DESCARTES KOORD A SÍKON

$X=\begin{bmatrix}x\\y\end{bmatrix}: S \leftrightarrow \mathbb{R}^2$

$d(p,q) := \{[x(p)-x(q)]^2+[y(p)-y(q)]^2\}^{1/2}$

Igaz a Pythagoras tétel:

$d(p,q)^2+d(p,r)^2=d(q,r)^2$ derékszögű △ p-nél

$\mathbb{R}^2$ műveletei

$\begin{bmatrix}\xi_1\\\eta_1\end{bmatrix}+\begin{bmatrix}\xi_2\\\eta_2\end{bmatrix}=\begin{bmatrix}\xi_1+\xi_2\\\eta_1+\eta_2\end{bmatrix},$

$\alpha\begin{bmatrix}\xi\\\eta\end{bmatrix}=\begin{bmatrix}\alpha\xi\\\alpha\eta\end{bmatrix},$

$\left\langle\begin{bmatrix}\xi_1\\\eta_1\end{bmatrix}\middle|\begin{bmatrix}\xi_2\\\eta_2\end{bmatrix}\right\rangle=\xi_1\xi_2+\eta_1\eta_2$

PONTBELI VEKTOR: $\vec{pq}$ $\quad X(\vec{pq})=\begin{bmatrix}x(q)-x(p)\\y(q)-y(p)\end{bmatrix}$

$d(p,q)^2=\langle X(\vec{pq})\rangle^2$

T: $\Leftrightarrow \langle X(\vec{pq})|X(\vec{pr})\rangle=0$

D-koordináták kapcsolata

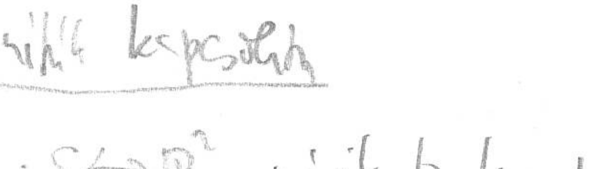
$\bar X=\begin{bmatrix}\bar x\\\bar y\end{bmatrix}: S\leftrightarrow\mathbb{R}^2$ másik D-koord

Kérdés: A $\begin{bmatrix}\xi\\\eta\end{bmatrix}$ X-koordinátájú pont $\bar X$-szerinti koord. milyen képletű

Def DET $[o,p_1,p_2]$ rendezett

L: $o,p,q\in S$, $\vec e=X(\vec{op})$ $\vec f:=X(\vec{oq})$

$[o,p,q]$ DET $\Leftrightarrow \exists\alpha,\beta$ $\alpha^2+\beta^2=1$ $X(\vec e)=\begin{bmatrix}\alpha\\\beta\end{bmatrix}$ $X(\vec f)=\pm\begin{bmatrix}-\beta\\\alpha\end{bmatrix}$



$X(o)$

K: $[o,p,q]$ DET $\Leftrightarrow \left[X(\vec{op}), X(\vec{oq})\right]=\begin{bmatrix}\alpha & \mp\beta\\\beta & \pm\alpha\end{bmatrix}$ ORT MÁTRIX

I $X: S \leftrightarrow \mathbb{R}^2$ adott D-koord

$\bar X: S \leftrightarrow \mathbb{R}^2$ PONTOSAN AKKOR D-koord, HA

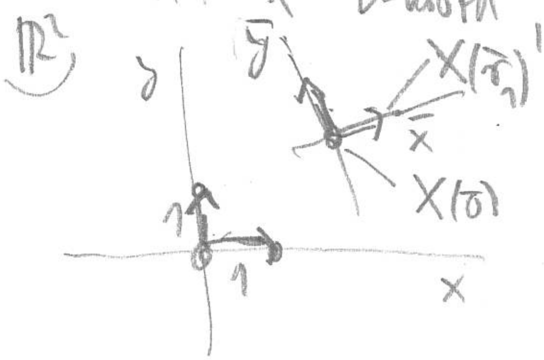
$\exists [\bar o, \bar r_1, \bar r_2]$ DET $\bar X(p) = F(X(p))$ $(p \in S)$, AHOL

$$F\left(\begin{bmatrix}\xi\\ \eta\end{bmatrix}\right) = A^T\left(\begin{bmatrix}\xi\\ \eta\end{bmatrix} - X(\bar o)\right) \qquad A = \left[X(\overrightarrow{\bar o \bar r_1}), X(\overrightarrow{\bar o \bar r_2})\right]$$

Különböző DET-ekhez különböző $\bar X$ tartozik

B $o := X^{-1}\left(\begin{bmatrix}0\\0\end{bmatrix}\right)$ $r_1 := X^{-1}\begin{bmatrix}1\\0\end{bmatrix}$ $r_2 := X^{-1}\begin{bmatrix}0\\1\end{bmatrix}$ ADJ DET

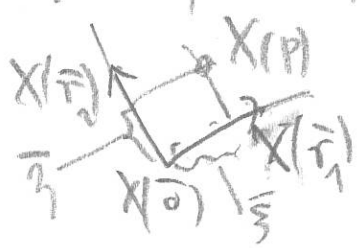
HA $\bar X$ D-koord, $[\bar o, \bar r_1, \bar r_2] := X^{-1}\left[\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right]$ DET



$u_0 := X(\bar o)$ $v_k := X(\overrightarrow{\bar o \bar r_k})$ $(k = 1,2)$

$A := [v_1, v_2] = \begin{bmatrix}\alpha & \sigma(-\beta)\\ \beta & \sigma\alpha\end{bmatrix}$ ORT MÁTRIX $\sigma \in \{\pm 1\}$

Legyen $\begin{bmatrix}\bar\xi\\ \bar\eta\end{bmatrix} = \bar X(p)$. Keressük $X(p)$ koordinátáit



$$X(p) = X(\bar o) + \bar\xi X(\overrightarrow{\bar o \bar r_1}) + \bar\eta X(\overrightarrow{\bar o \bar r_2}) =$$

$$= X(\bar o) + A\begin{bmatrix}\bar\xi\\ \bar\eta\end{bmatrix} \quad !!!$$

$$X(p) = X(\bar o) + A\bar X(p)$$

$$A\bar X(p) = X(p) - X(\bar o) \qquad A \text{ ORT}$$

$$\bar X(p) = A^T(X(p) - X(\bar o))$$

Észrevétel: HA A ORT, $u_0 \in \mathbb{R}^2$. $F_{A,u_0}: \begin{bmatrix}\xi\\ \eta\end{bmatrix} \mapsto A\left(\begin{bmatrix}\xi\\ \eta\end{bmatrix} - u_0\right)$

ESETÉN $F_{A,u_0}(X) =: \bar X$ D-koord.

MÉG KELL a Tételhez = **CSAK** $F_{A,u_0}(X)$ ALAKÚ D-KOORD. VAN

$\Leftarrow$ Legyen $\bar{X}$ D-KOORD, alappontjai $[\bar{o}, \bar{\pi}_1, \bar{\pi}_2] = \bar{X}^{-1}\left[\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right]$

$\tilde{X} := F_{A,u_0}(X)$ UGYANEZZEL AZ ALAPPONTTAL

Áll $\tilde{X} = \bar{X}$ $\Leftarrow$ $d(p,o)^2 = \langle \tilde{X}(\vec{op})\rangle^2 = \langle \bar{X}(\vec{op})\rangle^2$

$d(p,\pi_k)^2 = \langle \tilde{X}(\vec{\pi_k p})\rangle^2 = \langle \bar{X}(\vec{\pi_k p})\rangle^2$ $(k=1,2)$

$\tilde{X}(o) = \bar{X}(o) = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{0}$ $\tilde{X}(\pi_k) = \bar{X}(\pi_k) = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2}$

$\tilde{z} = \tilde{X}(p) = \begin{bmatrix}\tilde\xi\\\tilde\eta\end{bmatrix}$ $\bar{z} = \bar{X}(p) = \begin{bmatrix}\bar\xi\\\bar\eta\end{bmatrix}$

$\langle \tilde{z}-0 | \tilde{z}-0\rangle = \langle \bar{z}-0 | \bar{z}-0\rangle$ $\quad \langle\tilde{z}\rangle^2 = \langle\bar{z}\rangle^2$

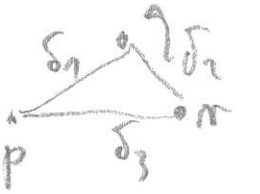
$\langle \tilde{z}-e_1 | \tilde{z}-e_1\rangle = \langle \bar{z}-e_1 | \bar{z}-e_1\rangle$ $\quad \langle\tilde{z}\rangle^2 - 2\langle\tilde{z}|e_k\rangle + \overbrace{\langle e_k\rangle^2}^{1} =$

$\langle \tilde{z}-e_2 | \tilde{z}-e_2\rangle = \langle \bar{z}-e_2 | \bar{z}-e_2\rangle$ $\quad = \langle\bar{z}\rangle^2 - 2\langle\bar{z}|e_k\rangle + \underbrace{\langle e_k\rangle^2}_{1}$

$\langle\tilde{z}|e_k\rangle = \langle\bar{z}|e_k\rangle$ $\quad \underbrace{\left\langle \begin{bmatrix}\tilde\xi\\\tilde\eta\end{bmatrix} \middle| \begin{bmatrix}1\\0\end{bmatrix}\right\rangle}_{\tilde\xi} = \underbrace{\left\langle \begin{bmatrix}\bar\xi\\\bar\eta\end{bmatrix} \middle| \begin{bmatrix}1\\0\end{bmatrix}\right\rangle}_{\bar\xi}$

ugyanígy $\begin{bmatrix}0\\1\end{bmatrix}$-gyel $\tilde\eta = \bar\eta$ **QED**

Δ-egyenlőtlenség



$d(p,q) + d(q,r) \geq d(p,r)$

EGYSZERŰ ESET

OLYAN KOORDIN., AHOL

$q = \bar{X}^{-1}\begin{pmatrix}\xi\\\eta\end{pmatrix}$

$p = \bar{X}^{-1}\begin{pmatrix}0\\0\end{pmatrix}$ $r = \bar{X}^{-1}\begin{pmatrix}\delta_3\\0\end{pmatrix}$

$d(p,q) \geq |\xi|$

$d(q,r) \geq |\delta_3 - \xi|$

Görbék

$G:[\alpha,\beta]\to S$

$G\sim\bar{G}$ EKVIVALENCIA : $\exists T:[\alpha,\beta]\nearrow[\bar{\alpha},\bar{\beta}]$

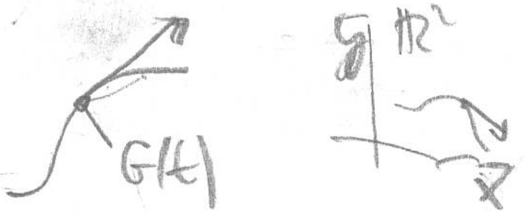
$\bar{G}:[\bar{\alpha},\bar{\beta}]\to S$

$\bar{G}(T(t))=G(t)\quad\forall t$

$T(\alpha)=\bar{\alpha}\ ,\ T(\beta)=\bar{\beta}$

Hossz: $L(G):=\sup\left\{\sum_{k=1}^{n} d(G(t_{k-1}),G(t_k)) : \alpha=t_0\le t_1\le\dots\le t_n=\beta\right\}$

Sebesség:



X D-koord szerint $\underbrace{\frac{d}{dt}X(G(t))}_{\dot{G}_X(t)}=\lim_{h\to0}\begin{bmatrix}\frac{1}{h}X(G(t+h))-X(G(t))\\ \frac{1}{h}[G(t+h)-G(t)]\end{bmatrix}$

T: Ha $t\mapsto G(t)$ FOLYT DIFF, $\Rightarrow$

$$L(G)=\int_{t=\alpha}^{\beta}\|\dot{G}_X(t)\|\,dt$$

$[\dot{G}_X(t)$ HOSSZA$]=[\langle G_X(t)\rangle^2]^{1/2}$

AKÁRMILYEN X D-koord mellett

$G(t)\longrightarrow G(t)+X^{-1}(\dot{G}_X(t))$ UAZ

Egyenesek

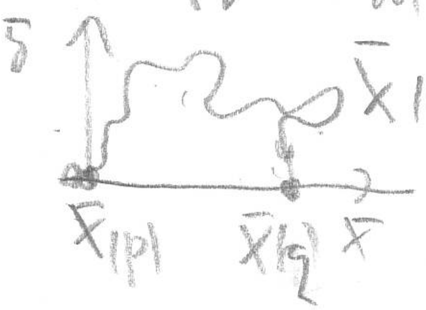
„Két pont között legrövidebb út az egyenes" (szakasz konkrétan)

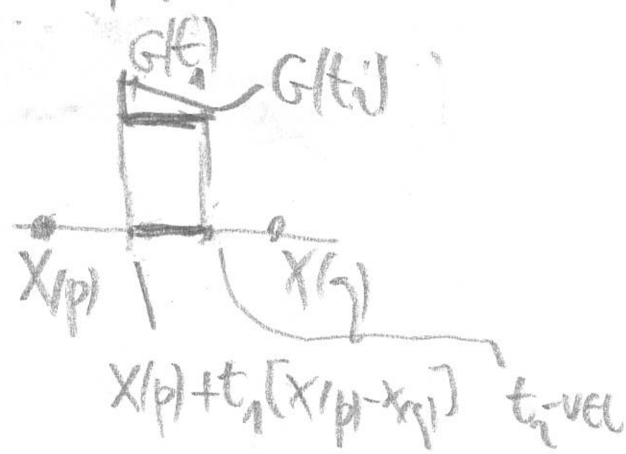
T Ha $G:[\alpha,\beta]\to S$ görbe, $p=G(\alpha)$, $q=G(\beta)$ akkor $L(G)\ge d(p,q)=L([t\mapsto p_t])$ ahol

$$p_t = X^{-1}\left(X(p)+t\,[X(q)-X(p)]\right),\quad 0\le t\le 1$$

tetsz. D koord

B olyan $\bar X$ D-koord-t veszünk, hogy





Def $M\subset S$ egyenes, ha $\exists u,v\in\mathbb{R}^2$ $v\neq 0$ $\exists X$ D-koord

$$M=\{X^{-1}(u+tv): t\in\mathbb{R}\}$$

T Az egyenesek szakaszai a legrövidebbek utak

Tetsz. D-koord szerint ugyanolyan képlet (más u,v-vel)

Kör

„Adott ponttól adott távolságra lévő pontok" (halmaza)

$$K = \{ p \in S : d(p,O) = \rho \}$$

(O: középp., ρ: sugár)

$\begin{bmatrix} x \\ y \end{bmatrix} = X$ D-koord.

$$K = \{ p : [x(p) - x(O)]^2 + [y(p) - y(O)]^2 = \rho^2 \} =$$

$$= \{ p : \langle X(p) - X(O) \mid X(p) - X(O) \rangle = \rho^2 \} =$$

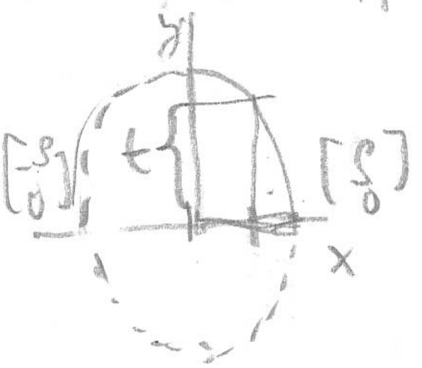
$$= \{ p : \langle X(\vec{Op}) \rangle^2 = \rho^2 \} = X^{-1} \{ u \in \mathbb{R}^2 : \langle u - X(O) \rangle^2 = \rho^2 \}$$

KOORD FELT KÉPLET

Kérdés: MIÉRT GÖRBE A KÖR?

$\frac{1}{4}$-kör : $\exists X$ D-koord (AHOL $X(O) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

$$\tilde{K} = \{ p : \langle X(p) \rangle^2 = \rho^2 ,\ x(p) y(p) \geq 0 \}$$



$t \in [-\rho, \rho]$

$$K^+ = \{ p \in K : y(p) = t,\ x(p) \geq 0 \}$$

$$= X^{-1} \left\{ \begin{bmatrix} \sqrt{\rho^2 - t^2} \\ t \end{bmatrix} : 0 \leq t \leq \rho \right\}$$

$$G : t \mapsto X^{-1} \begin{bmatrix} \sqrt{\rho^2 - t^2} \\ t \end{bmatrix}$$

$\underline{\underline{\pi}}$ (az Euklideszi geom. szerint analízis)

1-sugarú félkör hossza

2× $\frac{1}{4}$-edkör hossza

$$\frac{\pi}{2} = \int_{t=0}^{1} \left\| \frac{d}{dt}\begin{bmatrix}\sqrt{1-t^2}\\ t\end{bmatrix} \right\| dt = \int_{t=0}^{1} \left\| \begin{bmatrix}\frac{-t}{\sqrt{1-t^2}}\\ 1\end{bmatrix} \right\| dt =$$

$$= \int_{t=0}^{1} \left\{ \frac{t^2}{1-t^2} + 1 \right\}^{1/2} dt = \int_{t=0}^{1} \frac{1}{\sqrt{1-t^2}} dt =$$

$$= \int_{t=0}^{1} \left[1-t^2\right]^{-1/2} dt = \int_{t=0}^{1} \sum_{n=0}^{\infty} \binom{-1/2}{n} t^{2n} dt =$$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{1}{2n+1} = 1 + \frac{(-1/2)}{1}\cdot\frac{1}{3} + \frac{(-1/2)(-1/2-1)}{2!}\cdot\frac{1}{5} + \dots$$

$$= 1 - \frac{1}{6} + \frac{3}{40} \mp \dots + (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot\frac{1}{2n+1} + \dots$$

Ívhossz szerinti paraméterezés

$G : [\alpha, \beta] \to S \qquad t \mapsto X(G(t))$ görbe $\quad L(G) < \infty$

$\hat{G} : [0, L(G)] \to S$

$\qquad L(G|_{[\alpha,t]}) \mapsto G(t)$

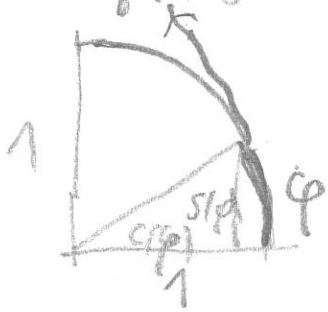
$\underline{\underline{L}}$ Ha $t \mapsto X(G(t))$ folyt. diff., $\frac{d}{dt}X(G(t)) \neq 0$, akkor

$\qquad \sigma \mapsto X(\hat{G}(\sigma))$ is folyt. diff. $\left\|\frac{d}{d\sigma} X(\hat{G}(\sigma))\right\| = 1$

1-sebességgel megyünk végig G-n

Sin, Cos

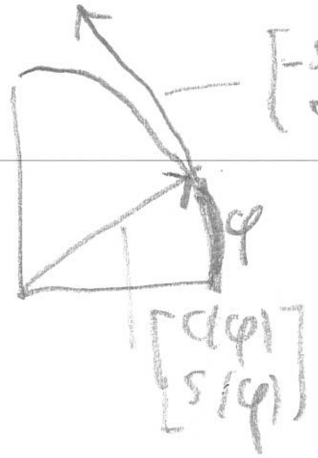
$\frac{1}{4}$-ed 1 sugarú kör



$s(\varphi)=t$

$c(\varphi)=\sqrt{1-t^2}$

$$\frac{d}{dt}X\begin{pmatrix}\sqrt{1-t^2}\\ t\end{pmatrix}=\begin{bmatrix}-\frac{t}{\sqrt{1-t^2}}\\ 1\end{bmatrix}\perp\begin{bmatrix}\sqrt{1-t^2}\\ t\end{bmatrix}$$



$$\left.\begin{array}{ll}\frac{d}{d\varphi}c(\varphi)=-s(\varphi), & c(0)=1\\ \frac{d}{d\varphi}s(\varphi)=c(\varphi), & s(0)=0\end{array}\right\}$$

DIFF EGY: EGYÉRT. ∞-szer diffható

$$\frac{d^2}{d\varphi^2}c(\varphi)=-\frac{d}{d\varphi}s(\varphi)=-c(\varphi) \qquad c(0)=1,\ c'(0)=s(0)=0$$

$$\frac{d^2}{d\varphi^2}s(\varphi)=\frac{d}{d\varphi}c(\varphi)=-s(\varphi) \qquad s(0)=0,\ s'(0)=c(0)=1$$

SORFEJTÉS:

$$s(\varphi)=\sum_{n=0}^{\infty}\alpha_n\varphi^n=-s''(\varphi)=-\sum_{n=2}^{\infty}\alpha_n\cdot n\cdot(n-1)\varphi^{n-2}$$

$$\alpha_0=0 \quad \alpha_1=1 \quad \alpha_{n+2}=-\frac{1}{n(n-1)}\alpha_n,\ \Rightarrow \alpha_{2k}=0 \quad \alpha_{2k+1}=\frac{(-1)^k}{(2k+1)!}$$

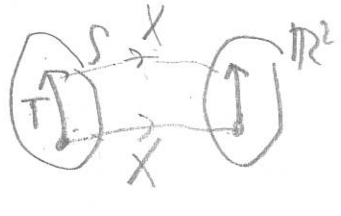
$$s(\varphi)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\varphi^{2k+1}$$

HASONLÓAN

$$c(\varphi)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\varphi^{2k}$$

Transzformációk

$T: S \to S$



T reprezentációja az $X: S \leftrightarrow \mathbb{R}^2$ koordinátában

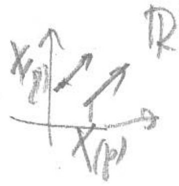
$$X^{\#}T \doteq X \circ S \circ X^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$$

T távolságtartó (izometria): $d(T(p),T(q)) = d(p,q)$

$\underline{T}$ (Láttuk már úgy fogalmazásban) $\bar{X}$ D-koord $\Leftrightarrow \exists T$ izom $\overline{\bar{X} = T \circ X \circ T^{-1}}$

Egy izometriát 3 pont meghatároz ($\exists$ET-n láttuk)

Eltolás: $T_{\vec{or}}$: $X(T_{\vec{or}}(p)) = X(p) + \underbrace{X(\vec{or})}_{X(r)-X(o)}$



$$T_{\vec{or}} = X^{-1} \circ \left[u \mapsto u + X(\vec{or})\right] \circ X$$

Forgatás: $R_{o,\varphi}$: forgatás „bele" o körül φ szöggel

$$X(\overrightarrow{o R_{o,\varphi}(p)}) = \begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix} X(\vec{op}) \qquad R_{o,\varphi}(o) = o$$

$$R_{o,\varphi} = X^{-1} \circ \left[u \mapsto [\;](u - X(o)) + X(o)\right] \circ X$$

Ha $X(o) = \begin{pmatrix}0\\0\end{pmatrix}$, egyszerűbb: $R_\varphi = X^{-1} \circ \left[\begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix} \cdot \right] \circ X$

$\underline{T}$ $T_{\vec{rs}} \circ T_{\vec{or}} = T_{\vec{os}}$

$T_1 \circ T_2 = T_2 \circ T_1$ két eltolásra

$R_{\varphi+\psi} = R_\varphi \circ R_\psi$ közös o pont körül

$\underline{K}$ $\begin{bmatrix}\cos(\varphi+\psi) & -\sin(\varphi+\psi) \\ \sin(\varphi+\psi) & \cos(\varphi+\psi)\end{bmatrix} = \begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}\cos\psi & -\sin\psi \\ \sin\psi & \cos\psi\end{bmatrix}$

$\cos(\varphi+\psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi \qquad \sin(\varphi+\psi) = \cos\varphi\sin\psi + \sin\varphi\cos\psi$

Tükrözés $T = X^{-1}\left[\begin{pmatrix}\xi\\ \eta\end{pmatrix} \mapsto \begin{pmatrix}-\xi\\ \eta\end{pmatrix}\right] \circ X$ valamilyen X D-koord mellett

Terület

$Vol_2 : X^{-1}[0,1)^2 \mapsto 1$
$(0,1)^2$

EGYBEVÁGÓ ALAKZATOK Vol_2-je UAZ (≥ 0) $\quad G_1 = \underset{T\ \text{izom}}{T}(G_2)$

DISZJUNKT ALAKZATOK U-ján összegződik

I $\quad a,b,c \in S \quad \Delta_{a,b,c} = X^{-1}\{\lambda_1 X(a) + \lambda_2 X(b) + \lambda_3 X(c) : \lambda_1+\lambda_2+\lambda_3=1,\ \lambda_i \geq 0\}$

$Vol_2\, \Delta_{a,b,c} = \pm\frac{1}{2} \det [X(\vec{ab})\ X(\vec{ac})] \quad \forall X$ D-koord

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N-DIM EUKL GEOM D-KOORDINÁTÁKKAL

$X : S \leftrightarrow \mathbb{R}^N$ D-koord: $d(p,q)^2 = \sum_{k=1}^{N} [x_k(p) - x_k(q)]^2$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad x_k : S \to \mathbb{R}^N$

M $\mathbb{R}^N$ is ugyanolyan S-nek: $x_k : \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \mapsto \xi_k$ KANONIKUS KOORD

I $\quad T : S \leftrightarrow S$ IZOMETRIA : TETSZ X D-KOORD $\exists u \in \mathbb{R}^N,\ A \in ORT(N)$

$X(T(p)) = A X(p) + u \quad (p \in S)$

$T = X^{-1}[v \mapsto Av + u] \circ X$

N+1 pont képe egyért. meghatározza, $\bar{X} = X \circ \underbrace{T \circ X^{-1}}$, izom

Térfogat $\quad K \leq N \quad G := \left\{ F\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} : 0 \leq t_1, \ldots, t_k \leq 1 \right\} \quad F : [0,1]^k \to S$

X TETSZ D-KOORD

$$Vol_k(G) = \int_{t_1=0}^{1} \cdots \int_{t_k=0}^{1} \sqrt{\det\left([XF'(t)]^T [XF'(t)]\right)}\, dt_k \ldots dt_1$$

$$XF(t) = \begin{bmatrix} x_1(F(t)) \\ \vdots \\ x_N(F(t)) \end{bmatrix} \qquad \underset{\begin{pmatrix} x_1 \circ F \\ \vdots \\ x_N \circ F \end{pmatrix}}{\phi'}(t) = \begin{bmatrix} \partial\phi_1/\partial t_1 & \cdots & \partial\phi_1/\partial t_k \\ \vdots & \ddots & \vdots \\ \partial\phi_N/\partial t_1 & \cdots & \partial\phi_N/\partial t_k \end{bmatrix}$$

M $\quad Vol_k$ DEF METRIKUS TÉRBEN KÉRDÉS