On the structure of C_0-semigroups of holomorphic Carathéodory isometries

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MÖBIUS TRANSFORMATIONS

[MÖBIUS TRF] = $\Big[$hol. aut. of the unit ball $\mathbf{B}$ of a JB*-triple $(\mathbf{E}, \{\ldots\})\Big]$

Canonical form [**Kaup**, 1983]: $\Phi = M_a \circ U$

$M_a(x) = A + B(a)^{1/2}[1 + L(x,a)]^{-1}x$, U surj.lin $\mathbf{E}$-isom.

$$B(a) = \mathrm{Bergman}(a) = 1 - 2L(a,a) + Q(a)^2$$
$$L(a,b)x := \{abx\}, \quad Q(a)x := \{axa\}$$

Hol. extension to a nbh. of $\overline{\mathbf{B}}$ $\longrightarrow \Phi \in \mathrm{Aut}(\overline{\mathbf{B}})$.

CARATHÉODORY ISOMETRIES

$d_{\mathbf{B}}$: $\mathrm{Aut}(\mathbf{B})$-inv. distance on $\mathbf{B}$

$$d_{\mathbf{B}}(0,x) = \|x\| + o(\|x\|) \quad (x \to 0)$$

$d_{\mathbf{B}}$-isometries: $M_a \circ U\big|\mathbf{B}$, $U : \mathbf{E} \to \mathbf{E}$ lin. isometry [Vesentini]

$[\Phi^t : t \in \mathbb{R}_+]$ one-parameter C_0-semigroup in $\mathrm{Iso}(d_{\mathbf{B}})$

$$\Phi^t \circ \Phi^h = \Phi^{t+h} \ (t,h \geq 0), \quad \Phi^0 = \mathrm{Id}, \quad t \mapsto \Phi^t(x) \text{ cont. } \forall\, x \in \mathbf{B}$$

One-parameter C_0-groups (str.cont.1prg) analogous def.

Example: Hille-Yosida theory for LINEAR isometries

INFINITESIMSIMAL GENERATORS

$\mathrm{gen}[\Phi^t : t \in \mathbb{R}_+] = \Phi'(x) = \frac{d}{dt}\Big|_{t=0+} \Phi^t(x)$

$\mathrm{dom}(\Phi') = \{x : \ t \mapsto \Phi^t \text{ diff.}\}$

Hille-Yosida: LIN. $\Longrightarrow \mathrm{dom}(\Phi')$ dense in **B** (or **E** with lin.ext.)

$\Phi' \longleftrightarrow [\Phi^t : t \in \mathbb{R}_+]$ NO EXP in general

Reich – Shoikhet 1996:

Imitate H-Y resolvents with hol. vector fields $\underbrace{\mathbf{D}}_{\text{conv.}} \longrightarrow \underbrace{\mathbf{U}}_{\text{Banach}}$

Drawback: In lin. case *bounded* generator

Kaup's bded generators: If $[\Phi^t : \ t \in \mathbb{R}]$ **unif.**cont.1prg

$\Phi' : \mathbf{B} \ni x \mapsto a - \{xax\} + iAx, \quad A$ lin.herm.

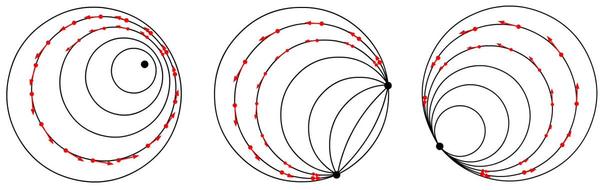
UNIT DISC

Example: $\mathbf{E} = \mathbb{C}$, $\mathbf{B} = \{z : |z| < 1\}$

$M_a(z) = \frac{z+a}{1+\bar{a}z}$, $Uz = \kappa z$ with $|\kappa| = 1$

$[\Phi^t : t \in \mathbb{R}_+]$ str.cont.1prsg $\iff$ extends to $[\Phi^t : t \in \mathbb{R}]$ unif.cont.1prg.

$\Phi'(z) = a - z\bar{a}z + i\alpha z$, $|a| < 1$, $\alpha \in \mathbb{R}$



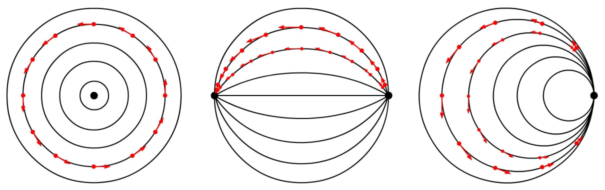
MÖBIUS EQUIVALENCE

Definition: $[\Phi^t : t \in \mathbb{R}_+], [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}}$ are *Möbius equiv.* if $\exists\ \Theta \in \mathrm{Aut}(\mathbf{B}) \quad \Psi^t = \Theta \circ \Phi^t \circ \Theta^{-1} \quad (t \in \mathbb{R}_+)$

Example. In 1 DIM. alternatives up to Möbius equiv:

(1) $\Phi^t = e^{i\alpha t} x$; (2) $\Phi^t = \frac{x + \tanh(\alpha t)}{1 + x \tanh(\alpha t)}$

(3) $\Phi^t = \frac{1 + i\alpha t}{1 - i\alpha t} \frac{x + \alpha t/(1 + i\alpha t)}{1 + x\alpha t/(1 - i\alpha t)}$



FRACTIONAL LIN. APPROACH

$\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$

$$\mathcal{F}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \; : \quad \mathbf{E}X \mapsto (AX+B)(CX+D)^{-1}, \qquad X \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$$

Vesentini [Adv. Math. 1987, $+ \cdots$]
Khatshkevich – Reich – Shoikhet 2001

$\Phi \in \mathrm{Iso}(d_{\mathbf{B}}) \iff \exists\, T \;\; \Phi = \mathcal{F}(T),\; T^*\mathrm{diag}(1,-1)T = T$

Str.cont.1-prsg. $[\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}}),\; \dim(\mathbf{H}_2) < \infty$

Problem: Repr. up to unit scalars: $\Phi = \mathcal{F}(\kappa T) \;\; |\kappa| = 1$

Minor error in [Vesentini 87, end p.283] (Hilbert space, $\mathbf{H}_2 = \mathbb{C}$)

HILBERT BALL

$(\mathbf{H}, \langle .|.\rangle)$ Hilbert space $\qquad \mathbf{H} \simeq \mathcal{L}(\mathbf{H}, \mathbb{C})$ with $h \equiv [\zeta \mapsto \zeta h]$

$\Phi^t = M_{a(t)} \circ U_t = \mathcal{F}(\mathcal{M}_{a(t)} \mathcal{U}_t)$ $\quad$ Kaup's type form

$$\mathcal{M}_a = \begin{bmatrix} Q_a & a \\ a^* & 1 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q_a = P_a + \beta_a(1 - P_a), \;\; P_a = [Proj. \text{ onto } \mathbb{C}a], \quad \beta_a = \sqrt{1 - \|a\|^2}$$

ADJUSTMENT WITH FIXED POINT

$\exists$ **joint fixed point**: $M_{a(t)} U_t \overline{e} = \overline{e} \quad (t \in \mathbb{R}_+)$

USE : $\qquad \mathcal{P}^t = \kappa(t) \mathcal{M}_{a(t)} \mathcal{U}_t, \quad \kappa(t) = 1/[1 + \langle U_t \overline{e} | a(t) \rangle]$

$\Phi^t = \mathcal{F}(\mathcal{P}^t), \quad [\mathcal{P}^t : t \in \mathbb{R}_+]$ str.cont.1prsg. $\qquad \mathcal{P}^t [\overline{e}\ 1]^{\mathrm{T}} = [\overline{e}\ 1]^{\mathrm{T}}$

$\mathrm{dom}(\mathcal{P}') = [\mathbf{Z}\ 0]^{\mathrm{T}} + \mathbb{C}[\overline{e}\ 1]^{\mathrm{T}}, \quad \mathrm{dom}(\Phi') = (\overline{e} + \mathbf{Z}) \cap \mathbf{B},$

$$\Phi' x = [\Lambda(\overline{e} - x)] x + B(x - \overline{e})$$

$Bz = \left[\mathcal{P}'[z\ 0]^{\mathrm{T}}\right]_{\mathbf{H}}, \quad \Lambda z = \left[\mathcal{P}'[z\ 0]^{\mathrm{T}}\right]_{\mathbb{C}} \quad (z \in \mathbf{Z})$

Remark. MATRIX FORM $\iff \overline{e} \in \mathbf{Z} \iff 0 \in \mathrm{dom}(\Phi')$

MAIN RESULT

$\mathcal{T} = \begin{bmatrix} 1 & \overline{e} \\ 0 & 1 \end{bmatrix}$ prjective version of $T : x + \overline{e}$

Theorem. Any str.cont.1prsg. of hol. Hilbert ball Carath.isometries is Móbius equiv. to some $[\Phi^t : t \in \mathbb{R}_+]$ with joint fixed point $\overline{e}$ whose projective generator $\mathcal{P}'$ is of $[\mathbf{H} \ominus (\mathbb{C}\overline{e})] \oplus [\mathbb{C}\overline{e}] \oplus \mathbb{C}$ matrix form and $\mathcal{T}^{-1}\mathcal{P}'\mathcal{T}$ UPPER TRIANGULAR.

Remark. Hence, in physicist's terminology, $[\Phi^t : t \in \mathbb{R}_+]$ is an INTEGRABLE DYN. SYSTEM

Finite formulas with functions of unbded self-adjoint(!) op.s

DILATION

J.A. Deddens 1969:
Tf $[U^t : t \in \mathbb{R}_+]$ str.cont.1prsg. of LIN. ISOM. in $\mathbf{H}$
$\Longrightarrow \exists\,[\widehat{U}^t : t \in \mathbb{R}]$ str.cont,1prg. in $\widehat{\mathbf{H}} \supset \mathbf{H}$ with $U^t = \widehat{U}^t\big|\mathbf{H}$

Remark. Φ^t permutes the discs $\mathbf{B} \cap [\overline{e} + \mathbb{C}v]$.

Theorem. $[\Phi^t : t \in \mathbb{R}_+]$ admits a str.cont dilation 1prg. $[\widehat{\Phi}^t : t \in \mathbb{R}]$.

Alternatives up to Möbius equivalence

$\exists\ [\widehat{\Phi}^t : t \in \mathbb{R}] \sim [\Psi^t : t \in \mathbb{R}]$ such that

ALTERNATIVES ($\sim$classical)

(1) If $\mathbf{B} \cap \bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\Phi^t) \neq \emptyset$,

$$\widehat{\Phi}^t(x) = \exp(itA)x.$$

(2) If $\bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\Phi^t) = \{2 \text{ points in } \partial\mathbf{B}\}$,

$$\Psi^t(\xi\overline{e} + z) = M_{\alpha t}^{(2)}(\xi)\overline{e} + \gamma^{(2)}(\alpha t)\exp(itC)z \quad (z \perp \overline{e}).$$

(3a) if $\bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\Phi^t) = \{\overline{e}\}$ and $\exists\ [\Phi^t : t \in \mathbb{R}_+]$-fixed disc $\mathbf{B} \cap [\overline{e} + \mathbb{C}v]$,

$$\Psi^t(\xi\overline{e} + z) = M_{\alpha t}^{(2)}(\xi)\overline{e} + \gamma(\alpha t)\exp(itC)z \quad (z \perp \overline{e}).$$

ALTERNATIVES (non-class.)

(3b) If $\bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\Phi^t) = $ [one point in $\partial\mathbf{B}$] and
$\not\exists$ $[\Phi^t : t \in \mathbb{R}_+]$-fixed disc $\mathbf{B} \cap [\overline{e} + \mathbb{C}v]$,

$$\Psi^t(\xi\overline{e} + z) = \varphi\overline{e} + \psi b + \vartheta V^t z \quad (z \perp \overline{e})$$

where $V^t = \exp(itC)$, φ, ψ, ϑ fract.lin.expr. in ξ with parameters

$\mu \in \mathbb{R},\ b \perp \overline{e}, \int_0^t V^s ds\, b,\ \int_0^t \int_0^s V^h dh\, ds\, b$

Remark. IF $\dim(\mathbf{H}) < \infty$ ONLY (1),(2),(3a).

Case (3) is possible [Stachó JMAA 2016, Example 4.2]

PROOF ingredients

We may assume: Kaup's type (up to Möb.equiv.)

$0 \in \mathrm{dom}(\Phi')$, $\overline{e} \in \partial\mathbf{B} \cap \bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\overline{\Phi^t})$

$\Phi'(x) = b - \langle x|b\rangle + iRx$, $b = \frac{d}{dt}\Big|_{t=0+} a(t)$

$\Phi^t = F(\mathcal{U}^t)$, $[\mathcal{U}^t : t \in \mathbb{R}_+]$ str.cont.1prsg in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$,

$\mathcal{R} := \mathrm{gen}[\mathcal{U}^t : t \in \mathbb{R}_+] = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix}$, $\mathrm{dom}(\mathcal{R}) = \mathrm{dom}(R) \oplus \mathbb{C}$

$iR = W' = \mathrm{gen}[W^t : t \in \mathbb{R}_+]$ str.cont.1prsg of $\mathcal{L}(\mathbf{H})$-isometries

$\overline{e} \oplus 1$ joint eigenvector $[\mathcal{U}^t : t \in \mathbb{R}_+]$

$$\overline{e} \in \mathrm{dom}(R), \quad \mathcal{R}[\overline{e} \oplus 1] = \nu[\overline{e} \oplus 1], \quad b = \nu\overline{e} - iR\overline{x}, \quad \nu = \langle \overline{e}|b\rangle$$

PROOF: triangularization

With the projective shift $\mathcal{T} : x \oplus \xi \mapsto (x + \xi\overline{e}) \oplus \xi$

$$\mathcal{V}^t := e^{-\nu t}\mathcal{T}^{-1}\mathcal{U}^t\mathcal{T}, \quad \mathcal{B} = \mathcal{T}^{-1}\mathcal{A}\mathcal{T}$$

$\mathcal{B} = \mathrm{gen}[\mathcal{V}^t : t \in \mathbb{R}_+]$ bded perturb of $(iR) \oplus 1$, $\mathrm{dom}(\mathcal{B}) = \mathrm{dom}(R) \oplus \mathbb{C}$.

$$\mathcal{B} = \begin{bmatrix} I & -\overline{e} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} iR - \nu I & b \\ b^* & -\nu \end{bmatrix} \begin{bmatrix} I & \overline{e} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B & 0 \\ b^* & 0 \end{bmatrix}, \qquad \begin{aligned} B &:= iR - \nu I - \overline{x}b^* \\ &= \mathrm{gen}[V^t : t \in \mathbb{R}_+] \end{aligned}$$

Lemma of Triangular Generators $\Longrightarrow$

$$\mathcal{V}^t = \begin{bmatrix} V^t & 0 \\ b^* \int_0^\tau V^\tau d\tau & 1 \end{bmatrix}$$

PROOF: second triangularization

Quadrature formula for V^t with space decomposition

$$\mathbf{H}_0 := \mathbf{H} \ominus (\mathbb{C}\overline{x}), \quad P := \mathrm{P}_{\mathbb{C}\overline{x}} = \overline{x}\,\overline{x}^*, \quad P_0 := \mathrm{P}_{\mathbf{H}_0} = I - P,$$

By setting $\lambda := \mathrm{Re}(\nu)$, $\mu := \mathrm{Im}(\nu) = \langle \overline{x} \big| R\overline{x} \rangle / 2$, $S := R - \mu I$,
$b_0 := P_0 b = -iP_0 R\overline{x} = b - \nu\overline{x}$, $S_0 := P_0 S P_0 \big| \mathbf{H}_0$, $I_0 := \mathrm{Id}_{\mathbf{H}_0}$

in terms of $\mathbf{H}_0 \oplus (\mathbb{C}\overline{x})]$-matrices,

$$B = \begin{bmatrix} P_0[iS - \lambda I] \big| \mathbf{H}_0 & iP_0 S\overline{x} \\ 0 & -2\lambda \end{bmatrix} = \begin{bmatrix} iS_0 + \lambda I_0 & -b_0 \\ 0 & 0 \end{bmatrix} - 2\lambda \begin{bmatrix} I_0 & 0 \\ 0 & 1 \end{bmatrix}$$

$P_0[iS]P_0 = \big[\text{bded } iR\text{-pert}\big]$, $\Rightarrow iS_0 \big| \mathbf{H}_0 = \mathrm{gen}[V_0^t : t \in \mathbb{R}_+]$ $\mathbf{H}_0$-isometries

Lemma of Tri. Gen. $\Longrightarrow$

$$V^t = e^{-2\lambda t} \begin{bmatrix} e^{\lambda t} V_0^t & \int_0^t e^{\lambda\tau} V_0^\tau b_0 \, d\tau \\ 0 & 1 \end{bmatrix}$$

WORK IN PROGRESS

$\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ adjusted cont. of proj. repr.

Problem. For $\Phi^t = \mathcal{F}(\mathcal{A}_t)$ we known only: $\mathcal{A}_t \mathcal{A}_h = \lambda(t,h) \mathcal{A}_{t+h}$.
IS $[\mathcal{A}_t : t \in \mathbb{R}_+]$ ABELIAN?

Remark. $A : e_n \mapsto e_{n+1}$ bilat.shift, $B : e_n \mapsto e^{in} \Longrightarrow$
$AB(e_n) = e^{in} e_{n+1}$, $BA(e_n) = e^{i(n+1)} e_{n+1}$, $BA = e^i AB$.

If $1 < \dim(\mathbf{H}_2) < \infty$, trace argument $\Rightarrow$ YES $\Rightarrow$ adj.cont.
Structure with finite spectral decomposition.

BEYOND LINEARITY

$\mathbf{E}, \{\ldots\}$ JB*-triple, $\quad \Phi^t = M_{a_t} \circ U_t$

Proved: $0 \in \mathrm{dom}(\Phi'), \implies \mathrm{dom}(\Phi') = \mathbf{B} \cap \mathbf{F}$ with $\mathbf{F} = \{x : t \mapsto U_t x$ diff.$\}$ subtriple

Problem: Is $\mathrm{dom}(\Phi') = \emptyset$ possible?

Theorem. Let $[\Phi^{\cdot} t \in \mathbb{R}]$ str.cont.1par.$\underline{\mathrm{group}}$, $\quad 0 \in \mathrm{dom}(\Phi')$, $e \in \bigcap_{t \in \mathbb{R}} \mathrm{Fix}(\Phi^t)$. Then

(1) $[\mathrm{D}_e \Phi^t : t \in \mathbb{R}]$ str.cont.1prg in $\mathrm{GL}(\mathbf{E})$; $\quad$ (2) $\mathbf{F}$ is dense in $\mathbf{E}$;

(3) Φ' ∼Kaup's type: $\quad \Phi'(z) = b - \{zbz\} + iAz$,
iA lin. and tangent to $\partial \mathbf{B}$

Application to SPIN FACTORS, reflexive JB*-triples

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