INTEGRÁLÁS KOORD SZERINT

$X : S \leftrightarrow \mathbb{R}^N$ <u>Descartes</u> koord

d TÁVOLSÁG $\longrightarrow \mathrm{Vol}_K$ k-dim TÉRFOGAT (tehát $K \geq 0$!)

$$\int_{p \in A} f(p) \mathrm{Vol}_N(dp) := \int_{\begin{bmatrix} x_1 \\ x_N \end{bmatrix} \in A} {}^X f(x_1, \dots, x_N)\, dx_1 \cdots dx_N$$

<u>T</u> $Y : \underset{S}{\underset{\cap}{A}} \leftrightarrow G \subset \mathbb{R}^N$ új koord A-n

$$\int_{p \in A} f(p) \mathrm{Vol}_N(dp) = \int_{\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \in G} {}^Y f(y_1, \dots, y_N) \left| \det \frac{\partial X}{\partial Y} \right| dy_1 \cdots dy_N$$

<u>Pl</u> $\mathcal{I} := \int_{x=-\infty}^{\infty} e^{-x^2} dx$ kiszámítás

$$\mathcal{I}^2 = \int_{x=-\infty}^{\infty} e^{-x^2} dx \int_{y=-\infty}^{\infty} e^{-y^2} dy = \int_{(x,y) \in \mathbb{R}^2} e^{-(x^2+y^2)} dx\, dy$$

$P = \begin{bmatrix} r \\ \varphi \end{bmatrix}$ polárkoord $\quad x = r\cos\varphi,\ y = r\sin\varphi \quad \frac{\partial X}{\partial P} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$

$$\mathcal{I}^2 = \int_{p \in \mathbb{R}^2} e^{-(x^2+y^2)(p)} \mathrm{Vol}_2(dp) = \int_{\substack{r>0 \\ -\pi<\varphi<\pi}} e^{-r^2} \underbrace{\left| \det \frac{\partial X}{\partial P} \right|}_{r} dr\, d\varphi =$$

$$= 2\pi \int_{r=0}^{\infty} e^{-r^2} \cdot r\, dr \underset{\substack{s=r^2 \\ ds = 2r dr}}{=} 2\pi \int_{s=0}^{\infty} e^{-s} \underbrace{\cdot \frac{1}{2} ds}_{1/2} = \underline{\pi}$$

$$\underline{\underline{\mathcal{I} = \sqrt{\pi}}}$$

A PARC. DERIVÁLTAK A TELJES KOORD-TÓL FÜGGNEK

$X : S \leftrightarrow G \subset \mathbb{R}^N$, $Y : S \leftrightarrow H \subset \mathbb{R}^N$

Y_{x_i} , X_{y_j} folyt diff

Tudjuk: $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial X} = \frac{\partial f}{\partial Y}\left[\frac{\partial X}{\partial Y}\right]^{-1}$

Előfordulhat: $x_1 = y_1$ (ált. $x_{i_0} = y_{j_0}$)

Ennek ellenére ÁLTALÁBAN $\frac{\partial f}{\partial x_1} \neq \frac{\partial f}{\partial y_1}$ HA X, ill Y szerint számoljuk

$$\frac{\partial f}{\partial x_1} = \left[\frac{\partial f}{\partial Y}\frac{\partial Y}{\partial X}\ 1.\text{ tagja}\right] = \frac{\partial f}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \dots + \frac{\partial f}{\partial y_N}\frac{\partial y_N}{\partial x_1}$$

Pl $X : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ $x_1\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \xi$, $x_2\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \eta$

$Y : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ $y_1\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \xi$, $y_2\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \xi + \eta$ $x_1 = y_1$, $y_2 = x_1 + x_2$

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f}{\partial y_2}\frac{\partial y_2}{\partial x_1} = \frac{\partial f}{\partial y_1}\cdot 1 + \frac{\partial f}{\partial y_2}\cdot 1$$

$f := x_2 = y_2 - y_1 \Rightarrow \frac{\partial f}{\partial x_1} = \frac{\partial x_2}{\partial x_1} = \underline{\underline{0}}$, $\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial y_2} = 0 - 1 = \underline{\underline{-1}}$

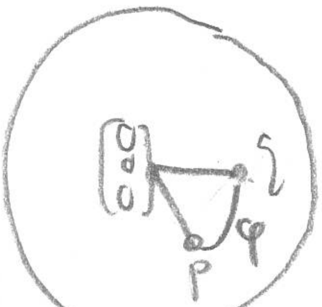
GÖMBI GEOMETRIA

$S := \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3 : \xi^2 + \eta^2 + \zeta^2 = 1 \right\} = (x^2 + y^2 + z^2 = 1),$

ahol $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \mathbb{R}^3 \leftrightarrow \mathbb{R}^3$ ALAPKOORD $x \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \xi$ STB.

d_S távolság S-en:

$d_S(p,q) := \begin{bmatrix} \text{legrövidebb } p\text{-t } q\text{-val összekötő} \\ S\text{-beli görbe hossza} \end{bmatrix} =$



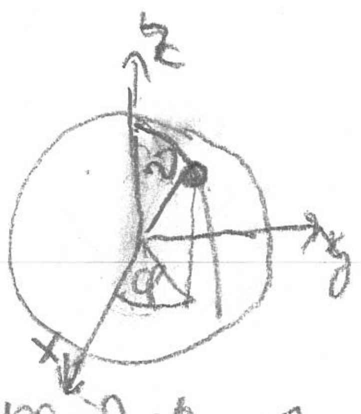
$= \arccos \langle X(p), X(q) \rangle =$

$= \arccos [x(p)x(q) + y(p)y(q) + z(p)z(q)]$

M NEM TRIV!

Euler koord

$E : \begin{bmatrix} \cos\vartheta \\ \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \end{bmatrix} \mapsto \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix}$



$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X : \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$

$\frac{\partial X}{\partial E} = \begin{bmatrix} \partial x/\partial\vartheta & \partial x/\partial\varphi \\ \partial y/\partial\vartheta & \partial y/\partial\varphi \\ \partial z/\partial\vartheta & \partial z/\partial\varphi \end{bmatrix}$

$\frac{\partial X}{\partial E} = \begin{bmatrix} -\sin\vartheta & 0 \\ \cos\vartheta\cos\varphi & -\sin\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & \sin\vartheta\cos\varphi \end{bmatrix}$

K-dim FELÜLET K-dim TERÜLETE

$X : \mathbb{R}^N \leftrightarrow \mathbb{R}^N$ természetes D-koord $X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = Id$

d természetes TÁV $\mathbb{R}^N$-en

$S \subset \mathbb{R}^N$ K-dim felület

$S = \left\{ F\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} : \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in G \right\}$ $F : G \leftrightarrow S$

$F^{-1} : S \leftrightarrow G$ <u>K-dim KOORD</u> <u>F TERKÉP</u> $\begin{bmatrix} F_1(t_1, \dots t_k) \\ \vdots \\ F_N(t_1, \dots t_k) \end{bmatrix}$

$x_1, \dots, x_N | S$ megszorítás N db koord fgv

$\underline{\underline{T}}$ $f : S \to \mathbb{R}$ $f \circ F$ folyt DIFF

$F_i = {}^F(x_i|S)$ folyt diff, $\frac{\partial X}{\partial F} = \left[\frac{\partial F_i}{\partial t_j} \right]_{i=1\, j=1}^{N\ \ K}$ foká rangja $=k$

$$\int_{p \in S} f(p)\, d\,\mathrm{Vol}_k(p) =$$

$$= \int_{(t_1, \dots, t_k)^T \in G} {}^F f(t_1, \dots, t_k) \left\{ \det \left(\left[\frac{\partial X}{\partial F} \right]^T \left[\frac{\partial X}{\partial F} \right] \right) \right\}^{1/2} dt_1 \dots dt_k$$

<u>Megjegyzés</u> vehető $X|S$ megszorítás is, mint „TÁV KOORDINÁTÁZÁS"

<u>PL</u> EGYSÉG HÁROMSZÖG TERÜLETE

(Nagyon nem triv) <u>Egyszerűbb 3-ad GÖMB</u>

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \cos\varphi_0 \\ \sin\theta_0 \sin\varphi_0 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{bmatrix} \perp$$

$$0 = \cos\theta_0\cos\theta + \sin\theta_0\cos\varphi_0\sin\theta\cos\varphi + \sin\theta_0\sin\varphi_0\sin\theta\sin\varphi =$$
$$= \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\left[\cos\varphi_0\cos\varphi + \sin\varphi_0\sin\varphi\right] =$$
$$= \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi-\varphi_0)$$

$0 < \varphi < \Phi$ FIXED

$\theta := \theta(\varphi)$ $\quad \cos\theta_0\cos\theta(\varphi) + \sin\theta_0\sin\theta(\varphi)\cos(\varphi-\varphi_0) = 0$

$$1 + \operatorname{tg}\theta_0 \operatorname{tg}\theta(\varphi)\cos(\varphi-\varphi_0) = 0$$

$$\operatorname{tg}\theta(\varphi) = \frac{\operatorname{ctg}\theta_0}{\cos(\varphi-\varphi_0)}$$

$$T^{-1}: \begin{bmatrix} \theta \\ \varphi \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{bmatrix}$$

$$\frac{\partial X}{\partial T} = \begin{bmatrix} -\sin\theta & 0 \\ \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \end{bmatrix} \qquad \left[\frac{\partial X}{\partial T}\right]^T \left[\frac{\partial X}{\partial T}\right] = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

$$\underline{\underline{\text{Area}}} = \int_{\varphi=0}^{\Phi} \int_{\theta=0}^{\theta(\varphi)} \sin\theta \, d\theta \, d\varphi = \int_{\varphi=0}^{\Phi} \left[\cos\theta(\varphi) - 1\right] d\varphi =$$

$$= \int_{\varphi=0}^{\Phi} \frac{d\varphi}{\sqrt{1 + \frac{\operatorname{ctg}^2\theta_0}{\cos^2(\varphi-\varphi_0)}}} = \underline{\underline{\left(\Sigma \text{ SZÖGEK}\right) - \pi}}$$

KOMPLEX KOORD $\mathbb{C}$-N

M Vegyük a „komplexifikált koord"

$X=\begin{bmatrix}x\\y\end{bmatrix}:\mathbb{C}\leftrightarrow\mathbb{R}^2$ $\left.\begin{matrix}x(\xi+i\eta)=\xi\\ y(\xi+i\eta)=\eta\end{matrix}\right\}$ $d(p,q)=|p-q|$ Táv.

$Z:\begin{bmatrix}z\\ \bar z\end{bmatrix}:\mathbb{C}^2\leftrightarrow\mathbb{C}^2$

$z:=x+iy$ $\bar z:=x-iy$

Értelmezés: FORMÁLISAN $\frac{\partial}{\partial z}z^k\bar z^l=k\,z^{k-1}\bar z^l$

Tehát $P(x,y)$ polinom $\frac{\partial}{\partial \bar z}z^k\bar z^l=l\,z^k\bar z^{l-1}$

$\tilde P(z,\bar z)$ sthh

$x=(z+\bar z)/2$ $y=(z-\bar z)/(2i)$

LEHET-E

$$\frac{\partial f}{\partial Z}=\frac{\partial f}{\partial X}\frac{\partial X}{\partial Z}=\frac{\partial f}{\partial X}\left(\frac{\partial Z}{\partial X}\right)^{-1} \quad \text{ALAKÚ}$$

$$\frac{\partial f}{\partial Z}=\begin{bmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar z}\end{bmatrix}$$ — KIFEJEZHETŐ x,y-nal

$$\frac{\partial Z}{\partial X}=\begin{bmatrix}\partial z/\partial x & \partial z/\partial y\\ \partial\bar z/\partial x & \partial\bar z/\partial y\end{bmatrix}=\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}$$

$$\left(\frac{\partial Z}{\partial X}\right)^{-1}=\frac{1}{(-2i)}\begin{bmatrix}-i & -i\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1/2 & 1/2\\ -i/2 & +i/2\end{bmatrix}$$

Bizonyítás: $$\frac{\partial X}{\partial Z}=\begin{bmatrix}\partial(\frac12 z+\frac12\bar z)/\partial z & \partial(\frac12 z+\frac12\bar z)/\partial\bar z\\ \partial(\frac{1}{2i}z-\frac{1}{2i}\bar z)/\partial z & \partial(\frac{1}{2i}z-\frac{1}{2i}\bar z)/\partial\bar z\end{bmatrix}=\begin{bmatrix}1/2 & 1/2\\ -i/2 & i/2\end{bmatrix}$$

$$\boxed{\frac{\partial f}{\partial z}=\frac12\frac{\partial f}{\partial x}-\frac{i}{2}\frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial \bar z}=\frac12\frac{\partial f}{\partial x}+\frac{i}{2}\frac{\partial f}{\partial y}}$$

A GÖMB KONFORM (SZTEREOGRAFIKUS) TÉRKÉPE

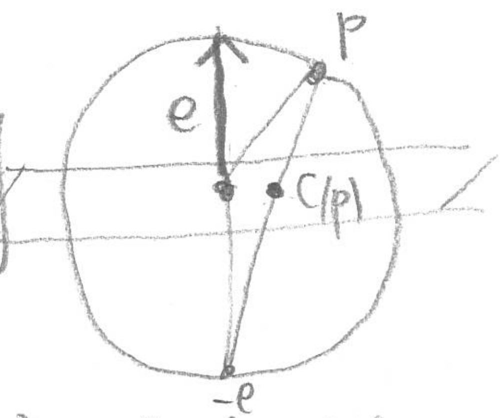
$S = \left\{ \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \in \mathbb{R}^3 : \xi_1^2+\xi_2^2+\xi_3^2 = 1 \right\} \subset \mathbb{R}^3$

$X : \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \mapsto \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}$ (!) D-koord $x_k : \begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} \mapsto \xi_k$

$e = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $C : S\setminus\{e\} \longleftrightarrow \mathbb{R}^2$

$p \mapsto \left[\frac{p+e}{1+\langle p|e\rangle} - e\right]$ 1-2 koord

(3. koord $\equiv 0$)



T: $t\mapsto p_1(t)$, $t\mapsto p_2(t)$ $(-\varepsilon,\varepsilon)\to S$ C^1-görbék,

$p \mapsto x_k(p(t))$ FOLYT DIFF, $p_1(0) = p_2(0) = p_0$ közös pont

$q_1(t) := C(p_1(t))$, $q_2(t) := C(q_2(t))$, $\dot p_j, \dot q_j$ $\frac{d}{dt}p_j$ $\frac{d}{dt}q_j$

Ekkor $\sphericalangle(\dot p_1(0), \dot p_2(0)) = \sphericalangle(\dot q_1(0), \dot q_2(0))$

B: $\sphericalangle(u,v) = \frac{\langle u|v\rangle}{\|u\|\cdot\|v\|}$ $\mathbb{R}^2, \mathbb{R}^3$-ban $\|u\|^2 = \langle u|u\rangle = \langle u\rangle^2$

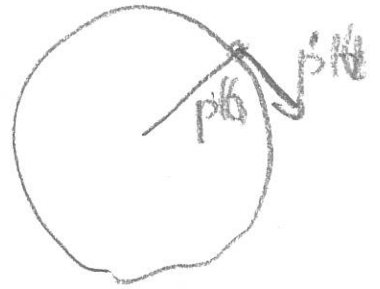
Bizonyítás: $\frac{\langle \dot p_1(0), \dot p_2(0)\rangle}{\langle \dot p_1(0)|\dot p_1(0)\rangle^{1/2} \langle \dot p_2(0)|\dot p_2(0)\rangle^{1/2}} =$ [uez p-helyett q-val]

$q(t) := C(p(t))$, $p(t) \in S$

$\begin{bmatrix}\dot q(t)\\ 0\end{bmatrix} = \frac{d}{dt}\left[\frac{p(t)+e}{1+\langle p(t)|e\rangle} - e\right] = \frac{\dot p(t)}{1+\langle p(t)|e\rangle} - \frac{p(t)+e}{(1+\langle p(t)|e\rangle)^2}\langle \dot p(t)|e\rangle$

$$= \frac{(1+\langle p(t)|e\rangle)\dot p(t) - \langle \dot p(t)|e\rangle (p(t)+e)}{(1+\langle p(t)|e\rangle)^2}$$

$$\begin{bmatrix}\dot q(t)\\ 0\end{bmatrix} = \frac{(1+\langle p_0|e\rangle)\,\dot p(0) - \langle \dot p(0)|e\rangle(p_0+e)}{(1+\langle p_0|e\rangle)^2}$$



Ersichtlich: $\dot p(t) \perp p(t)$

$$\left[\text{bis: } \frac{d}{dt}\langle p(t)|p(t)\rangle = \frac{d}{dt}1 = 0\right]$$
$$\frac{d}{dt}\langle p(t)|p(t)\rangle = 2\langle \dot p(t)|p(t)\rangle$$

$p \rightsquigarrow p_1, p_2 \qquad q \rightsquigarrow q_1, q_2$

$$\langle \dot q_i(0)|\dot q_j(0)\rangle = \frac{\langle (1+\langle p_0|e\rangle)\dot p_i(0) - \langle \dot p_i(0)|e\rangle(p_0+e) \,|\, \text{und } i\leftrightarrow j\rangle}{(1+\langle p_0|e\rangle)^4}$$

$$\dot p_1(0),\ \dot p_2(0) \perp p_0 = p_1(0) = p_2(0)$$

$$[\langle \dot q_i(0)|\dot q_j(0)\rangle \text{ [illegible]}] =$$

$$= \langle (1+\langle p_0|e\rangle)\dot p_i(0) - \langle \dot p_i(0)|e\rangle(p_0+e) \,|\, (1+\langle p_0|e\rangle)\dot p_j(0) - \langle \dot p_j(0)|e\rangle(p_0+e)\rangle =$$

$$= (1+\langle p_0|e\rangle)^2\langle \dot p_i(0)|\dot p_j(0)\rangle + \langle \dot p_i(0)|e\rangle\langle \dot p_j(0)|e\rangle \overbrace{\langle p_0+e\rangle^2}^{1+2\langle p_0|e\rangle+1} -$$

$$-(1+\langle p_0|e\rangle)\left[\langle \dot p_i(0)|e\rangle\langle \dot p_j(0)|e\rangle + \langle \dot p_j(0)|e\rangle\langle \dot p_i(0)|e\rangle\right] =$$

$$= \underline{(1+\langle p_0|e\rangle)^2\langle \dot p_i(0)|\dot p_j(0)\rangle}$$

$$\frac{\langle \dot q_i(0)|\dot q_j(0)\rangle}{[\langle \dot q_i(0)\rangle^2\cdot\langle \dot q_j(0)\rangle^2]^{1/2}} = \frac{(1+\langle p_0|e\rangle)^2\langle \dot p_i(0)|\dot p_j(0)\rangle}{(1+\langle p_0|e\rangle)^2\,[\langle \dot p_i(0)\rangle^2\langle \dot p_j(0)\rangle^2]^{1/2}}$$

$$\Longrightarrow \frac{\langle \dot q_1(0)|\dot q_2(0)\rangle}{\|\dot q_1(0)\|\cdot\|\dot q_2(0)\|} = \frac{\langle \dot p_1(0)|\dot p_2(0)\rangle}{\|\dot p_1(0)\|\cdot\|\dot p_2(0)\|} \qquad \text{Qu. e. d.}$$