A PARC. DERIVÁLTAK A TELJES KOORD-TÓL FÜGGNEK

$X: S \leftrightarrow G \subset \mathbb{R}^N$, $Y: S \leftrightarrow H \subset \mathbb{R}^N$

Y_{x_i}, X_{y_j} folyt diff

Tudjuk: $\dfrac{\partial f}{\partial X} = \dfrac{\partial f}{\partial Y}\dfrac{\partial Y}{\partial X} = \dfrac{\partial f}{\partial Y}\left[\dfrac{\partial X}{\partial Y}\right]^{-1}$

Előfordulhat: $x_1 = y_1$ (ill. $x_{i_0} = y_{j_0}$)

Ennek ellenére ÁLTALÁBAN $\dfrac{\partial f}{\partial x_1} \neq \dfrac{\partial f}{\partial y_1}$ HA X, ill Y szerint számolunk

$$\underset{\sim}{\frac{\partial f}{\partial x_1}} = \left[\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial X}\ 1.\text{ tagja}\right] = \frac{\partial f}{\underset{\sim}{\partial y_1}}\frac{\partial y_1}{\partial x_1} + \frac{\partial f}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \dots + \frac{\partial f}{\partial y_N}\frac{\partial y_N}{\partial x_1}$$

Pl $X: \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ $\quad x_1\binom{\xi}{\eta} = \xi$, $x_2\binom{\xi}{\eta} = \eta$

$Y: \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ $\quad y_1\binom{\xi}{\eta} = \xi$, $y_2\binom{\xi}{\eta} = \xi + \eta$ $\quad x_1 = y_1$, $y_2 = x_1 + x_2$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f}{\partial y_2}\frac{\partial y_2}{\partial x_1} = \frac{\partial f}{\partial y_1}\cdot 1 + \frac{\partial f}{\partial y_2}\cdot 1$$

$$f := \underset{y_2 - y_1}{\overset{\parallel}{x_2}} \Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial x_2}{\partial x_1} = \underline{\underline{0}}, \quad \frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial y_2} = 0 - 1 = \underline{\underline{-1}}$$

Emlékeztető (X,d) metr. tér $\alpha \geq 0$

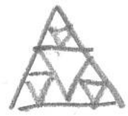
$$\mathrm{vol}_{\alpha,\varepsilon} A := \inf\left\{ \sum_{k=1}^{\infty} (\mathrm{diam}\, A_k)^{\alpha} : \bigcup_{k=1}^{\infty} A_k \supset A,\ \mathrm{diam}\, A_k \leq \varepsilon \right\}$$

$$\mathrm{vol}_{\alpha\varepsilon} A \nearrow \underline{\mathrm{vol}_{\alpha} A} \quad (\varepsilon \searrow 0,\ A \subset X).$$

Hausdorff mérték α-dim-ben (!)

1) $\exists!\ \alpha (= \dim A)$ $\quad \mathrm{vol}_{\beta} A = \infty$ ha $\beta < \alpha$

$\mathrm{vol}_{\beta} A = 0$ ha $\beta > \alpha$

Pl  $A_0 = \underset{1}{\Delta}$ $\quad A_n =$  $\quad A := \bigcap_{n=1}^{\infty} A_n$

$\dim A$: $\quad A_n = \dot{\bigcup} \left\{ 3^n \text{ db } \frac{1}{2^n} \text{ diam } \Delta \right\}$

$$\mathrm{vol}_{\alpha, 1/2^n} A = 3^n \cdot (1/2^n)^{\alpha} = \begin{cases} \infty & 3^n/2^{\alpha n} \geq 1 \\ 0 & 3^n/2^{\alpha n} < 1 \end{cases}$$

$$\dim A = [\alpha : 3^n/2^{\alpha n} = 1] = \underline{\frac{\log 3}{\log 2}}$$

2) Carathéodory tétele: A kompakt $\Rightarrow$ A vol_{α}-mérhető

$[\mathrm{vol}_{\alpha} A = \mathrm{vol}_{\alpha}(A \cap S) + \mathrm{vol}_{\alpha}(A \setminus S) \quad \forall S \subset X$

3) Hausdorff tétele: $X = \mathbb{R}^N$ $d =$ (EUKL TÁV) $\Rightarrow$

$$\mathrm{vol}_N A = \frac{\omega_N}{2^N} \mathrm{Vol}_N A$$ [HAGYOMÁNYOS N DIM TÉRFOGAT]

$\omega_N = [N\text{-dim EGYSÉGGÖMB TÉRFOGATA}] =$

$$= \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})} \qquad \Gamma(z) = \int_{t=0}^{\infty} t^{z-1} e^{-z}\, dz \quad \text{(EULER)}$$

$$\Gamma(n) = (n-1)! \quad (n=1,2,\dots)$$

<u>Integrálás</u> $S \subset X$, $f : S \to \mathbb{R}$ (S kompakt, f FOLYT)

$$\int_{p\in S} f(p)\, \underbrace{\mathrm{Vol}^N}_{\frac{2^N}{\omega_N}\cdot \mathrm{vol}_N}(dp) = \lim_{\substack{S=\bigcup_k A_k \\ \operatorname{diam} A_k < \varepsilon \\ p_k \in A_k}} \sum_k f(p_k)\, \mathrm{Vol}_N(A_k)$$

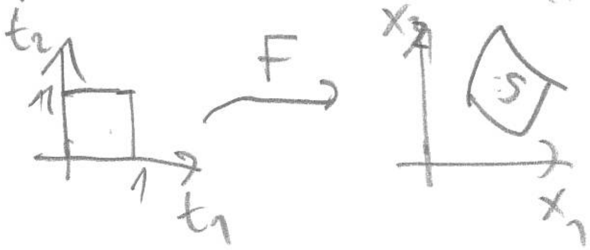
<u>Tétel</u> $X = \mathbb{R}^N$, $d = (\text{Eukl})$

$S = F([0,1]^N)$ $\quad F : [0,1]^N \leftrightarrow S$ [illegible] $\subset \mathbb{R}^N$ felület

$$\int_{p\in S} f(p)\, d\mathrm{Vol}_N(p) = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} f(F(t_1,\dots,t_N)) \cdot \mathrm{Jac}(F) \cdot dt_1 \cdots dt_N$$

$$\mathrm{Jac}(F) : \begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix} \mapsto \det\left[\frac{\partial F_i}{\partial t_j}\right]_{i,j=1}^{N} \quad \left[\text{a } \begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix} \text{ HELYEN}\right]$$

KOORD-INTERPRETÁCIÓ:



S-en KOORD:

$$T : p = F\begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix} \mapsto \begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix}$$

$$\underline{T = F^{-1}}$$

$\underline{\underline{\text{I}}}$ $\quad \operatorname{Jac}(F) = \dfrac{\partial X}{\partial T} \quad \left(X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad x_k : \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} \mapsto \xi_k \right)$

$\underline{\underline{b}}$ $\quad p \in S$-re $\quad X(p) = \begin{bmatrix} x_1(p) \\ \vdots \\ x_N(p) \end{bmatrix} = \begin{bmatrix} F_1(t_1(p), \dots, t_N(p)) \\ \vdots \\ F_N(t_1(p), \dots, t_N(p)) \end{bmatrix}$

$\underline{\underline{\text{Köv}}}$ $\quad \displaystyle\int_{p \in S} f(p)\, \mathrm{Vol}_N(dp) = \int_{t \in [0,1]^N}^{T} f(t_1, \dots, t_N) \left(\frac{\partial X}{\partial T} \right)^{T} \underbrace{dt}_{dt_1 \cdots dt_N}$

FELSZÍN - FORMULA

$\underline{\underline{\text{Emlékeztető}}}$ $\quad$ Ha is: $K \le N$, $F : (0,1)^K \to \mathbb{R}^N$ folyt. diff.

$F : (0,1)^K \leftrightarrow S \quad \operatorname{rank} \left[\dfrac{\partial F_i}{\partial t_j} \right]_{i=1\, j=1}^{N\quad K} = K$ (MAX)

$f : S \to \mathbb{R}$ folyt

$$\int_{p \in S} f(p)\, d\mathrm{Vol}_K(p) = \int_{t_1=0}^{1} \cdots \int_{t_K=0}^{1} f(F(t_1, \dots, t_K))\, \underline{\operatorname{Jac}(F)}\, dt_K \cdots dt_1$$

$$\underline{\operatorname{Jac} F := \sqrt{\det \left\{ \left(\frac{\partial F}{\partial T} \right)^T \left[\frac{\partial F}{\partial T} \right] \right\}}} \qquad \frac{\partial F}{\partial T} = \left[\frac{\partial F_i}{\partial t_j} \right]_{i=1\, j=1}^{N\quad K}$$

$\underline{\underline{\text{Koord-vezérl}}}$ $\quad \displaystyle\int_{p \in S} f(p)\, \mathrm{Vol}_K(dp) = \int_{t \in [0,1]^N}^{T} f(t)^{T} \left\{ \det \left(\frac{\partial X}{\partial T} \right)^T \left(\frac{\partial X}{\partial T} \right) \right\}^{1/2} dt$

$\underline{\underline{\text{Megjegyzés}}}$ NEMCSAK $S = (0,1)^K$, HANEM $S = A \subset \mathbb{R}^K$ $\underline{\text{nyílt halmaz is}}$

INTEGRÁLÁS KOORD SZERINT

$X : S \leftrightarrow \mathbb{R}^N$ <u>Descartes</u> koord

d TÁVOLSÁG $\longrightarrow Vol_K$ k-dim TÉRFOGAT (tetsz $K>0$!)

$$\int_{p\in A} f(p) Vol_N(dp) := \int_{\begin{bmatrix}x_1\\ \vdots \\ x_N\end{bmatrix}\in A} {}^X f(x_1,\dots,x_N)\, dx_1 \dots dx_N$$

<u>T</u> $Y : \underset{\substack{\cap \\ S}}{A} \leftrightarrow G \subset \mathbb{R}^N$ nyílt koord A-n

$$\int_{p\in A} f(p)\, Vol_N(dp) = \int_{\begin{bmatrix}y_1\\ \vdots \\ y_N\end{bmatrix}\in G} {}^Y f(y_1,\dots,y_N) \left|\det \frac{\partial X}{\partial Y}\right| dy_1 \dots dy_N$$

<u>Pl</u> $J := \int_{x=-\infty}^{\infty} e^{-x^2} dx$ kiszámítása

$$J^2 = \int_{x=-\infty}^{\infty} e^{-x^2} dx \int_{y=-\infty}^{\infty} e^{-y^2} dy = \int_{(x,y)\in\mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy$$

$P = \begin{bmatrix} r \\ \varphi \end{bmatrix}$ polárkoord $x = r\cos\varphi,\ y = r\sin\varphi$ $\quad \frac{\partial X}{\partial P} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$

$$J^2 = \int_{p\in\mathbb{R}^2} e^{-(x^2+y^2)(p)}\, Vol_2(dp) = \int_{\substack{r>0 \\ -\pi<\varphi<\pi}} e^{-r^2} \underbrace{\left|\det \frac{\partial X}{\partial P}\right|}_{r} dr\, d\varphi =$$

$$= 2\pi \int_{r=0}^{\infty} e^{-r^2} \cdot r\, dr \underset{\substack{s=r^2 \\ ds = 2r dr}}{=} 2\pi \int_{s=0}^{\infty} \underbrace{e^{-s} \cdot \frac{1}{2} ds}_{1/2} = \underline{\pi}$$

$$\underline{\underline{J = \sqrt{\pi}}}$$

K-dim FELÜLET K-dim TERÜLETE

$X : \mathbb{R}^N \leftrightarrow \mathbb{R}^N$ termszetes D-koord $X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = Id$

d termszetes TÁU $\mathbb{R}^N$-en

$S \subset \mathbb{R}^N$ k-dim felület

$S = \left\{ F\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} : \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in G \right\}$ $F : G \leftrightarrow S$

$F^{-1} : S \leftrightarrow G$ K-dim KOORD   F TÉRKÉP $\begin{bmatrix} F_1(t_1, \ldots t_k) \\ \vdots \\ F_N(t_1, \ldots t_k) \end{bmatrix}$

$x_1, \ldots, x_N \,|\, S$ megszorítás N db koord fgv

T $f : S \to \mathbb{R}$ $f \circ F$ folyt DIFF

$F_i = {}^F(x_i|S)$ folyt diff, $\frac{\partial X}{\partial F} = \left[\frac{\partial F_i}{\partial t_j}\right]_{i=1\, j=1}^{N\ \ k}$ folk rang $= k$

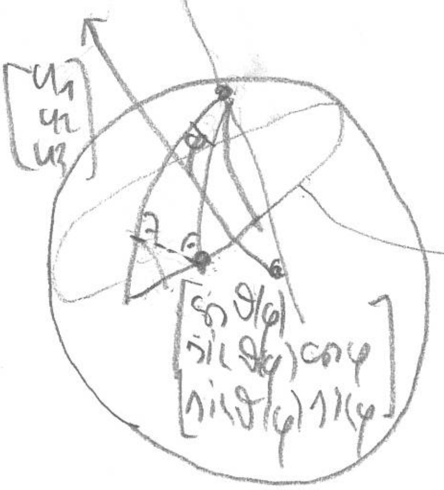
$$\int_{p\in S} f(p)\, d\,Vol_k(p) =$$

$$= \int_{(t_1, \ldots, t_k)^T \in G} {}^F f(t_1, \ldots, t_k) \left\{ \det\left(\left[\frac{\partial X}{\partial F}\right]^T \left[\frac{\partial X}{\partial F}\right] \right) \right\}^{1/2} dt_1 \ldots dt_k$$

Megjegyzés vehető $X|S$ megszorítás is, mint „TÚLKOORDINÁTÁZÁS"

PE EGYSZERHÁROMSZÖG TERÜLETE

(Nagyon nem triv)   Egyszerűbb 3-ad GEMB



$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos\vartheta_0 \\ \sin\vartheta_0 \cos\varphi_0 \\ \sin\vartheta_0 \sin\varphi_0 \end{bmatrix}$$

$$\begin{bmatrix} \cos\vartheta \\ \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \end{bmatrix} \perp$$

$$0 = \cos\vartheta_0 \cos\vartheta + \sin\vartheta_0 \cos\varphi_0 \sin\vartheta \cos\varphi + \sin\vartheta_0 \sin\varphi_0 \sin\vartheta \sin\varphi =$$

$$= \cos\vartheta_0 \cos\vartheta + \sin\vartheta_0 \sin\vartheta \left[\cos\varphi_0 \cos\varphi + \sin\varphi_0 \sin\varphi\right] =$$

$$= \cos\vartheta_0 \cos\vartheta + \sin\vartheta_0 \sin\vartheta \cos(\varphi - \varphi_0)$$

$0 < \varphi < \Phi$ FIXED

$$\vartheta := \vartheta(\varphi) \qquad \cos\vartheta_0 \cos\vartheta(\varphi) + \sin\vartheta_0 \sin\vartheta(\varphi) \cos(\varphi - \varphi_0) = 0$$

$$1 + \operatorname{tg}\vartheta_0 \operatorname{tg}\vartheta(\varphi) \cos(\varphi - \varphi_0) = 0$$

$$\operatorname{tg}\vartheta(\varphi) = \frac{\operatorname{ctg}\vartheta_0}{\cos(\varphi - \varphi_0)}$$

$$T^{-1} : \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} \mapsto \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \end{bmatrix}$$

$$\frac{\partial X}{\partial T} = \begin{bmatrix} -\sin\vartheta & 0 \\ \cos\vartheta \cos\varphi & -\sin\vartheta \sin\varphi \\ \cos\vartheta \sin\varphi & \sin\vartheta \cos\varphi \end{bmatrix} \qquad \left[\frac{\partial X}{\partial T}\right]^T \left[\frac{\partial X}{\partial T}\right] = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{bmatrix}$$

$$\underline{\underline{\text{Area}}} = \int_{\varphi=0}^{\Phi} \int_{\vartheta=0}^{\vartheta(\varphi)} \sin\vartheta \, d\vartheta \, d\varphi = \int_{\varphi=0}^{\Phi} \left[\cos\vartheta(\varphi) - 1\right] d\varphi =$$

$$= \int_{\varphi=0}^{\Phi} \frac{d\varphi}{\sqrt{1 + \frac{\operatorname{ctg}^2\vartheta_0}{\cos^2(\varphi - \varphi_0)}}} = \underline{\underline{\left(\sum \text{szögek}\right) - \pi}}$$

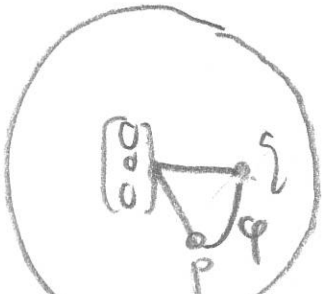
GÖMBI GEOMETRIA

$$S := \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3 : \xi^2+\eta^2+\zeta^2=1 \right\} = \left(x^2+y^2+z^2=1 \right),$$

ahol $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \mathbb{R}^3 \leftrightarrow \mathbb{R}^3$ ALAPKOORD $x\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \xi$ STB.

d_S távolság S-en:

$$d_S(p,q) := \left[\begin{array}{c} \text{legrövidebb } p\text{-t } q\text{-val összekötő} \\ S\text{-beli görbe hossza} \end{array} \right] =$$



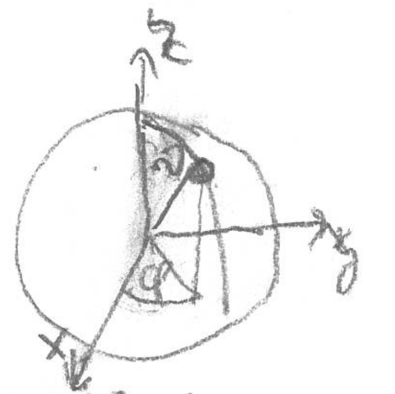
$$= \arccos \langle X(p), X(q) \rangle =$$

$$= \arccos \left[x(p)x(q) + y(p)y(q) + z(p)z(q) \right]$$

M NEM TRIV ↑

Euler koord

$$E : \begin{bmatrix} \cos\vartheta \\ \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \end{bmatrix} \mapsto \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix}$$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X : \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

$$\frac{\partial X}{\partial E} = \begin{bmatrix} \partial x/\partial\vartheta & \partial x/\partial\varphi \\ \partial y/\partial\vartheta & \partial y/\partial\varphi \\ \partial z/\partial\vartheta & \partial z/\partial\varphi \end{bmatrix}$$

$$\frac{\partial X}{\partial E} = \begin{bmatrix} -\sin\vartheta & 0 \\ \cos\vartheta\cos\varphi & -\sin\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & \sin\vartheta\cos\varphi \end{bmatrix}$$

N-DIM POLÁRKOORD

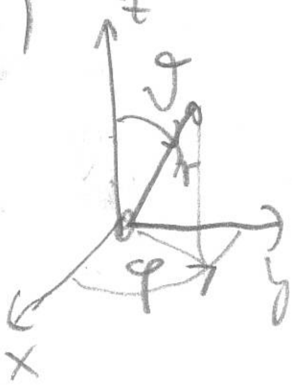
Emlékeztető $X : S \leftrightarrow \mathbb{R}^2$ DESCARTES KOORD

S/K

$P : S \setminus X^{-1}((-\infty,0]\times\{0\}) \to \mathbb{R}^2_{++}\times(-\pi,\pi)$

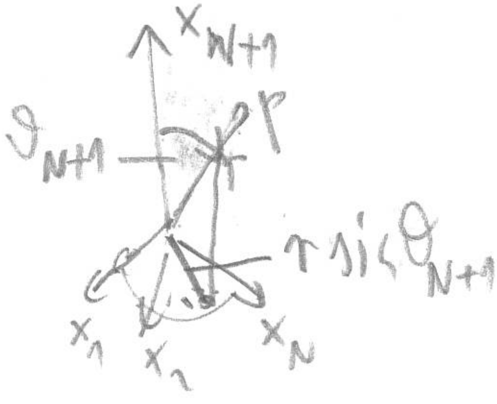
$$x = r\cos\varphi \qquad P = \begin{pmatrix} r \\ \varphi \end{pmatrix}$$
$$y = r\sin\varphi$$

3D (Euler-koord)



$$x = [r\sin\varphi]\cdot\sin\vartheta$$
$$y = [r\cos\varphi]\sin\vartheta$$
$$z = r\cos\varphi$$
$$0<\vartheta<\pi\,, \quad -\pi<\varphi<\pi$$

Konstrukció N-dim $\to$ (N+1)-dim polárkoord



$$x_{N+1} = r\cos\vartheta_{N+1} \to \text{új}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = (\sin\vartheta_{N+1})\cdot\begin{bmatrix} \text{RÉGI} \\ \vdots \\ \text{KOORD} \\ \text{N-DIM} \end{bmatrix}$$

Rekurzió $\to$ 4D-ben

$$x_1 = r\sin\vartheta_4\,[\sin\vartheta_3\cos\varphi]$$
$$x_2 = r\sin\vartheta_4\,[\sin\vartheta_3\sin\varphi]$$
$$x_3 = r\sin\vartheta_4\,[\cos\vartheta_3]$$
$$x_4 = r\cos\vartheta_4$$

5D-ben $x_1 = r\cdot\sin\vartheta_5\,[\sin\vartheta_4\,(\sin\vartheta_3\cdot\cos\varphi)]$

$x_2 = r\cdot\sin\vartheta_5\,[\sin\vartheta_4\,(\sin\vartheta_3\cdot\sin\varphi)]$

$x_3 = r\cdot\sin\vartheta_5\,[\sin\vartheta_4\cdot(\cos\vartheta_3)]$

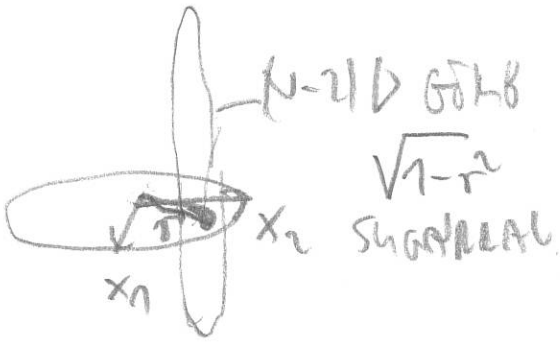
$x_4 = r\cdot\sin\vartheta_5\,[\cos\vartheta_4]$

$x_5 = r\cdot\cos\vartheta_5$

Jobb lenne: $(\cos\varphi, \sin\varphi)$ HELYETT $(\sin\vartheta_2, \cos\vartheta_2)$

VAGY $\sin\vartheta_h \leftrightarrow \cos\vartheta_h$ CSERE — GPS koord.

AZ N-DIM GÖMB TÉRFOGATA ω_N



$$\underline{\omega_N} = \int_{q\in(2D\text{ gömb})} \mathrm{Vol}_{N-2}(\sqrt{1-r^2}\text{ sug. }(N-2)D\text{ gömb})\,\mathrm{Vol}_N(dq) =$$

$$= \int_{r=0}^{1}\int_{\varphi=-\pi}^{\pi} \omega_{N-2}\cdot\left[\sqrt{1-r^2}\right]^{N-2} \underbrace{r\,d\varphi\,dr}_{dx_1dx_2} = \left|\begin{array}{l} s = 1-r^2 \\ ds = -2r\,dr \end{array}\right.$$

Jac $= r$ POLÁRK.

$$= 2\pi\cdot\int_{s=1}^{0} \omega_{N-2}\cdot s^{\frac{N-2}{2}}\cdot\left(-\frac{1}{2}\,ds\right) =$$

$$= \pi\,\omega_{N-2}\int_{s=0}^{1} s^{\frac{N-2}{2}}\,ds = \underline{\frac{\pi}{N/2}\cdot\omega_{N-2}}$$

Rekurzió: $\omega_1 = 2$, $\omega_2 = \pi$, $\pi_3 = \frac{\pi}{3/2}\cdot\omega_1 = \frac{4\pi}{3}$, $\pi_4 = \frac{\pi}{2}\cdot\omega_2 = \frac{\pi^2}{2}$

A GÖMB KONFORM (SZTEREOGRAFIKUS) TÉRKÉPE

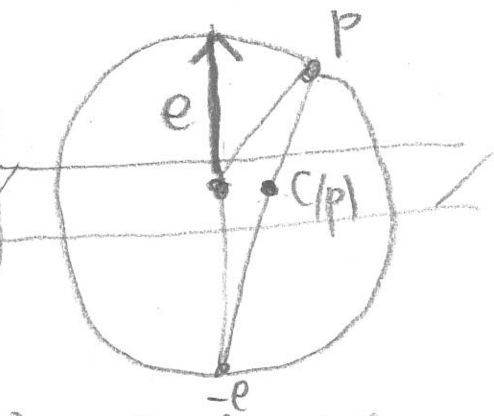
$$S = \left\{ \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \in \mathbb{R}^3 : \xi_1^2+\xi_2^2+\xi_3^2 = 1 \right\} \subset \mathbb{R}^3$$

$$X : \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \mapsto \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \; (!) \quad \text{D-koord} \quad x_k : \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \mapsto \xi_k$$

$$e = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \qquad C : S\setminus\{e\} \longleftrightarrow \mathbb{R}^2$$

$$p \mapsto \left[\frac{p+e}{1+\langle p|e\rangle} - e \quad \text{1-2 koord} \right]$$

(3. koord $\equiv 0$)



$\underline{\underline{T}}$ $t \mapsto p_1(t)$, $t \mapsto p_2(t)$ $(-\varepsilon,\varepsilon) \to S$ C^1-görbék, $p \mapsto x_k(p(t))$ FOLYT DIFF, $p_1(0) = p_2(0) = p_0$ közös pont

$q_1(t) := C(p_1(t))$, $q_2(t) := C(p_2(t))$, $\dot p_j, \dot q_j$ $\frac{d}{dt}p_j$ $\frac{d}{dt}q_j$

Ekkor $\measuredangle(\dot p_1(0), \dot p_2(0)) = \measuredangle(\dot q_1(0), \dot q_2(0))$

$\underline{\underline{D}}$ $\measuredangle(u,v) = \frac{\langle u|v\rangle}{\|u\|\cdot\|v\|}$ $\mathbb{R}^2, \mathbb{R}^3$-ban $\quad \|u\|^2 = \langle u|u\rangle = \langle u\rangle^2$

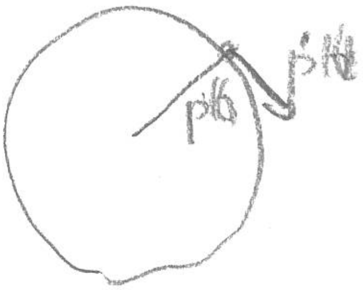
Belátandó: $\dfrac{\langle \dot p_1(0), \dot p_2(0)\rangle}{\langle \dot p_1(0)|\dot p_1(0)\rangle^{1/2} \langle \dot p_2(0)|\dot p_2(0)\rangle^{1/2}} = \Big[$ u.a. p-helyett q-val $\Big]$

$q(t) := C(p(t))$, $p(t) \in S$

$$\begin{bmatrix}\dot q(t)\\ 0\end{bmatrix} = \frac{d}{dt}\left[\frac{p(t)+e}{1+\langle p(t)|e\rangle} - e\right] = \frac{\dot p(t)}{1+\langle p(t)|e\rangle} - \frac{p(t)+e}{(1+\langle p(t)|e\rangle)^2}\langle \dot p(t)|e\rangle$$

$$= \frac{(1+\langle p(t)|e\rangle)\dot p(t) - \langle \dot p(t)|e\rangle (p(t)+e)}{(1+\langle p(t)|e\rangle)^2}$$

$$\begin{bmatrix}\dot q(t)\\ 0\end{bmatrix} = \frac{(1+\langle p_0|e\rangle)\,\dot p(0) - \langle \dot p(0)|e\rangle(p_0+e)}{(1+\langle p_0|e\rangle)^2}$$



Érvényes: $\dot p(t) \perp p(t)$

[biz: $\frac{d}{dt}\langle p(t)|p(t)\rangle = \frac{d}{dt}1 = 0$; $\frac{d}{dt}\langle p(t)|p(t)\rangle = 2\langle \dot p(t)|p(t)\rangle$]

$p \rightsquigarrow p_1, p_2 \qquad q \rightsquigarrow q_1, q_2$

$$\langle \dot q_i(0)|\dot q_j(0)\rangle = \frac{\langle (1+\langle p_0|e\rangle)\dot p_i(0) - \langle \dot p_i(0)|e\rangle(p_0+e) \,|\, \text{u.ez } i\leftrightarrow j\rangle}{(1+\langle p_0|e\rangle)^4}$$

$\dot p_1(0), \dot p_2(0) \perp p_0 = p_1(0) = p_2(0)$

[$\langle \dot q_i(0)|\dot q_j(0)\rangle$ számlálója] =

$= \langle (1+\langle p_0|e\rangle)\dot p_i(0) - \langle \dot p_i(0)|e\rangle(p_0+e) \,|$

$| (1+\langle p_0|e\rangle)\dot p_j(0) - \langle \dot p_j(0)|e\rangle(p_0+e)\rangle =$

$= (1+\langle p_0|e\rangle)^2\langle \dot p_i(0)|\dot p_j(0)\rangle + \langle \dot p_i(0)|e\rangle\langle \dot p_j(0)|e\rangle \overbrace{\langle p_0+e\rangle^2}^{1+2\langle p_0|e\rangle+1} -$

$-(1+\langle p_0|e\rangle)\left[\langle \dot p_i(0)|e\rangle\langle \dot p_j(0)|e\rangle + \langle \dot p_j(0)|e\rangle\langle \dot p_i(0)|e\rangle\right] =$

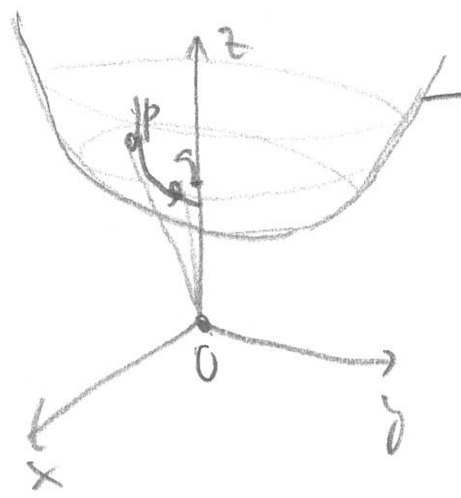
$= \underline{(1+\langle p_0|e\rangle)^2\langle \dot p_i(0)|\dot p_j(0)\rangle}$

$$\frac{\langle \dot q_i(0)|\dot q_j(0)\rangle}{\left[\langle \dot q_i(0)\rangle^2\cdot\langle \dot q_j(0)\rangle^2\right]^{1/2}} = \frac{(1+\langle p_0|e\rangle)^2\langle \dot p_i(0)|\dot p_j(0)\rangle}{(1+\langle p_0|e\rangle)^2\left[\langle \dot p_i(0)\rangle^2\langle \dot p_j(0)\rangle^2\right]^{1/2}}$$

$$\Rightarrow \frac{\langle \dot q_1(0)|\dot q_2(0)\rangle}{\|\dot q_1(0)\|\cdot\|\dot q_2(0)\|} = \frac{\langle \dot p_1(0)|\dot p_2(0)\rangle}{\|\dot p_1(0)\|\cdot\|\dot p_2(0)\|}$$

Qu. e. d.

HIPERBOLIKUS GEOMETRIA



$S = \left(z^2 - (x^2+y^2) = 1,\ z \geq 0\right)$

$= \{p \in \mathbb{R}^3 : z(p)^2 - x(p)^2 - y(p)^2 = 1,\ z(p) \geq 0\}$

$[p, q \in S$ egyenese$] =$

$= [\langle 0, p, q\rangle$ sík$] \cap S$

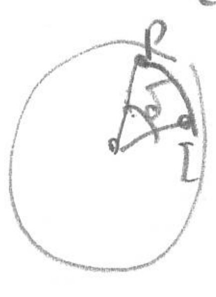
<u>Megj</u> gömbi geom-nál: $S = (z^2 + (x^2+y^2) = 1)$ volt,

$(p, q$ egyenese$) = [\langle 0, p, q\rangle$ sík$] \cap S$

Euler-koord:

$$\left.\begin{aligned} z(p) &= \cos\vartheta(p) \\ x(p) &= \sin\vartheta(p)\cos\varphi(p) \\ y(p) &= \sin\vartheta(p)\sin\varphi(p) \end{aligned}\right\} p \in S$$

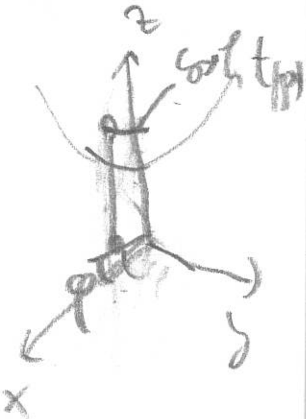
$[p, q$ távolsága$] = [p, q$ - ív hossza (x, y, z) szerint$] =$

$=$ 

$= \delta(p, q)$ (szög) $= \arccos\langle p | q\rangle$

<u>Hip-geom</u> Minkowski-koord ($\sim$ Euler gömbön)

$$\left.\begin{aligned} z(p) &= \cosh t(p) \\ x(p) &= \sinh t(p) \cdot \cos\varphi(p) \\ y(p) &= \sinh t(p) \cdot \sin\varphi(p) \end{aligned}\right\} p \in S$$



$$[p,q \text{ TÁVOLSÁGA}] = [p,q \text{ -hossza } S\text{-en}] =$$

$\langle\cdot|\cdot\rangle_M$ skalárszorzattal

Ez >0 az első koordináta

$$= \operatorname{arcosh} \langle p|q\rangle_{M}$$

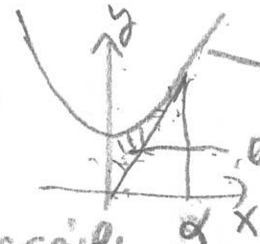
Minkowski

- $\langle p|q\rangle_M = z(p)\,z(q) - \big(x(p)\,x(q) + y(p)\,y(q)\big)$

- $\left\langle \begin{bmatrix}\xi_1\\ \eta_1\\ \zeta_1\end{bmatrix} \middle| \begin{bmatrix}\xi_2\\ \eta_2\\ \zeta_2\end{bmatrix}\right\rangle_M = \zeta_1\zeta_2 - (\xi_1\xi_2 + \eta_1\eta_2) \quad (\xi_i, \eta_i, \zeta_i \in \mathbb{R})$

$\langle p|p\rangle_M = 1 \quad (p\in S)$ sőt $S = \{p\in\mathbb{R}^3 : \langle p|p\rangle_M = 1,\ z(p)>0\}$

M AREA függvény:



$(y^2 - x^2 = 1) = (y = \sqrt{1+x^2})$

$\operatorname{arsinh}\alpha$ = TER. 2

$$\int_{x=0}^{\alpha} \sqrt{1+x^2}\,dx = \int_{t=0}^{\operatorname{arsinh}\alpha} \cosh^2 t\,dt = \circledast$$

$x = \sinh t$

$dx = \cosh t\,dt$

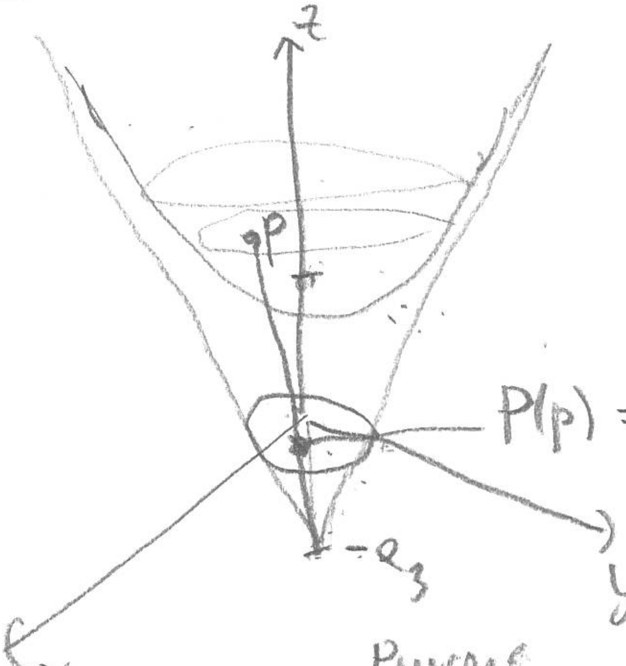
$\cosh^2 t - \sinh^2 t = 1$

$$\circledast = \int_{t=0}^{\operatorname{arsinh}\alpha} \frac{1}{2}(1+\cosh 2t)\,dt = \frac{1}{2}\operatorname{arsinh}\alpha + \frac{1}{4}\sinh(2\operatorname{arsinh}\alpha) =$$

$$= \frac{1}{2}\operatorname{arsinh}(\alpha) + \frac{1}{4}\cdot 2\sinh(\operatorname{arsinh}\alpha)\cosh(\operatorname{arsinh}\alpha) =$$

$$= \frac{1}{2}\operatorname{arsinh}(\alpha) + \frac{1}{2}\alpha\sqrt{1+\alpha^2}$$

Poincaré térkép



$P(p) = \left[\{p, -e_3\} \text{ egyenes} \cap xy \text{ sík}\right] =$

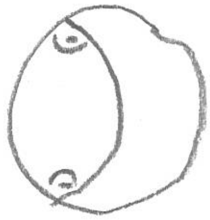
$= \left[\frac{p+e_3}{\cosh t(p)+1} - e_3 \quad xy \text{ KOORD}\right]$

$P = \begin{bmatrix} x_p \\ y_p \end{bmatrix}$ POINCARÉ

$$x_p = \frac{\sinh t \cdot \cosh\varphi}{1+\cosh t}$$

$$y_p = \frac{\sinh t \ \sinh\varphi}{1+\cosh t}$$

$\underline{\underline{T}}$ $P(S) = $ (NYITOTT EGYSÉGKÖR) $= \{q \in \mathbb{R}^2 : x(q)^2 + y(q)^2 < 1\}$

P(EGYENES) =  (egyenes $\perp$ körív)

KOMPLEX KOORD $\mathbb{C}$-N

M Vagyjelleg „komplexifikált koord"

$X = \begin{bmatrix} x \\ y \end{bmatrix} : \mathbb{C} \leftrightarrow \mathbb{R}^2$ $\left. \begin{matrix} x(\xi+i\eta) = \xi \\ y(\xi+i\eta) = \eta \end{matrix} \right\}$ $d(p,q) = |p-q|$ TÁV.

$Z : \begin{bmatrix} z \\ \bar{z} \end{bmatrix} : \mathbb{C}^2 \leftrightarrow \mathbb{C}^2$

$z := x+iy$ $\bar{z} := x-iy$

Értelmezés: FORMÁLISAN $\frac{\partial}{\partial z} z^k \bar{z}^l = k z^{k-1} \bar{z}^l$

Ha $P(x,y)$ polinom $\frac{\partial}{\partial \bar{z}} z^k \bar{z}^l = l z^k \bar{z}^{l-1}$

$\tilde{P}(z,\bar{z})$ ...

$x = (z+\bar{z})/2$ $y = (z-\bar{z})/(2i)$

LEHET-E

$$\frac{\partial f}{\partial Z} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial Z} = \frac{\partial f}{\partial X} \left(\frac{\partial Z}{\partial X}\right)^{-1} \quad \text{ÁLTALÁBAN}$$

$$\frac{\partial f}{\partial Z} = \begin{bmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar{z}} \end{bmatrix}$$ — KIFEJEZEMÉNY x,y-nal

$$\frac{\partial Z}{\partial X} = \begin{bmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial \bar{z}/\partial x & \partial \bar{z}/\partial y \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$\left(\frac{\partial Z}{\partial X}\right)^{-1} = \frac{1}{(-2i)} \begin{bmatrix} -i & -i \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -i/2 & +i/2 \end{bmatrix}$$

Direkt: $$\frac{\partial X}{\partial Z} = \begin{bmatrix} \partial(\frac{1}{2}z+\frac{1}{2}\bar{z})/\partial z & \partial(\frac{1}{2}z+\frac{1}{2}\bar{z})/\partial \bar{z} \\ \partial(\frac{1}{2i}z-\frac{1}{2i}\bar{z})/\partial z & \partial(\frac{1}{2i}z-\frac{1}{2i}\bar{z})/\partial \bar{z} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{bmatrix}$$

$$\boxed{\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}}$$