y} + \sum_{i=1}^n z_0 \psi(q_i) \underbrace{(x_i - x_{i-1})}_{\Delta_i x}$$



superficie

Derivata della funzione  $Q(t)$  rispetto al tempo  $t$  è  $\frac{dQ(t)}{dt}$  che è uguale a  $\frac{dQ(t)}{dt}$  all'istante  $t$

$$\Delta_i x \approx \frac{\partial Q}{\partial t}(q_i) \cdot \Delta_i t \rightarrow W \approx \sum_{i=1}^n x_0 \psi(q_i) \frac{\partial Q}{\partial t}(q_i) \Delta_i t + \dots$$

↓

$$W = \int_{t=a}^b x_0 \psi(q(t)) \cdot \frac{\partial Q}{\partial t} dt$$

$$W = \int_{t=a}^b \left[ x_0 \psi(q(t)) \cdot \frac{\partial Q}{\partial t} dt \right] + \int_{t=a}^b \left[ y_0 \psi(q(t)) \cdot \frac{\partial Q}{\partial t} dt \right] + \int_{t=a}^b \left[ z_0 \psi(q(t)) \cdot \frac{\partial Q}{\partial t} dt \right]$$



$1 - w(1) \leq \frac{\Delta_i x}{\frac{\partial Q}{\partial t} \Delta t} \leq 1 + w(1)$

$\rightarrow E_{\text{ext}} \rightarrow \sum_{i=0}^n \left[ x_0^i(Q(t)) \cdot \Delta_i x + y_0^i(Q(t)) \Delta_i y + z_0^i(Q(t)) \Delta_i z \right] \rightarrow$

$\rightarrow \int_{t=a}^b x_0^i(Q(t)) \frac{\partial Q}{\partial t} dt + \int_{t=a}^b y_0^i(Q(t)) \frac{\partial Q}{\partial t} dt + \dots$

Relais:  $\int_Q [(x_0^i) dx + (y_0^i) dy + (z_0^i) dz]$

$\text{la } \vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$

$\int_Q [t_1 dx + t_2 dy + t_3 dz]$

Def  $Q: [a, b] \rightarrow \mathbb{R}^N$  1-dim  $C^1$ -kurve  $t \mapsto Q(t) = \begin{bmatrix} Q_1(t) \\ \vdots \\ Q_N(t) \end{bmatrix}$

$f_1, \dots, f_N: \mathbb{R}^N \rightarrow \mathbb{R}$

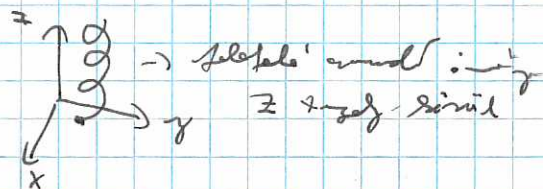
$Q_i(t) = x_i \circ Q(t)$   
 1st. diff

$\int_Q [f_1 dx_1 + f_2 dx_2 + \dots + f_N dx_N] := *$

$* = \int_{t=a}^b f_1(Q(t)) \frac{\partial Q_1}{\partial t} dt + \int_{t=a}^b f_2(Q(t)) \frac{\partial Q_2}{\partial t} dt + \dots + \int_{t=a}^b f_N(Q(t)) \frac{\partial Q_N}{\partial t} dt$

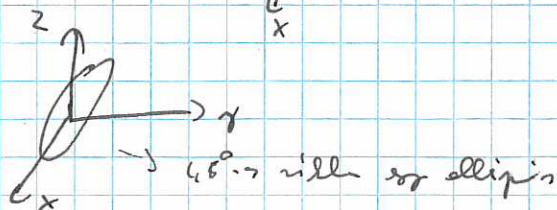


RL (3)  $\vec{r}(x, y, z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$



$$Q(t) = \begin{bmatrix} \cos t \\ \sin t \\ -\cos t \end{bmatrix}$$

$$t \in [0, 2\pi]$$



$$\int_Q [x \, dx + y \, dy + z \, dz] = ?$$

$$\int_Q [-y \, dx + x \, dy + dz] = \int_{t=0}^{2\pi} [-\sin t (-\sin t) dt + \cos t \cdot \cos t dt + \sin t dt] = *$$

$$x = Q_1(t)$$

$$dx = \frac{\partial Q_1}{\partial t} dt = \frac{\partial \cos t}{\partial t} dt = -\sin t dt$$

$$y = Q_2(t)$$

$$dy = \frac{\partial Q_2}{\partial t} dt = \frac{\partial \sin t}{\partial t} dt = \cos t dt$$

$$z = Q_3(t)$$

$$dz = \frac{\partial Q_3}{\partial t} dt = \frac{\partial -\cos t}{\partial t} dt = \sin t dt$$

$$* = \int_{t=0}^{2\pi} [\sin^2 t + \cos^2 t + \sin t] dt = \int_0^{2\pi} [1 + \sin t] dt = 2\pi + 0 = 2\pi$$



②

$$Q(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^4 \end{bmatrix} \quad t \in [-1, 1]$$

$$f_1 = \frac{x_4^2}{2}$$

$$f_2 = x_2^2$$

$$f_3 = x_3^2$$

$$f_4 = x_1^2$$

$$\int_Q [f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + f_4 dx_4] = *$$

Heskeln:

$$x_1 = 1$$

$$x_2 = t$$

$$x_3 = t^2$$

$$x_4 = t^4$$

$$dx_1 = 0 \cdot dt$$

$$dx_2 = 1 \cdot dt$$

$$dx_3 = 2t \cdot dt$$

$$dx_4 = 4t^3 \cdot dt$$

$$f_1 = t^8$$

$$f_2 = t^2$$

$$f_3 = t^2$$

$$f_4 = 1$$

$$* = \int_{-1}^1 (t^8 \cdot 0 + t^2 \cdot 1 \cdot dt + t^2 \cdot 2t \cdot dt + 4t^3 \cdot dt) = \int_{-1}^1 (t^8 + t^2 + 2t^3 + 4t^3) dt =$$

$$= \frac{2}{5}$$



$$Q: t \mapsto \begin{bmatrix} t^2 \\ t \\ t^3 \end{bmatrix}, [0,1] \rightarrow \mathbb{R}^3$$

$$f: (x,y,z) \mapsto \begin{bmatrix} x^2 y^3 \\ z^2 x^4 \\ y^5 z^6 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$x = t^2$$

$$y = t$$

$$z = t^3$$

$$dx = 2t dt$$

$$dy = dt$$

$$dz = 3t^2 dt$$

$$f_1 = t^{15}$$

$$f_2 = t^{14}$$

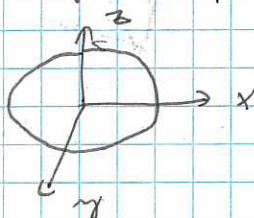
$$f_3 = t^{27}$$

$$\int_Q f_1 dx + f_2 dy + f_3 dz = \int_{t=0}^1 (2t^{16} + t^{14} + 3t^{25}) dt =$$

$$= \left[ \frac{2}{17} t^{17} + \frac{t^{15}}{15} + \frac{3}{26} t^{26} \right]_0^1 = \frac{2}{17} + \frac{1}{15} + \frac{3}{26}$$

— 0 —

Polymerat pelda:



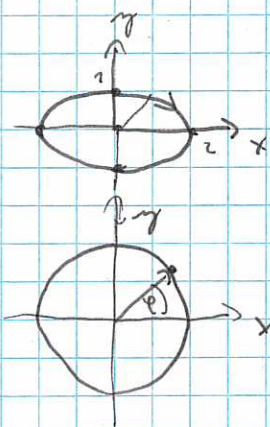
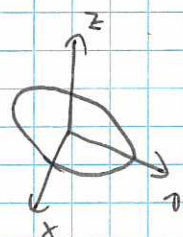
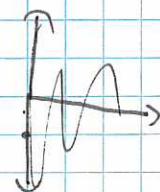
Ellipsis x-y síkban.

x tengelyen 2 a. l. van, (f. i. d. s. g. d.)

y tengelyen 1 a. l. van, (mind f. i. d. s. g. d.)

$$f(x,y,z) = \begin{bmatrix} 4z \\ 3x \\ x^2 y^2 \end{bmatrix}$$

A zárt A @ O f. i. d. s. g. d.?



$$Q: \begin{bmatrix} \sin u \\ \cos u \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \sin u \\ \frac{1}{2} \cos u \\ 0 \end{bmatrix}, u \in [0, 2\pi]$$

$$x = \sin u, dx = \cos u du, f_1 = 0$$

$$y = \frac{1}{2} \cos u, dy = -\frac{1}{2} \sin u du, f_2 = 2 \sin u$$

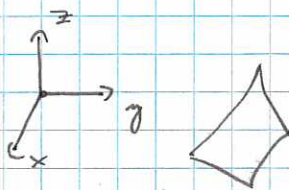
$$z = 0, dz = 0, f_3 = \sin^2 u \cos^2 u$$

$$\int_0^{2\pi} (4z dx + 2x dy + x^2 y^2 dz) = \int_{u=0}^{2\pi} (0 + 2 \sin u \cos u du + 0) = \frac{2}{2} \int_{u=0}^{2\pi} \sin^2 u du = 0$$

$$2 \sin u \cos u = \sin 2u$$



# A'afogis / Fluxus

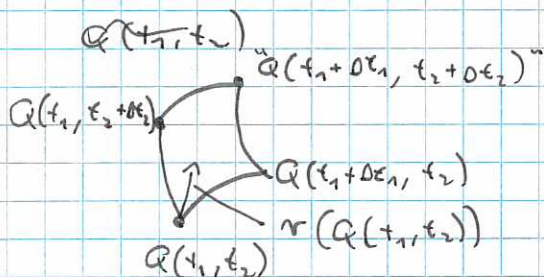


$$Q: [0,1]^2 \rightarrow \mathbb{R}^3$$

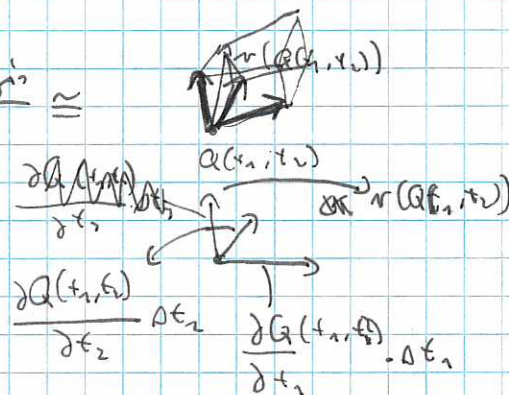
Mindein punktken  $r$  salen'gi fagade'l.

$$r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1(x,y,z) \\ r_2(x,y,z) \\ r_3(x,y,z) \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

Meji niz egg a't 1 misodpne det  $Q [0,1]^2$ -a?



$$ds: \frac{A'afogis}{\text{area}} \approx$$



Ennel a Verfagade  $\text{Vol}_3 \left[ r(Q(t_1, t_2)), \frac{\partial r(Q(t_1, t_2))}{\partial t_1} \Delta t_1, \frac{\partial r(Q(t_1, t_2))}{\partial t_2} \Delta t_2 \right]$   
 elin parallelepiped (Verfagade) =

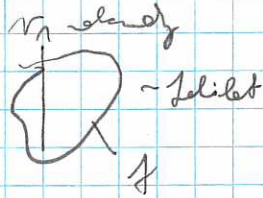
$$= r_1 \cdot \frac{\partial r_2}{\partial t_1} \Delta t_1 \cdot \frac{\partial r_2}{\partial t_2} \Delta t_2 + r_2 \cdot \frac{\partial r_2}{\partial t_1} \Delta t_1 \frac{\partial r_2}{\partial t_2} \Delta t_2 +$$

$$+ r_3 \cdot \frac{\partial r_1}{\partial t_1} \Delta t_1 \frac{\partial r_2}{\partial t_2} \Delta t_2$$



4 ritikete xyz síla

Superpozíció



$$[A' d f o g h] = \overset{x o r}{r_a} \cdot Vol_2 [P_{xy}^z(1)] + \\ y o r \cdot Vol_2 [P_{zx}(1)] + \\ z o r \cdot Vol_2 [P_{yz}(1)]$$

$$\begin{bmatrix} \partial^1 x \\ \partial^1 y \\ \partial^1 z \end{bmatrix} \rightarrow \begin{bmatrix} \phantom{x} \\ \phantom{y} \\ \phantom{z} \end{bmatrix} \xrightarrow{A^1} \begin{bmatrix} \partial^1 x \\ \partial^1 y \\ \partial^1 z \end{bmatrix} \quad z = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$$f_x [\partial^1 y \partial^2 z - \partial^2 y - \partial^1 z] + f_y [\partial^1 z \partial^2 x - \partial^2 z - \partial^1 x] + \\ + f_z [\partial^1 x \partial^2 y - \partial^2 x - \partial^1 y]$$

idei jövele = átfogó

Matr. átváltás

$$\int_{t_1, t_2=0}^1 r_x(Q(x_1, t_1)) \cdot \det \begin{bmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} \\ \frac{\partial y}{\partial t_1} & \frac{\partial y}{\partial t_2} \\ \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} \end{bmatrix} dt_1 dt_2 + \dots$$

$$\int_{t_1, t_2=0}^1 \left[ r_1 \det \begin{bmatrix} \frac{\partial Q_1}{\partial t_1} & \frac{\partial Q_1}{\partial t_2} \\ \frac{\partial Q_2}{\partial t_1} & \frac{\partial Q_2}{\partial t_2} \end{bmatrix} dt_1 dt_2 + r_2 \det \begin{bmatrix} \frac{\partial Q_1}{\partial t_1} & \frac{\partial Q_1}{\partial t_2} \\ \frac{\partial Q_2}{\partial t_1} & \frac{\partial Q_2}{\partial t_2} \end{bmatrix} dt_1 dt_2 + r_3 \det \begin{bmatrix} \frac{\partial Q_1}{\partial t_1} & \frac{\partial Q_1}{\partial t_2} \\ \frac{\partial Q_2}{\partial t_1} & \frac{\partial Q_2}{\partial t_2} \end{bmatrix} dt_1 dt_2 \right]$$

$\det \begin{vmatrix} d_1 y & d_2 y \\ d_1 z & d_2 z \end{vmatrix}$ 
 $\det \begin{vmatrix} d_1 z & d_2 z \\ d_1 x & d_2 x \end{vmatrix}$ 
 $\det \begin{vmatrix} d_1 x & d_2 x \\ d_1 y & d_2 y \end{vmatrix}$

$$x = Q_1, y = Q_2, z = Q_3$$

Nehézségek:  $dx \rightsquigarrow \frac{\partial Q_1}{\partial t_1} dt_1 + \frac{\partial Q_1}{\partial t_2} dt_2$

$dy \rightsquigarrow \frac{\partial Q_2}{\partial t_1} dt_1 + \frac{\partial Q_2}{\partial t_2} dt_2$

$dz \rightsquigarrow \frac{\partial Q_3}{\partial t_1} dt_1 + \frac{\partial Q_3}{\partial t_2} dt_2$



Hogor + 11h

$$Q: [a, 1]^N \rightarrow \mathbb{R}^N$$

$$r = \begin{bmatrix} r_1 \\ \vdots \\ r_N \end{bmatrix} : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

~~↪ r = b - r~~

Qda tipini integrallat helleal:

$$\int_Q r(x_1, \dots, x_N) \det \begin{bmatrix} d_1 x_{11} & d_2 x_{12} & \dots & d_N x_{1N} \\ d_1 x_{21} & d_2 x_{22} & \dots & d_N x_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ d_1 x_{N1} & d_2 x_{N2} & \dots & d_N x_{NN} \end{bmatrix} dx_1 \dots dx_N$$

ifad iige

Küör algebra jölölö'neinil iij jölölö'neinil

$$\det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} = \det \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{NN} \end{bmatrix} = a_{11} a_{22} \dots a_{NN}$$

↓ a<sub>ii</sub> r  
↓ Küör rüüt

$$\int_Q r(x_1, \dots, x_N) dx_{11} \wedge dx_{12} \wedge \dots \wedge dx_{1N}$$

Söw. Kä'jes cüw'ije

$$(1) a_1 \wedge a_2 \wedge \dots \wedge a_N \Leftrightarrow a_1 \wedge a_2 \wedge \dots \wedge a_{i-1} \wedge a_{i+1} \wedge \dots \wedge a_N$$

$$(2) a_1 \wedge a_2 \wedge \dots \wedge a_{i-1} \wedge a_{i+1} \wedge \dots \wedge a_N = 0 \quad (\text{al rü 2 güt})$$

Distributiv lifte's



RL

$$r = x_1^2 \cdot x_2^3 x_3^5 x_4^2 x_5^3$$

$$Q: [0,1]^3 \rightarrow \mathbb{R}^5$$

$$Q_i = x_1^{2i} x_2^i x_3^{i+1} x_4^{4i} x_5^{6i}$$

$$Q_i = t_1^{2i} t_2^i t_3^{i+1}$$

$$i = 1 \dots 5$$

$$\int_Q r \, dx_1 dx_2 dx_3$$

$$x_1 = Q_1(t_1, t_2, t_3) = t_1^2 t_2 t_3^2, \quad dx_1 = 2t_1 t_2 t_3^2 dt_1 + t_1^2 t_3^2 dt_2 + 2t_1^2 t_2 t_3 dt_3$$

$$x_2 = t_1^4 t_2^2 t_3^3, \quad dx_2 = 4t_1^3 t_2^2 t_3^3 dt_1 + 2t_1^4 t_2 t_3^3 dt_2 + 3t_1^4 t_2^2 t_3^2 dt_3$$

$$x_3 = t_1^6 t_2^3 t_3^4, \quad dx_3 = 6t_1^5 t_2^3 t_3^4 dt_1 + 3t_1^6 t_2^2 t_3^4 dt_2 + 4t_1^6 t_2^3 t_3^3 dt_3$$

$$dx_1 \wedge dx_2 = [2t_1 t_2 t_3^2 dt_1 + \dots] \wedge [6t_1^5 t_2^3 t_3^4 dt_1 + \dots] = (\dots) dt_1 dt_2 +$$

$$(\dots) dt_1 dt_3 + (\dots) dt_2 dt_1 + (\dots) dt_2 dt_3 + (\dots) dt_2 dt_3 +$$
$$-(\dots) dt_1 dt_2$$

$$(\dots) dt_2 dt_1 + (\dots) dt_2 dt_3 + (\dots) dt_3 dt_1 = *$$
$$-(\dots) dt_1 dt_2 + \dots$$
$$-(\dots) dt_2 dt_3$$

$$* = (\dots) dt_1 dt_2 + (\dots) dt_1 dt_3 + (\dots) dt_2 dt_3$$



$$Q_i = t_1^{2i} t_2^i t_3^{i+1}$$

$$x_1 = t_1^2 t_2 t_3^2 \quad \neq \quad dx_1 = 2t_1 t_2 t_3^2 dt_1 + t_1^2 t_3^2 dt_2 + 2t_1^2 t_2 t_3 dt_3$$

$$x_2 = t_1^4 t_2^2 t_3^3, \quad dx_2 = 4t_1^3 t_2^2 t_3^3 dt_1 + 2t_1^4 t_2 dt_2 + 3t_1^4 t_2^2 t_3^2 dt_3$$

$$x_3 = t_1^6 t_2^3 t_3^4, \quad dx_3 = 6t_1^5 t_2^3 t_3^4 dt_1 + 3t_1^6 t_2^2 t_3^4 dt_2 + 4t_1^6 t_2^3 t_3^3 dt_3$$

$$x_4 = t_1^8 t_2^4 t_3^5, \quad dx_4 = 8t_1^7 t_2^4 t_3^5 dt_1 + 4t_1^8 t_2^3 t_3^5 dt_2 + 5t_1^8 t_2^4 t_3^4 dt_3$$

$$x_5 = t_1^{10} t_2^5 t_3^6, \quad dx_5 = 10t_1^9 t_2^5 t_3^6 dt_1 + 5t_1^{10} t_2^4 t_3^6 dt_2 + 6t_1^{10} t_2^5 t_3^5 dt_3$$

$$dx_3 \wedge dx_5 = *$$

Pl: lat rine polin invariante:

$$(1 + 7x^2 + 2x^3 + 7x^9) \cdot (1 + x + 4x^2 + 6x^4) = \underbrace{(5 \cdot 1)}_{\text{wavy line}} + \underbrace{(5)}_x + \underbrace{(9+20)}_{x^2} + \underbrace{(7+6)}_{x^3} + \dots$$

$$+ \underbrace{(12+20+2)}_{x^4} + \dots$$

$$(\quad) dt_2 dt_1 = -(\quad) dt_1 dt_2$$

$$dt_1 dt_1 = 0$$

\*

$$dx_3 \wedge dx_5 = (\quad) dt_1 dt_2 + (\quad) dt_1 dt_3 + (\quad) dt_2 dt_3 +$$

$$dx_3 \wedge dx_5 = \quad dt_1 dt_2$$

$$(20 t_1^{15} t_2^7 t_3^{10} - 20 t_1^{15} t_2^7 t_3^{10}) dt_1 dt_2 + (16 t_1^{14} t_2^8 t_3^9 - 40 t_1^{15} t_2^8 t_3^9) dt_1 dt_3$$

$$+ (18 t_1^{16} t_2^7 t_3^9 - 20 t_1^{16} t_2^7 t_3^9) dt_2 dt_3 = -4 t_1^{18} t_2^8 t_3^9 dt_1 dt_2 + 2 t_1^{16} t_2^8 t_3^9 dt_2 dt_3$$



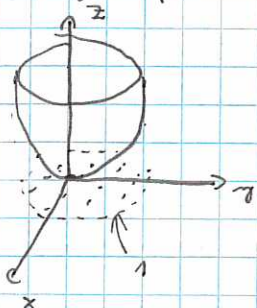
$$dx_1 \wedge dx_2 \wedge dx_3 = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) dt_1 dt_2 dt_3 = 0$$

$$dt_1 \cdot (dt_2 \wedge dt_3)$$

$$dt_2 \cdot (dt_1 \wedge dt_3)$$

parallel lines: -  
 pairs case: + / - -

find: paraboloid:



$$(z = x^2 + y^2 \leq 1) = \{ p \in \mathbb{R}^3 \mid z(p) = x(p)^2 + y(p)^2 \leq 1 \} =: P$$

Vol<sub>2</sub> P = ?

1, Parameteris

$$Q: [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$$

$$r(u) \mapsto \begin{bmatrix} r \cos u \\ r \sin u \\ r^2 \end{bmatrix}$$

$$x = r \cos u$$

$$y = r \sin u$$

$$z = x^2 + y^2 = r^2 (\cos^2 u + \sin^2 u) = r^2$$

$$0 \leq r \leq 1$$

$$0 \leq u \leq 2\pi$$

2, Integral

$$\textcircled{1} \text{ Vol}_2 P = \int_{r=0}^1 \int_{u=0}^{2\pi} \sqrt{\det(G)} \, du \, dr \rightarrow \text{exponential}$$

\textcircled{2}



$$\begin{aligned} \textcircled{2} \quad \text{Vol}_2 P &= \int_Q \sqrt{\|dx \wedge dy\|^2 + \|dx \wedge dz\|^2 + \|dy \wedge dz\|^2} = \\ &= \int_{r=0}^1 \int_{\varphi=0}^{2\pi} * \end{aligned}$$

$$x = r \cos \varphi \quad [Q - dx \wedge dy \wedge dz (x \otimes Q)]$$

$$y = r \sin \varphi$$

$$z = r^2$$

$$dx = \cos \varphi dr - r \sin \varphi d\varphi$$

$$dy = \sin \varphi dr + r \cos \varphi d\varphi$$

$$dz = 2r dr$$

$$dx \wedge dy = (\cos \varphi dr - r \sin \varphi d\varphi) \wedge (\sin \varphi dr + r \cos \varphi d\varphi) =$$

$$= r \cos^2 \varphi + r \sin^2 \varphi dr d\varphi = r dr d\varphi$$

$$\|dx \wedge dy\|^2 = r^2 \|dr \wedge d\varphi\|^2 = \underline{r^2 (dr d\varphi)^2}$$

dy

$$dx \wedge dz = (\cos \varphi dr - r \sin \varphi d\varphi) \wedge (2r dr) = 2r^2 \sin \varphi dr d\varphi$$

$$\|dx \wedge dz\|^2 = 4r^4 \sin^2 \varphi (dr d\varphi)^2$$

$$dy \wedge dz = -2r^2 \cos \varphi$$

$$\|dy \wedge dz\|^2 = 4r^4 \cos^2 \varphi (dr d\varphi)^2$$

$$\sqrt{\|dx \wedge dy\|^2 + \|dx \wedge dz\|^2 + \|dy \wedge dz\|^2} = \sqrt{r^2 + 4r^4 \sin^2 \varphi + 4r^4 \cos^2 \varphi} dr d\varphi =$$

$$= \sqrt{r^2 (1 + 4r^2)} dr d\varphi = r \sqrt{1 + 4r^2} dr d\varphi \quad *$$



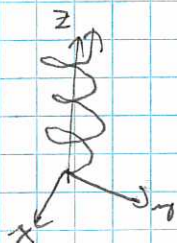
Szerel

$$\int_{r=0}^1 \int_{\varphi=0}^{2\pi} r \sqrt{1+4r^2} \, dr \, d\varphi = \frac{\pi}{6} (4\sqrt{125} - 1)$$

$$\tilde{P} = \text{[diagram of a sphere with a shaded cap]} \cup \text{[diagram of a ring]}$$

"fedettség"

Áramló víz (örvösítő víz)



$$r(x, y, z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$$

Melyi víz áramlik át  $\tilde{P}$  határain? 0, víz ingadozik.

↑

$$[\tilde{P} \text{ határain}] = [P + \text{fedés}]$$

P hely a víz

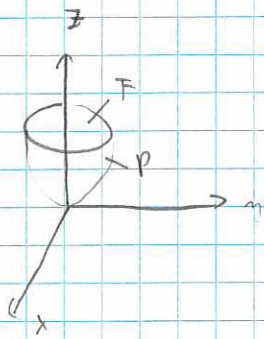
F hely a víz

felhatárain  
↓

$$[P \text{ felhatárain}] = \int_Q 20r \cdot ds \, dy + x \cdot r \, dy \, dz + y \cdot r \, dx \, dz$$

dxdz





$$Q: \begin{matrix} (r, \varphi) \\ \in \mathbb{R}^2 \end{matrix} \mapsto \begin{bmatrix} r \cos \varphi \\ r \sin \varphi \\ r^2 \end{bmatrix} \quad [0, 1] \times [0, 2\pi] \mapsto \mathbb{R}^3$$

$$RQ: \begin{matrix} (r, \varphi) \\ \in \mathbb{R}^2 \end{matrix} \mapsto \begin{bmatrix} r \cos \varphi \\ r \sin \varphi \\ 1 \end{bmatrix}$$

$$r: \mathbb{R}^3 \mapsto \mathbb{R}^1$$

$$r(x, y, z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$$

$$\begin{matrix} x \\ y \end{matrix} \rightarrow z$$

$$[r \text{ d'f. f. } P \text{ - } r] = \int_Q (x \circ r \, dy \, dz + y \circ r \, dz \, dx + z \circ r \, dx \, dy)$$

$$[r \text{ d'f. f. } z \text{ - } r] = \int_R (-1)$$

$$[r \text{ d'f. f. } P \text{ - } r] = \int_{r=0}^1 \int_{\varphi=0}^{2\pi} \int_{z=0}^1 (-r \sin \varphi \cdot (-2r^2 \cos \varphi) \, dr \, d\varphi + r \cos \varphi (-2r^2 \sin \varphi) \, dr \, d\varphi + r \, dr \, d\varphi) \cdot *$$

$$x = r \cos \varphi$$

$$dx = \cos \varphi \, dr - r \sin \varphi \, d\varphi$$

$$y = r \sin \varphi$$

$$dy = \sin \varphi \, dr + r \cos \varphi \, d\varphi$$

$$z = r^2$$

$$dz = 2r \, dr$$

$$dy \wedge dz = -2r^2 \cos \varphi \, dr \, d\varphi$$

$$dz \wedge dx = -2r^2 \sin \varphi \, dr \, d\varphi$$

$$dx \wedge dy = r \cos \varphi \, dr \, d\varphi + r \sin^2 \varphi \, dr \, d\varphi = r \, dr \, d\varphi$$

$$x \circ r = -y = -r \sin \varphi$$

$$y \circ r = x = r \cos \varphi$$

$$z \circ r = 1$$

$dr \, d\varphi$  hell. ist  
= integral  
ist, setzt es  
= 1



$$* \int_{r=0}^1 \int_{\varphi=0}^{2\pi} \left[ -r \sin \varphi \cdot (+2r^2 \cos \varphi) + r \cos \varphi \cdot 2r^2 \sin \varphi - r \right] d\varphi dr =$$

$$- \int_{r=0}^1 \int_{\varphi=0}^{2\pi} r d\varphi dr = - \int_{r=0}^1 2\pi r dr = -\pi$$

$\uparrow$   
 $\oplus \rightarrow \text{is it minus?}$

$$[\text{Algebra's } \nabla \cdot \mathbf{a}] = \int_{r=0}^1 \int_{\varphi=0}^{2\pi} [0 + 0 + -r] d\varphi dr = -\pi$$

$\uparrow$   
 $\oplus \quad ???$

Hábor  $\ominus$ , mibor  $\oplus$  a dőjel?



$\oplus$  - amikor beágy  
 $\ominus$  - amikor kiágy

$[\text{Algebra's } \nabla \cdot \mathbf{a}]$  kifejezés a víz  $\Rightarrow \ominus$

$[\text{Algebra's } \nabla \cdot \mathbf{a}]$  kifejezés a víz  $\Rightarrow \oplus$

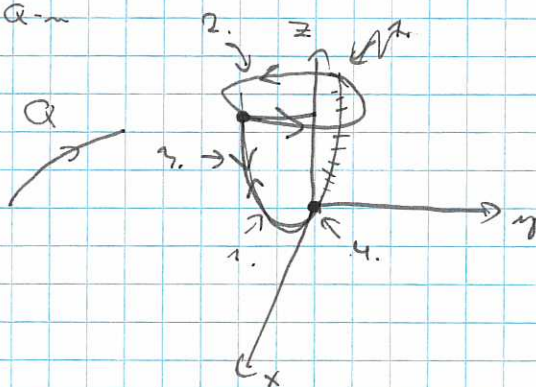
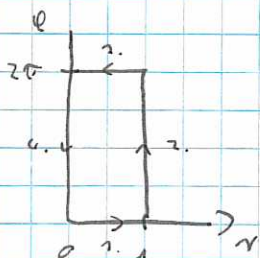


Paraméterek és előjelről



$$\int_{r=0}^1 \int_{\phi=0}^{2\pi} \dots d\phi dr \Rightarrow \int_{r=0}^1 \int_{\phi=0}^{2\pi} \dots d\phi dr$$

Határozzuk meg az integrál értékeit



1)  $\phi = 0, r = [0, 1]$

$$Q = \begin{bmatrix} r \cdot \cos \phi \\ r \cdot \sin \phi \\ r^2 \end{bmatrix} \rightarrow \begin{bmatrix} r \\ 0 \\ r^2 \end{bmatrix}$$

2)  $\phi = [0, 2\pi], r = 1$

$$Q = \begin{bmatrix} r \cdot \cos \phi \\ r \cdot \sin \phi \\ r^2 \end{bmatrix} \rightarrow \begin{bmatrix} \cos \phi \\ \sin \phi \\ 1 \end{bmatrix}$$

3)  $\phi = 2\pi, r = [0, 1]$

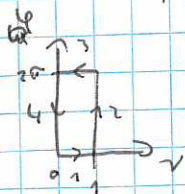
$$Q = \begin{bmatrix} r \cdot \cos \phi \\ r \cdot \sin \phi \\ r^2 \end{bmatrix} \rightarrow \begin{bmatrix} r \\ 0 \\ r^2 \end{bmatrix}$$

4)  $\phi = 2\pi \rightarrow 0, r = 0$

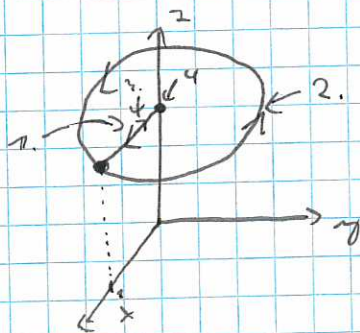
$$Q = \begin{bmatrix} r \cdot \cos \phi \\ r \cdot \sin \phi \\ r^2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



átfogó F-én



→



$$1, u=0, v=0 \rightarrow 1$$

$$R = \begin{bmatrix} \cos u \\ \sin u \\ 1 \end{bmatrix} = \begin{bmatrix} v \\ u \\ 1 \end{bmatrix}$$

$$2, u=2\pi, v=0 \rightarrow 1$$

$$4, u=2\pi \rightarrow 0, v=0$$

$$2, u=0 \rightarrow 2\pi, v=1$$

$$R = \begin{bmatrix} \cos u \\ \sin u \\ 1 \end{bmatrix}$$

A lelt felvétel "első 1. lépés", az első fordítási lépés (0, 1) irányítás.

⇒

P-él, hogy a felvétel a vektor-érő

"P-él a vektor-érő első lépés" "első lépés"



???

A lüti, le'pallale

$$\frac{r^2}{\sqrt{1+r^2}}$$

Anchisedes: - nem kezdol

- a rálatti sül öse e'pp fent fent
- rüga a rüga rüga

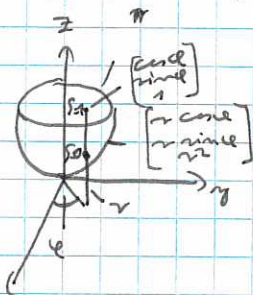
tötrünl fel e, nyuggal



a rálatti sül öse e'gi, a rüga rüga

rüga ⇒ rüga: fent rüga - rüga rüga

A fúja polár paraméterezése

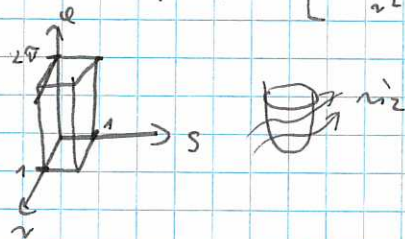


Sollene e'gi 1 paraméter

$$\int \text{paraméter: } (1-s) \begin{bmatrix} r \cos u \\ r \sin u \\ r^2 \end{bmatrix} + s \begin{bmatrix} \cos u \\ \sin u \\ 1 \end{bmatrix}$$

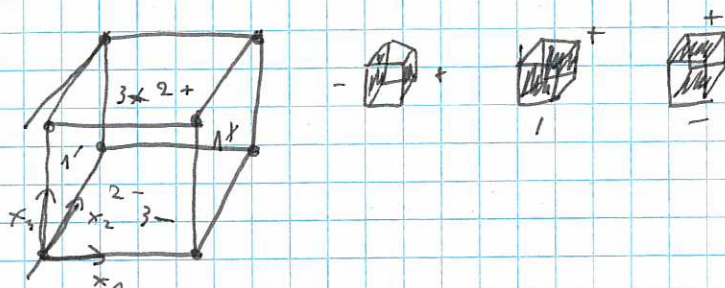
$$\tilde{a} : [0, 1] \times [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$$

$$\tilde{a}(r, s, u) = (1-s) \begin{bmatrix} r \cos u \\ r \sin u \\ r^2 \end{bmatrix} + s \begin{bmatrix} \cos u \\ \sin u \\ 1 \end{bmatrix}$$





Kocka iragított határa

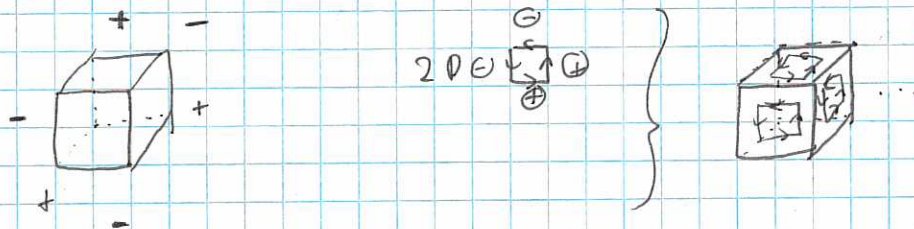


Integrálás iragított lapjai:

$$\int_{t_1=0}^1 \int_{t_2=0}^1 \int_{t_3=0}^1 f(x) dx_1 dx_2 dx_3 = \int_{t_1=0}^1 \int_{t_2=0}^1 \int_{t_3=0}^1 [\dots] dt_1 dt_2 dt_3$$

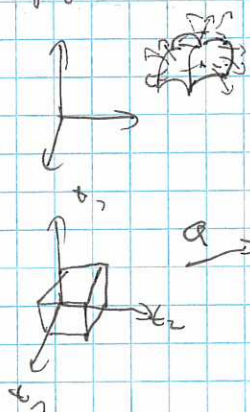
$2^{l-1}$  oldal előjele:  $(-1)^{l-1}$

$2^{l-1}$  oldal előjele:  $(-1)^l$



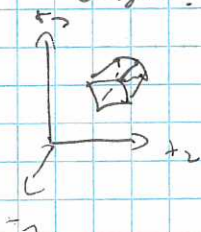
Alapfogalmak

Alapfogalmak:



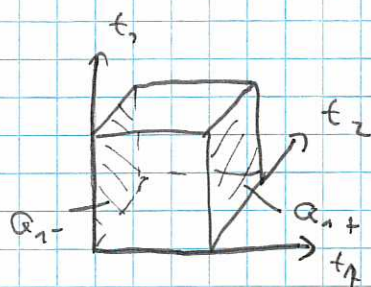
A deriváció az integrálás fordítottja.

Előjel?



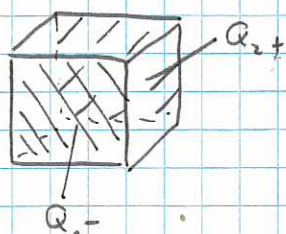
$$Q: [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \rightarrow \mathbb{R}^1$$





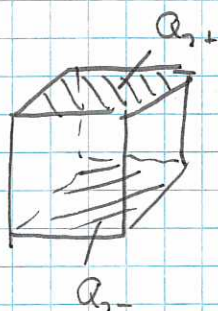
$$Q_{1+}: (t_1, t_2) \mapsto Q(t_1, t_2, t_3)$$

$$Q_{1-}: (t_1, t_2) \mapsto Q(a_1, t_2, t_3)$$



$$Q_{2+}: (t_1, t_3) \mapsto Q(t_1, b_2, t_3)$$

$$Q_{2-}: (t_1, t_3) \mapsto Q(t_1, a_2, t_3)$$



$$Q_{3+}: (t_1, t_2) \mapsto Q(t_1, t_2, b_3)$$

$$Q_{3-}: (t_1, t_2) \mapsto Q(t_1, t_2, a_3)$$

$\int \frac{\partial Q}{\partial Q}$   
 $\uparrow$

Alle Flächen sind -line  
 $\left( \int_{Q_{1+}} - \int_{Q_{1-}} \right) - \left( \int_{Q_{2+}} - \int_{Q_{2-}} \right) + \left( \int_{Q_{3+}} - \int_{Q_{3-}} \right) \dots$   
 $\dots \underbrace{\left( r_1 dx_2 \wedge dx_3 + r_2 dx_3 \wedge dx_1 + r_3 dx_1 \wedge dx_2 \right)}_{=}$

Alternatives set:  $k$ -dim  $C_1^1$  hoch  $\mathbb{R}^N$ -ban ( $k \leq N$ )

$$Q: [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k] \mapsto \mathbb{R}^N$$

$$\left( \int \frac{\partial Q}{\partial Q} \sum_{i=1}^k (-1)^{i-1} \left( \int_{Q_{i+}} - \int_{Q_{i-}} \right) \right)$$

$$Q_{1+}: (t_2, t_3, \dots, t_k) \mapsto Q(b_1, t_2, \dots, t_k)$$

$$Q_{1-}: (t_2, t_3, \dots, t_k) \mapsto Q(a_1, t_2, \dots, t_k)$$

$$Q_{2+}: (t_1, t_3, \dots, t_k) \mapsto Q(t_1, b_2, t_3, \dots, t_k)$$



$$Q_{d+}: (t_1, t_2, \dots, t_{d-1}, t_{d+1}, \dots, t_n) \rightarrow Q(t_1, \dots, t_d, \dots, t_n)$$

$$Q_{d-}: (t_1, \dots, t_{d-1}, t_{d+1}, \dots, t_n) \rightarrow Q(t_1, \dots, t_d, \dots, t_n)$$

$$\int_Q \sum_{d=1}^n (-1)^{d-1} \left( \int_{Q_{d+}} - \int_{Q_{d-}} \right) \left( \sum_{i_1 < \dots < i_{n-1}} \frac{\partial f}{\partial x_{i_d}} dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}} \right)$$

d-Kalkülus

pl:  $f = x_1^2 x_2^4 x_3^3 x_4$   $N=4$   
 $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 + \frac{\partial f}{\partial x_4} dx_4$

$$f: \mathbb{R}^N \rightarrow \mathbb{R}$$

$$df = \sum_{j=1}^N \frac{\partial f}{\partial x_j} dx_j$$

$$\sum_{i=1}^N f dx_i$$

$$d \left[ f \cdot dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}} \right] = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}} = \sum_{j=1}^N \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{n-1}}$$

pl:  $N=3$

(Maxwell - 1870)

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

$$\left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \rightarrow f \text{ gradient}$$

$$d(v_1 dx_2 \wedge dx_3 + v_2 dx_3 \wedge dx_1 + v_3 dx_1 \wedge dx_2) = \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) dx_1 dx_2 dx_3$$

u rotations

r divergenz (potenziell)

$$d(u_1 dx_2 + u_2 dx_3 + u_3 dx_1) = \left( \frac{\partial u_1}{\partial x_1} - \frac{\partial u_3}{\partial x_2} \right) dx_1 \wedge dx_2 + \left( \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_3} \right) dx_2 \wedge dx_3 + \left( \frac{\partial u_3}{\partial x_3} - \frac{\partial u_2}{\partial x_1} \right) dx_3 \wedge dx_1$$



Tétel: Stokes-tétel

$$\int_{\partial\Omega} w = \int_{\Omega} dw$$

(kér: 2D-ben, 6 integrál is lehet  
1 integrál is lehet integrálni)

Tétel:  $d(dw) = 0$

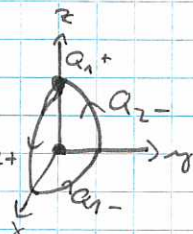
Tétel (Poincaré lemma):

$$dw = 0 \Rightarrow \exists \bar{w}, w = d\bar{w} \quad (\text{vektor potenciál létezése})$$

pl (Példák)

Gömbfelület

$$Q: [0, \pi/2] \times [0, 2\pi] \rightarrow \mathbb{R}^3$$



$$Q(\varphi, \psi) = \begin{bmatrix} \cos\varphi \cdot \cos\psi \\ \cos\varphi \cdot \sin\psi \\ \sin\varphi \end{bmatrix}$$

$$u: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$u(x, y, z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$$

$$\int_{dQ} \overbrace{u_1 dx + u_2 dy + u_3 dz}^w = ?$$

$$\textcircled{1} \text{ Kiszámolás: } \int_{\partial Q} = \left( \int_{Q_{1+}} - \int_{Q_{1-}} \right) - \left( \int_{Q_{2+}} - \int_{Q_{2-}} \right) w$$

$$\int_{Q_{1+}} w =$$

$$Q_{1+}: \varphi \mapsto Q(\pi/2, \varphi) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q_{1-}: \varphi \mapsto Q(0, \varphi) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix}$$

$$Q_{2+}: \varphi \mapsto Q(\varphi, \pi/2) = \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix}$$

$$Q_{2-}: \varphi \mapsto Q(\varphi, 0) = \begin{bmatrix} \cos\varphi \\ 0 \\ \sin\varphi \end{bmatrix}$$



$$\int_{Q_1} w = \int_{\varphi=0}^{\pi/2} [0] d\varphi = 0$$

$$x=0 \quad dx=dy=dz=0$$

$$y=0$$

$$z=1$$

$$\int_{Q_2} w = \int_{\varphi=0}^{\pi/2} (\sin^2 \varphi + \cos^2 \varphi) d\varphi = \int_{\varphi=0}^{\pi/2} d\varphi = \pi/2$$

$$x = \cos \varphi$$

$$dx = -\sin \varphi d\varphi$$

$$u_1 = -\sin \varphi$$

$$y = \sin \varphi$$

$$dy = \cos \varphi d\varphi$$

$$u_2 = \cos \varphi$$

$$z=0$$

$$dz=0$$

$$u_3 = 1$$

$$\int_{Q_2} w$$

$$d\varphi$$

② Stokess Theorem

$$\int_{\partial Q} w = \int_Q dw$$

$$w = u_1 dx + u_2 dy + u_3 dz$$

$$w = -y dx + x dy + dz$$

$$dw = d(-y dx + x dy + dz)$$

$$= d(-y dx) + d(x dy) + \overset{0}{d dz}$$

$$= d(-y) \wedge dx + dx \wedge dy$$

$$= -dy \wedge dx + dx \wedge dy = dx \wedge dy + dx \wedge dy$$

$$= 2 dx \wedge dy$$

$$\int_Q dw = \int_Q 2 dx \wedge dy = \int_{\varphi=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} \text{[scribbles]} d\varphi d\varphi$$

$$x = \cos \varphi \cos \psi$$

$$dx = -\sin \varphi \cos \psi d\varphi - \cos \varphi \sin \psi d\psi$$

$$y = \cos \varphi \sin \psi$$

$$dy = -\sin \varphi \sin \psi d\varphi + \cos \varphi \cos \psi d\psi$$

$$z = \sin \varphi$$

$$dz = \cos \varphi d\varphi$$

( )



$$dx \wedge dy = \underbrace{\left( \cos \varphi \sin \varphi \sin^2 \varphi + \cos \varphi \sin \varphi \cos^2 \varphi \right)}_{\cos \varphi \sin \varphi} d\varphi \wedge d\varphi$$

$$\int_Q 2 \, dx \wedge dy = 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi \, d\varphi \, d\varphi = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \sin 2\varphi \, d\varphi = \frac{\pi}{8} \left[ -\cos 2\varphi \right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

Koordináta szer

$S$  halmaza  $x_1, \dots, x_n: S \rightarrow \mathbb{R}$

$N$ -dimenziós koordináták  $S$ -nél, ha megkülönböztethetők a pontjaik.

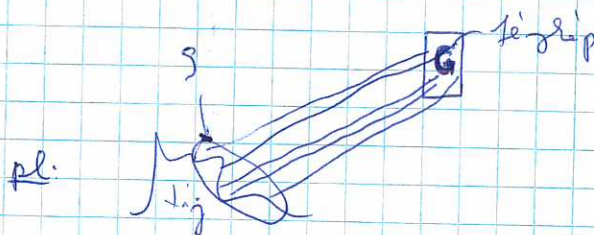
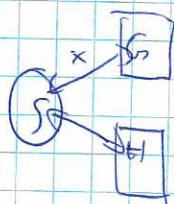
$$p, q \in S \quad p \neq q \in S \quad \exists j \, x_j(p) \neq x_j(q)$$

Közvetlen koordináták:  $\exists G \text{ nyílt halmaz } \subset \mathbb{R}^N$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = p \mapsto \begin{bmatrix} x_1(p) \\ \vdots \\ x_N(p) \end{bmatrix} \quad X: S \rightarrow G$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

közvetlen koordináták  $S$ -en



Tudjuk  $Y$  alapján hogyan nyílt halmaz  $U_j$   $X$ -et.

$$x_i = x_i(y_1, \dots, y_N), \dots, x_N = x_N(y_1, \dots, y_N)$$

inverz (F.2.)  $f: S \rightarrow \mathbb{R}$

$$\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_N} \text{ alapján hogyan}$$

$$\text{nyílt halmaz } U_j \text{ li } \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}$$



$$\underline{\underline{I}} \quad \underbrace{\left[ \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N} \right]}_{\frac{\partial f}{\partial \mathbf{x}}} = \underbrace{\left[ \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_N} \right]}_{\frac{\partial f}{\partial \mathbf{y}}} \cdot \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_N} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_N}{\partial x_1} & \frac{\partial y_N}{\partial x_2} & \dots & \frac{\partial y_N}{\partial x_N} \end{bmatrix}}_{\text{Jacobian}} = *$$

↑  
jacobische  
matrix

$$\underbrace{\frac{\partial \mathbf{x}}{\partial \mathbf{y}}}_{\text{Jacobian}} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial y_1} & \dots & \frac{\partial x_N}{\partial y_N} \end{bmatrix}}_{\text{Jacobian}} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial y_1} & \dots & \frac{\partial x_N}{\partial y_N} \end{bmatrix}^{-1}}_{\text{Jacobian}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{y}}$$

$$* = \underbrace{\left[ \frac{\partial f}{\partial y_1} \quad \frac{\partial f}{\partial y_N} \right]}_{\text{Jacobian}} \cdot \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_N}{\partial y_1} & \dots & \frac{\partial x_N}{\partial y_N} \end{bmatrix}}_{\text{Jacobian}}$$

Wz:  $df = \frac{\partial f}{\partial y_1} dy_1 + \dots + \frac{\partial f}{\partial y_N} dy_N$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_N} dx_N = \frac{\partial f}{\partial x_1} \left[ \frac{\partial x_1}{\partial y_1} dy_1 + \dots + \frac{\partial x_1}{\partial y_N} dy_N \right] + \dots +$$

$$+ \frac{\partial f}{\partial x_N} \left[ \frac{\partial x_N}{\partial y_1} dy_1 + \dots + \frac{\partial x_N}{\partial y_N} dy_N \right] =$$

$$= \left[ \right] dy_1 + \dots + \left[ \right] dy_N$$

$$\frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_1} + \dots + \frac{\partial f}{\partial x_N} \cdot \frac{\partial x_N}{\partial y_1}$$

$$\frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_N} + \dots + \frac{\partial f}{\partial x_N} \cdot \frac{\partial x_N}{\partial y_N}$$



$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_1} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial y_1} = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right] \cdot \begin{bmatrix} \frac{\partial x_1}{\partial y_1} \\ \frac{\partial x_2}{\partial y_1} \\ \vdots \\ \frac{\partial x_n}{\partial y_1} \end{bmatrix}$$

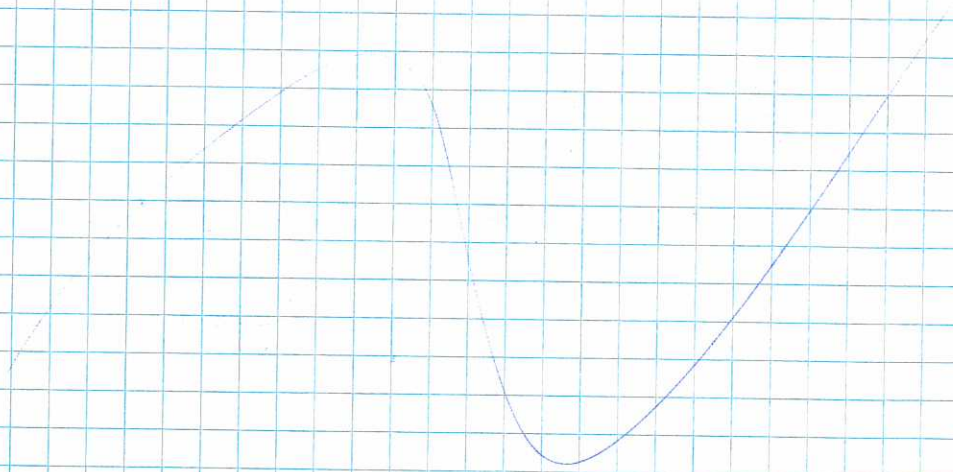
$$\frac{\partial f}{\partial y_2} = \left[ \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right] \cdot \begin{bmatrix} \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_2} \\ \vdots \\ \frac{\partial x_n}{\partial y_2} \end{bmatrix} = \frac{\partial f}{\partial x} \cdot \begin{bmatrix} \frac{\partial x}{\partial y_2} \end{bmatrix}$$

Ergebnis alle Ableitungen:

$$\left[ \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n} \right] = \left[ \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right] \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \dots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}$$

x-y Ableitung ist

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial y}{\partial x}$$





Lagrange-ansatz

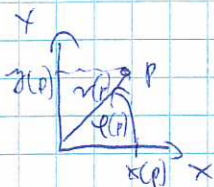
Δ polarkoordinaten algebra

$$D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$Df = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$c^2 = r^2$$



$$x(p) = r(p) \cos \varphi(p)$$

$$y(p) = r(p) \sin \varphi(p)$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$P = \begin{pmatrix} r \\ \varphi \end{pmatrix}$$

$$\Delta f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}$$

$$\begin{aligned} x &= r \cos \varphi & \frac{\partial x}{\partial r} &= \cos \varphi \\ y &= r \sin \varphi & \frac{\partial x}{\partial \varphi} &= -r \sin \varphi \\ & & \frac{\partial y}{\partial r} &= \sin \varphi \\ & & \frac{\partial y}{\partial \varphi} &= r \cos \varphi \end{aligned}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{pmatrix} \cdot \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det: r \cdot \cos^2 \varphi + r \cdot \sin^2 \varphi = r$$

$$\rightarrow \frac{1}{r} \begin{pmatrix} r \cos \varphi & r \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$$

→



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{1}{r} \cos \varphi + \frac{\partial f}{\partial \varphi} \cdot \frac{1}{r} (-\sin \varphi) = \cos \varphi \frac{\partial f}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial f}{\partial \varphi}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \cdot \frac{1}{r} r \sin \varphi + \frac{\partial f}{\partial \varphi} \cdot \frac{1}{r} \cos \varphi = \sin \varphi \frac{\partial f}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial f}{\partial \varphi}$$

also:  $\frac{\partial^2 f}{\partial x^2} = \frac{\partial(\frac{\partial f}{\partial x})}{\partial x}$ ,  $\frac{\partial^2 f}{\partial y^2} = \frac{\partial(\frac{\partial f}{\partial y})}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \cdot \underbrace{\left( \frac{\partial f}{\partial x} \right)}_g = \frac{\partial g}{\partial x} = \cos \varphi \frac{\partial g}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial g}{\partial \varphi} =$$

$$= \cos \varphi \frac{\partial}{\partial r} \left[ \cos \varphi \frac{\partial f}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial f}{\partial \varphi} \right] -$$

$$- \frac{1}{r} \frac{\partial}{\partial \varphi} \left[ \underbrace{\cos \varphi}_{\sin \varphi} \frac{\partial f}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial f}{\partial \varphi} \right] =$$

$$= \cos \varphi \left[ \cos \varphi \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \sin \varphi \frac{\partial f}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial^2 f}{\partial r \partial \varphi} \right] -$$

$$- \frac{1}{r} \sin \varphi \left[ -\sin \varphi \frac{\partial f}{\partial r} + \cos \varphi \frac{\partial^2 f}{\partial r \partial \varphi} - \frac{1}{r} \cos \varphi \frac{\partial f}{\partial \varphi} - \frac{1}{r} \sin \varphi \frac{\partial^2 f}{\partial \varphi^2} \right]$$

$$\frac{\partial^2 f}{\partial y^2} = \sin \varphi \left[ \sin \varphi \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \cos \varphi \frac{\partial f}{\partial \varphi} + \frac{1}{r} \cos \varphi \frac{\partial f}{\partial r \partial \varphi} \right] +$$

$$+ \frac{1}{r} \cos \varphi \left[ \cos \varphi \frac{\partial f}{\partial r} + \sin \varphi \frac{\partial^2 f}{\partial r \partial \varphi} + \frac{1}{r} \cos \varphi \frac{\partial f}{\partial \varphi} + \frac{1}{r} \sin \varphi \frac{\partial^2 f}{\partial \varphi^2} \right]$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (\cos^2 \varphi + \sin^2 \varphi) \frac{\partial^2 f}{\partial r^2} + 0 \cdot \frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r} (\cos^2 \varphi + \sin^2 \varphi) \frac{\partial f}{\partial r} + 0 \cdot \frac{\partial f}{\partial \varphi} + \frac{1}{r} (\sin^2 \varphi + \cos^2 \varphi) \frac{\partial f}{\partial \varphi}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2}$$



Figyelni kell:

Merítéskényszer:  $X = \begin{bmatrix} x \\ y \end{bmatrix}$   $u = \begin{bmatrix} x \\ u \end{bmatrix}$  (egyel koordinátán szorzó)  $x$

Ekkor a mérési egyenlet:  $u = \begin{bmatrix} x \\ u \end{bmatrix}$  jelöléssel