$\underline{\pi}$ – LEIBNIZ-formulája

$$\frac{\pi}{4} = \operatorname{arctg}(1) = \int_{x=0}^{1} \left(\frac{d}{dx}\operatorname{arctg} x\right) dx = \int_{x=0}^{1} \frac{1}{1+x^2}\, dx =$$

$$= \int_{x=0}^{1} \frac{1}{1-(-x^2)}\, dx = \int_{x=0}^{1} \sum_{n=0}^{\infty} (-x^2)^n \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_{x=0}^{1} x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = \underline{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

$\underline{M}$ miért $\frac{d}{dx}\operatorname{arctg} x = \frac{1}{1+x^2}$?

$g\ [= \operatorname{arctg}]$ $\quad g(\operatorname{tg} t) = t \quad \Big| \frac{d}{dt}$

$$g'(\operatorname{tg} t)(\operatorname{tg} t)' = 1$$

$$g'(\operatorname{tg} t)\underbrace{\left(\frac{\sin t}{\cos t}\right)'}_{} = 1$$

$$\frac{\cos t \cos t - \sin t(-\sin t)}{\cos^2 t} = 1 + \operatorname{tg}^2 t$$

$$g'(\operatorname{tg} t) = \frac{1}{1+\operatorname{tg}^2 t} \qquad x \sim \operatorname{tg} t$$

Newton LASSÚ [illegible]

Gyorsabb sor $\quad \frac{\pi}{6} = \arcsin\frac{1}{2} = \int_{t=0}^{1/2} \arcsin'(t)\, dt = \int_{t=0}^{1/2} \frac{dt}{\sqrt{1-t^2}} = \int_{t=0}^{1/2} \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n}\, dt =$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(1/2)^{2n+1}}{2n+1} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1/2)(-1/2-1)\cdots[-\frac{1}{2}-n+1]}{n!\,(2n+1)} \cdot \frac{(-1)^n}{2^{2n+1}}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n)} \cdot \frac{1}{2^{2n+1}} \cdot \frac{1}{2n+1}$$

AFFIN LEKÉPEZÉSEK, PARALLELOGRAMMÁK

$X: S \leftrightarrow \mathbb{R}^N$ $\bar{X}: \bar{S} \leftrightarrow \mathbb{R}^K$ D. koord.

Def $T: S \to \bar{S}$ affin: egyenes marad → egy. mar.
egy.-t osztópont → egy.-ti oszt.

L T affin $\Leftrightarrow$ s felezőpont (p, q) → $T(s)$ felezőpont ($T(p)$, $T(q)$)

B felezőpont inv → $0, \frac{1}{2}, 1$ inv → $k/2^n$ osztópont; $k/2^n$-inv

d folyt → λ, $1-\lambda$ inv

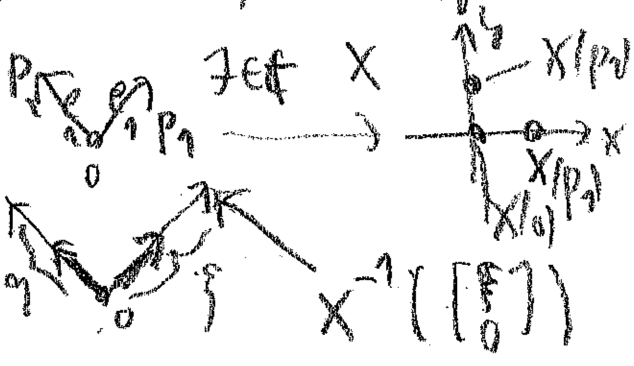
[**M** Ha T: $\frac{1}{2}, \frac{1}{2}$ → … ill. $\frac{1}{3}, \frac{2}{3}$ → … akkor m/f felezőpont inv "folyt." nélkül hiba]

Def $[a, b, c, d] \in S^4$ PARALELOGRAMMA: $\frac{1}{2}(X(a)+X(c)) = \frac{1}{2}(X(b)+X(d))$

L T affin → PAR $\xrightarrow{T}$ PAR

T T affin $\Rightarrow$ $\underline{X(T(p)) = A X(p) + b}$ $(p \in S)$ A tetsz. mátrix

B (2D-ben, de ugyanígy megy N dim-ben is)



$\left[X^{-1}\begin{bmatrix}0\\0\end{bmatrix}, X^{-1}\begin{bmatrix}\xi\\0\end{bmatrix}, X^{-1}\begin{bmatrix}\xi\\\eta\end{bmatrix}, X^{-1}\begin{bmatrix}0\\\eta\end{bmatrix}\right]$ PAR

$\xrightarrow{T}$ $\Rightarrow$ is PAR

innen algebra ismert

$\mathbb{R}^N$ mint eukl tér, ALG STRUKTÚRÁI

Kanonikus koord: $x_1 : \begin{bmatrix}\xi_1\\ \vdots\\ \xi_N\end{bmatrix} \mapsto \xi_1,\ x_2 : \begin{bmatrix}\xi_1\\ \vdots\\ \xi_N\end{bmatrix} \mapsto \xi_2, \dots, x_N : \begin{bmatrix}\xi_1\\ \vdots\\ \xi_N\end{bmatrix} \mapsto \xi_N$

$$d\left(\begin{bmatrix}\xi_1\\ \vdots\\ \xi_N\end{bmatrix}, \begin{bmatrix}\eta_1\\ \vdots\\ \eta_N\end{bmatrix}\right)^2 = \sum_{k=1}^{N} (\xi_k - \eta_k)^2 = \sum_{k=1}^{N} [x_k(\xi) - x_k(\eta)]^2$$

$X = \begin{bmatrix}x_1\\ \vdots\\ x_N\end{bmatrix} : \underbrace{\mathbb{R}^N}_{\xi} \to \mathbb{R}^N$ D-koord

Alappontok $= \{0, e_1, \dots, e_N\}$ — kanonikus bázis.

I (Tudjuk) távtartó leképezések: $\xi \mapsto \mathrm{ORT}\cdot\xi + b$

AFFIN leképezés $\xi \mapsto A\xi + b$

VEKTORTÉR LIN KOORD

V vektortér ($\dim V = N < \infty$)

$X : V \leftrightarrow \mathbb{R}^N$ lin koord: $X(\alpha u + \beta v) = \alpha X(u) + \beta X(v)$

$X : V \to \mathbb{R}^N$ LIN leképezés

I $X : V \to \mathbb{R}^N$ N-dim lin koord $\Leftrightarrow$

$\exists (v_1, \dots, v_N)$ bázis $\subset V$ $\quad X^{-1}\begin{bmatrix}\xi_1\\ \vdots\\ \xi_N\end{bmatrix} = \xi_1 v_1 + \dots + \xi_N v_N$

b LINEÁRIS izomorf

K $X = \begin{bmatrix}v_1^*\\ \vdots\\ v_N^*\end{bmatrix}$ $\quad v_1^*, \dots, v_N^*$ LINFUNKC lin $V \to \mathbb{R}$

I* $T : V \to W$ lin, $\exists X : V \leftrightarrow \mathbb{R}^K,\ Y : W \leftrightarrow \mathbb{R}^N$ lin koord

$\exists r \le K, N$ $\quad y_k(Tv) = x_k(v)$ $(k = 1, \dots, r)$

rank(T) $\quad = 0$ ha $k > r$

Áll T^{**} $V, \bar V$ vektortér $d, \bar d$ fv Eukl

(Eukl: $\exists\, X : V \leftrightarrow \mathbb{R}^N$ $d(p,q)^2 = \sum_{k=1}^{N} [x_k(p) - x_k(q)]^2$ D-koord)

$T: V \to \bar V$ lin $\Rightarrow$ $\exists\, Y, \bar Y$ D-koord, $\exists \lambda_1, \dots, \lambda_r > 0$ $r \le k, N$

$$y_k(Tv) = \lambda_k(v) \quad k \le r$$
$$= 0 \quad k > r$$

Def $T : S_1 \to \bar S_1$ $X : S_1 \leftrightarrow \mathbb{R}^k$ $\bar X : \bar S \leftrightarrow \mathbb{R}^N$ (koord)

T reprezentációja $X, \bar X$ szerint $\begin{bmatrix}\xi_1\\ \vdots \\ \xi_k\end{bmatrix} \mapsto \bar X\left(T\left(X^{-1}\begin{bmatrix}\xi_1\\ \vdots \\ \xi_k\end{bmatrix}\right)\right)$

$$\bar X \circ T \circ X^{-1} : \xi \xmapsto{X^{-1}} X^{-1}(\xi) \xmapsto{T} T(X(\xi)) \xmapsto{X} \bar X \circ T \circ X^{-1}(\xi)$$

szám$_n$ pontok pontok szám$_n$

T^*, T^{**} $\Rightarrow$ lin leképezés LEGEGYSZERŰBB REPR.

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \end{bmatrix} \xrightarrow{\text{lin}} \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & \end{bmatrix}$$ mátrix diag

P V vt. $X : V \leftrightarrow \mathbb{R}^N$ lin koord

$T_1, T_2 : V \to V$ lin $\Rightarrow$

$[T_1 T_2\ X\text{-repr}] = [T_1\ X\text{-repr}][T_2\ X\text{-repr}]$ mátrix szorzat

T (JORDAN) V vt. $T : V \to V$ lin $\Rightarrow$

$\exists\, X : V \leftrightarrow \mathbb{R}^N$ lin koord $[T\ X\text{-repr}] = \bigoplus \boxed{\text{Jordan blokkok}}$

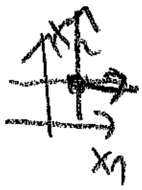
T DET KOORFGTL: $L : V \to V$ lin, $Y, X : V \leftrightarrow \mathbb{R}^N$ lin koord $\Rightarrow$

$$\det(XLX^{-1}) = \det\left((XY^{-1})YLY^{-1}(XY^{-1})^{-1}\right) = \det(YLY^{-1})$$

PARCIÁLIS DERIVÁLTAK

$X : S \leftrightarrow G \subset \mathbb{R}^N$ egyenes koord : G nyílt $\subset \mathbb{R}^N$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ $\quad p \in S$ -ből induló k-ik koord-vonal



$$t \mapsto p_t = \begin{bmatrix} x_1(p) \\ \vdots \\ x_{k-1}(p) \\ x_k(p)+t \\ x_{k+1}(p) \\ \vdots \\ x_N(p) \end{bmatrix}$$

Pl $S = \mathbb{R}^N$ $\quad x_k : \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \mapsto \xi_k$ $\quad$ 0-ból induló koordv. = TENGELYEK

$$f = x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} \qquad \frac{\partial}{\partial x_k} f = x_1^{n_1} \cdots x_{k-1}^{n_{k-1}} \left(n_k x_k^{n_k - 1} \right) x_{k+1}^{n_{k+1}} \cdots x_N^{n_N}$$

Észrevétel: $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N$ FIX mellett diff x_k szerint

$$[\text{Ez a } p \text{ helyen}] = \left.\frac{d}{dt}\right|_{t=0} [f(p_t)]$$

a p-ből induló k-ik koord vonalon

$$p_t = X^{-1}(X(p) + t e_k) \qquad e_k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow k$$

Def $X : S \leftrightarrow G \subset \mathbb{R}^N$ egyenes koord

$f : S \to \mathbb{R}$ fgv

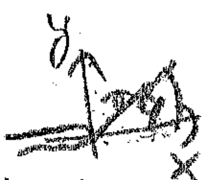
$$[f \text{ REPR: } X \text{ szerint}] := f \circ X^{-1} \qquad \text{szám}_N \xrightarrow{X^{-1}} \text{pont} \xrightarrow{f} \mathbb{R}$$

$$\frac{\partial f}{\partial x_k} = [f \circ X^{-1} \ k\text{-ik változó szerint diff}] =$$

$$= \left[p \mapsto \left.\frac{d}{dt}\right|_{t=0} f(p_t) \right] = \left[p \mapsto \left.\frac{d}{dt}\right|_{t=0} f \circ X^{-1}(X(p) + t e_k) \right]$$

Pl POLÁRKOORD

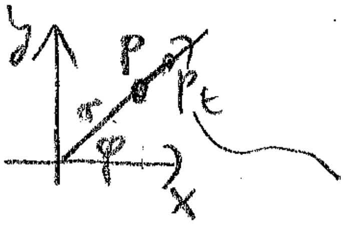
$\begin{bmatrix} x\\ y\end{bmatrix} = X : S \leftrightarrow \mathbb{R}^2$ D-koord



$\begin{bmatrix} r\\ \varphi\end{bmatrix} = P : S \setminus \{\text{NEGATÍV } x\text{-TENGELY}\} \leftrightarrow (0,\infty)\times(-\pi,\pi)$ polár koord

$\parallel P$ X-től függ

Hogyan kell kiszámítani $\begin{bmatrix}\partial f/\partial r\\ \partial f/\partial\varphi\end{bmatrix}$-t $\begin{bmatrix}\partial f/\partial x\\ \partial f/\partial y\end{bmatrix}$ alapján (ha ismert)?



ADOTT $\overset{x}{f} = f\circ X^{-1}$ f KIFEJEZVE $\binom{x}{y}$-nal

$$\frac{\partial f}{\partial r}(p) : \frac{d}{dt}\Big|_{t=0} \underbrace{f(p_t)}_{f\circ X^{-1}\circ X(p_t)} = \frac{d}{dt}\Big|_{t=0} \overset{x}{f}\begin{pmatrix}(r(p)+t)\cos\varphi(p)\\(r(p)+t)\sin\varphi(p)\end{pmatrix}$$

f KIF X-nel $\qquad p_t$ KIF X-nel

$D_1 =$ (1-ső vált. szerinti derivált), $D_2 =$ (2-ik ...)

$$\frac{\partial f}{\partial r}(p) = \underbrace{D_1\overset{x}{f}(p)}_{\frac{\partial f}{\partial x}(p)}\cos\varphi(p) + \underbrace{D_2\overset{x}{f}(p)}_{\frac{\partial f}{\partial y}(p)}\cdot\sin\varphi(p) = \cos\varphi(p)\frac{\partial f}{\partial x}(p) + \sin\varphi(p)\frac{\partial f}{\partial y}(p)$$

$$\frac{\partial f}{\partial r} = \cos\varphi\cdot\frac{\partial f}{\partial x} + \sin\varphi\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \varphi}(p) : \frac{d}{dt}\Big|_{t=0} f(q_t) = \frac{d}{dt}\Big|_{t=0}\overset{x}{f}\begin{pmatrix} r(p)\cos(\varphi(p)+t)\\ r(p)\sin(\varphi(p)+t)\end{pmatrix} =$$

$$= D_1\overset{x}{f}(p)[-r(p)\sin\varphi(p)] + D_2\overset{x}{f}(p)[r(p)\cos\varphi(p)]$$

$$\frac{\partial f}{\partial \varphi} = -\sin\varphi\cdot\frac{\partial f}{\partial x} + \cos\varphi\frac{\partial f}{\partial y}$$

$$\begin{bmatrix}\frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi}\end{bmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}\begin{bmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{bmatrix}$$

Kérdés: mi $\begin{bmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{bmatrix}$ jelentése?

$$\left.\begin{aligned} x(p) &= r(p)\cos\varphi(p)\\ y(p) &= r(p)\sin\varphi(p)\end{aligned}\right\} \qquad \left.\begin{aligned} x &= r\cos\varphi\\ y &= r\sin\varphi\end{aligned}\right\} \begin{bmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{bmatrix} = \begin{bmatrix}\partial x/\partial r & \partial x/\partial\varphi\\ \partial y/\partial r & \partial y/\partial\varphi\end{bmatrix}$$

Koordináta-transzformáció

$X: S \leftrightarrow G \subset \mathbb{R}^N$ $\quad Y: S \leftrightarrow H \subset \mathbb{R}^N$ rögzített koord.

Y kifejezése X-szel $\quad G \ni \text{szám}_N \xmapsto{X^{-1}} \underset{S\text{-ben}}{\text{pont}} \xmapsto{Y} \text{szám}_N \in H \;\Big|\; Y\circ X^{-1}$

Pl. $\underbrace{\text{D-koord}}_{X}$, $\underbrace{\text{Polár-koord}}_{P}$ $\quad \begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}x(p)\\ y(p)\end{bmatrix}$

$$X_P = \begin{bmatrix}X_r\\ X_\varphi\end{bmatrix} = \begin{bmatrix}\sqrt{x^2+y^2}\\ \operatorname{arctg} y/x\end{bmatrix} \qquad \left.\begin{aligned} x(p) &= r(p)\cos\varphi(p)\\ y(p) &= r(p)\sin\varphi(p)\end{aligned}\right\} \qquad \begin{aligned}\varphi(p) &= \operatorname{arctg}\frac{y(p)}{x(p)}\\ r(p) &= \sqrt{x(p)^2+y(p)^2}\end{aligned}$$

Ált is: $\dfrac{\partial y_i}{\partial x_k} = \underbrace{D_k\, {}^X y_i}$ k-ik vált. szerint diff

$f: S \to \mathbb{R}$ $\quad {}^X f = f\circ X^{-1}$

$$\frac{\partial f}{\partial x_k} = D_k\, {}^X f, \quad \frac{\partial f}{\partial y_k} = D_k\, {}^Y f = D_k\, f\circ Y^{-1} =$$

$$= D_k\left[(f\circ X^{-1})\circ(X\circ Y^{-1})\right] =$$

$$= D_k\left({}^X f \circ {}^Y X\right) =$$

$$= \frac{\partial}{\partial y_k}\, {}^X f\left(x_1(y_1,\dots,y_N),\dots,x_N(y_1,\dots,y_N)\right) =$$

$$= \frac{\partial\, {}^X f}{\partial x_1}\frac{\partial x_1}{\partial y_k} + \frac{\partial\, {}^X f}{\partial x_2}\frac{\partial x_2}{\partial y_k} + \dots + \frac{\partial\, {}^X f}{\partial x_N}\frac{\partial x_N}{\partial y_k}$$

Leibniz (ÖSSZETETT FGV DIFF)

$$\frac{\partial f}{\partial y_k} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_N} \end{bmatrix} \begin{bmatrix} \partial x_1 / \partial y_k \\ \partial x_2 / \partial y_k \\ \vdots \\ \partial x_N / \partial y_k \end{bmatrix}$$

<u>Jelölés</u> $\quad \frac{\partial}{\partial X} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_N} \end{bmatrix} \quad \Big|$ pl. $\frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_N} \end{bmatrix}$

$$\frac{\partial Y}{\partial X} = \begin{bmatrix} \partial y_1 / \partial X \\ \vdots \\ \partial y_N / \partial X \end{bmatrix} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_N \\ \vdots & & \ddots & \vdots \\ \partial y_N/\partial x_1 & \partial y_N/\partial x_2 & \cdots & \partial y_N/\partial x_N \end{bmatrix}$$

$$\boxed{\frac{\partial f}{\partial Y} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial Y}}$$

<u>Gyakran</u> $\frac{\partial X}{\partial Y}$ helyett $\frac{\partial Y}{\partial X}$ kellene

<u>I</u> $\quad \frac{\partial Y}{\partial X} = \left[\frac{\partial X}{\partial Y}\right]^{-1} \quad \Big| \quad \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial X}\frac{\partial X}{\partial Y}, \quad \frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y}\left[\frac{\partial X}{\partial Y}\right]^{-1} \quad \forall f$

<u>B</u> $\quad I = (\underbrace{X \circ Y^{-1}}_{F}) \circ (\underbrace{Y \circ X^{-1}}_{G}) \qquad (F\circ G)'(\xi) = F'(G(\xi))\, G'(\xi) = I$

$$\left[D_i (F\circ G)\, e_j\right]_{i,j=1,N}$$

<u>K</u> $\quad \boxed{\frac{\partial f}{\partial Y} = \frac{\partial f}{\partial X}\left(\frac{\partial Y}{\partial X}\right)^{-1}}$

<u>Pl</u> $\quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ polárkoord alakja

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \varphi/\partial x & \partial \varphi/\partial y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \partial x/\partial r & \partial x/\partial \varphi \\ \partial y/\partial r & \partial y/\partial \varphi \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\cos\varphi - \frac{\partial f}{\partial \varphi}\,\frac{1}{r}\sin\varphi \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\sin\varphi + \frac{\partial f}{\partial \varphi}\cdot\frac{1}{r}\cos\varphi$$

$$g := \frac{\partial f}{\partial x} \qquad \frac{\partial g}{\partial x} = \frac{\partial g}{\partial r}\cos\varphi - \frac{\partial g}{\partial \varphi}\cdot\frac{1}{r}\sin\varphi$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}\right]\cos\varphi - \frac{\partial}{\partial \varphi}\left[\frac{\partial f}{\partial x}\right]\cdot\frac{1}{r}\sin\varphi =$$

$$= \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial r}\cos\varphi - \frac{\partial f}{\partial \varphi}\cdot\frac{1}{r}\sin\varphi\right]\cos\varphi -$$

$$- \frac{\partial}{\partial \varphi}\left[\frac{\partial f}{\partial r}\cos\varphi - \frac{\partial f}{\partial \varphi}\cdot\frac{1}{r}\sin\varphi\right]\frac{1}{r}\sin\varphi =$$

$$= \left[\frac{\partial^2 f}{\partial r^2}\cos\varphi - \frac{\partial^2 f}{\partial r\,\partial \varphi}\cdot\frac{1}{r}\sin\varphi + \frac{\partial f}{\partial \varphi}\cdot\frac{1}{r^2}\sin\varphi\right]\cos\varphi -$$

$$- \left[\frac{\partial^2 f}{\partial \varphi\,\partial r}\cos\varphi - \frac{\partial f}{\partial r}\sin\varphi - \frac{\partial^2 f}{\partial \varphi^2}\frac{1}{r}\sin\varphi - \frac{\partial f}{\partial \varphi}\frac{1}{r}\cos\varphi\right]\frac{1}{r}\sin\varphi$$

$$= \frac{\partial^2 f}{\partial r^2}\cos^2\varphi - 2\,\frac{\partial^2 f}{\partial r\,\partial \varphi}\,\frac{1}{r}\sin\varphi\cos\varphi + \frac{\partial^2 f}{\partial \varphi^2}\,\frac{1}{r^2}\sin^2\varphi +$$

$$+ \frac{\partial f}{\partial r}\,\frac{1}{r}\sin^2\varphi + 2\,\frac{\partial f}{\partial \varphi}\,\frac{1}{r^2}\sin\varphi\cos\varphi$$

HF
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2}\sin^2\varphi + 2\,\frac{\partial^2 f}{\partial r\,\partial \varphi}\,\frac{1}{r}\sin\varphi\cos\varphi + \frac{\partial^2 f}{\partial \varphi^2}\,\frac{1}{r^2}\cos^2\varphi +$$

$$+ \frac{\partial f}{\partial r}\,\frac{1}{r}\cos^2\varphi - 2\,\frac{\partial f}{\partial \varphi}\,\frac{1}{r^2}\sin\varphi\cos\varphi$$

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r}\cdot\frac{1}{r} + \frac{\partial^2 f}{\partial \varphi^2}\cdot\frac{1}{r^2}$$

$$\boxed{\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}}$$