Möbius transformations

Hol. aut. of the unit ball. $\mathbf{B}$ of a JB*-triple $(\mathbf{E},\{\ldots\})$

$\Phi \in \mathrm{Aut}(\mathbf{B})$ extends holomorphically to a neighborhood of $\overline{\mathbf{B}}$.

Canonical form [Kaup MathZ. 1983]: $\quad \Phi = M_a \circ U$

$M_a(x) = A + \mathrm{Bergman}(a)^{1/2}[1 + L(x,a)]^{-1}x,\ \ U$ surj.lin $\mathbf{E}$-isom.

Faces: If $\mathbf{E}$ JBW*-triple and $\mathbf{F}$ is a (norm-exposed) face of $\partial\mathbf{B}$ then

$\exists\, e$ TRIP in $\mathbf{E}$ $\quad \mathbf{F} = \big\{x \in \partial\mathbf{B}:\ x - e \perp e\big\} = \big\{M_c(e):\ c \perp e,\ \|c\| \leq 1\big\}$.

Möbius equivalence: $\quad \Phi \sim \Psi \quad$ if $\exists\,\Theta$ Möbius trf. with $\Psi = \Theta \circ \Phi \circ \Theta$

Basic assumption:

(0) $\big[\Phi^t : t \in \mathbf{R}\big]$ str.cont.1prg. of Möbius trf's

Example: 1-dim. cases up to Möbius equ.

$\Phi^t = M_{a(t)} \circ U_t \qquad U_t \in \mathrm{Aut}(\mathbf{E},\{\ldots\})$ surj.lin.isom.

$t \mapsto a(t) = \Phi^t(0)$ cont. $\quad \Rightarrow \quad t \mapsto U_t x = (2\pi i)^{-1} \int_{|\zeta|=1} \zeta^{-1}\Phi(\zeta x)\, d\zeta$ cont. $\forall\, x$

Candidate of infinitesimal generator:

$\Phi' := \frac{d}{dt}\big|_{t=0+}\Phi^t, \quad \mathrm{dom}(\Phi') = \{x : \exists\, v \ \ \Phi^h(x) = x + hv + o(h)\}$

Lemma. $x \in \mathrm{dom}(\Phi') \iff t \mapsto \Phi^t(x)$ diff.

Proof. $\Phi^h(x) = x + hv + o(h) \implies \Phi^{t+h}(x) - \Phi^t(x) = \Phi^t\big(x + hv + o(h)\big) - \Phi^t(x) =$

$= h[D_{z=x}\Phi^t(z)]v + o(h)$ In particular $x \in \mathrm{dom}(\Phi') \Rightarrow x \in \mathrm{dom}\big(\frac{d}{ds}\big|_{s=t+0}\Phi^s\big)$.

$\Phi^{t-h}(x) - \Phi^t(x) = \Phi^{t-h}(x) - \Phi^{t-h}(x + hv + o(h)) =$

$= -[D_{z=x}\Phi^{t-h}(z)](hv+o(h)) + o(\|hv+o(h)\|) = -h[D_{z=x}\Phi^{t-h}(z)]v + o(h).$ Q. e. d.

Corollary. $x \in \mathrm{dom}(\Phi') \Longrightarrow \{\Phi^t(x) : x \in \mathbf{R}\} \subset \mathrm{dom}(\Phi')$.

Open problem: $\exists?$ $[\Phi^t : t \in \mathbf{R}]$ nowhere diff. in t?

Assumption: $\mathrm{dom}(\Phi') \cap \mathbf{B} \neq \emptyset$ or (up to Möbius equ.) $0 \in \mathrm{dom}(\Phi')$, $t \mapsto a(t)$ diff.

Lemma. $x \in \mathrm{dom}(\Phi') \iff t \mapsto U_t x$ diff. $\quad \big(U_h \in \mathrm{dom}(\Phi')\big)$.

Proof. $U_t x = M_{-a(t)} \underbrace{\Phi^t}_{M_{a(t)} \circ U_t} (x)$. $\quad (a, z) \mapsto M_a(z)$ real-anal.

$M_{c+hv+o(h)}(u + hw + o(h)) =$

$= (c+hv+o(h)) + B(c+hv+o(h))^{1/2}\big(1 + L(u+hw+o(h), c+hv+o(h))\big)^{-1}(u+hw+o(h)) =$

$= M_c(u) - h(L(w,c) + L(u,v))u + h\big(1 + L(u,c)\big)^{-1} w + o(h).$

Corollary. $\mathrm{dom}(\Phi')$ is closed to the Jordan-prod. $\{\ldots\}$

Proof. $x, y, z \in \mathrm{dom}(\Phi') \Rightarrow t \mapsto U_t\{xyz\} = \{(U_t x)(U_t y)(U_t z)\}$ diff.

Remark: In particular $\mathrm{dom}(\Phi') = \big[\text{Jordan subtriple}\big] \cap \overline{\mathbf{B}}$.
The orbit $\{\Phi^t(0) : t \in \mathbf{R}\} \subset \mathrm{dom}(\Phi')$.

Lemma. $x \in \mathrm{dom}(\Phi' \Longrightarrow U_h x \in \mathrm{dom}(\Phi'$ $(h \in \mathbf{R})$.

Proof. $U_h x \in \mathrm{dom}(\Phi' \iff t \mapsto U_h U_t x$ diff.

$\Phi^{t+h}(x) = \Phi^t \circ \Phi^h(x) = M_{a(t)} \circ U_t \circ M_{a(h)} \circ U_h x =^{U \circ M_a \circ U^{-1} = M_{Ua}}$

$= M_{a(t)} \circ M_{U_t a(h)} \circ U_t U_h x.$

$U_t U_h x = M_{-U_t a(h)} \circ M_{-a(t)} \circ \Phi^{t+h}(x),$ $\quad a(h) \in \mathrm{dom}(\Phi') \Rightarrow t \mapsto U_t a(h)$ diff.

$t \mapsto \Phi^t$ diff., $t \mapsto a(t)$ diff., $(a, b) \mapsto M_a \circ M_b$ real-anal. ; $\Longrightarrow t \mapsto U_t U_h x$ diff.

Notation: $\mathbf{D} := \overline{\mathrm{dom}(\Phi')}$ closure in $\mathbf{E}$, $\mathbf{F} := \mathrm{Span}(\mathbf{D})$

Proposition. We have seen: $\mathbf{F}$ closed JB*-subtriple in $\mathbf{E}$, $\mathbf{D} = \overline{\mathrm{Ball}(\mathbf{F})}$,

$$\{U_t|\mathbf{F} : t \in \mathbf{R}\} \subset \mathrm{Aut}(\mathbf{F}, \{\ldots\}), \quad \{M_{a(t)}|\mathbf{D} : t \in \mathbf{R}\} \subset \mathrm{Aut}_{\mathrm{hol}}(\mathbf{D}).$$

Remark. $[\Phi^t]^{-1} = \Phi^{-t} \iff U_t^{-1} M_{-a(t)} = M_{a(-t)} \circ U_{-t}$

$$\iff M_{-U_t^{-1}a(t)} \circ U_t^{-1} = M_{a(-t)} \circ U_{-t}$$
$$\iff U_t^{-1} = U_{-t} \text{ and } -U_t^{-1}a(t) = a(-t).$$

Lemma. $\mathbf{F}^{\perp \text{ Jordan}} = 0.$

Proof. $y \perp^{\text{Jordan}} x \in \mathbf{F}$, $x + y \in \mathbf{D} \implies M_{a(t)} U_t(x+y) = \underbrace{M_{a(t)} U_t(x)}_{\Phi^t(x)} + y$ diff. in t

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Assumumption: $e = \Phi^t(e) \ \ \forall\, t \in \mathbf{R}$ common fixed point

$\Lambda^t := \mathrm{D}_e \Phi^t$ $(: z \mapsto \frac{d}{dt}\big|_{t=0} \Phi^t(e + tz)$ Fréchet derivative

$\Lambda_t z = \int_{|\zeta|=1} \zeta^{-1} \Phi^t(e + \zeta z) d\zeta$

$[\Lambda^t : t \in \mathbf{R}]$ str.cont.1prg LIN $\mathbf{Z} := \mathrm{dom}(\Lambda')$ dense lin. in $\mathbf{E}$

$\Phi = M_a U$ $(= M_a \circ U)$ t FIX

$$w + e = \Phi(e + z) = M_a(Uz + Ue)$$

$$w + e = a + B(a)^{1/2}[1 + L(Ue + Ue, a)]^{-1}(Uz + Ue)$$

$$[1 + L(Uz + Ue, a)]B(a)^{-1/2}(w + (e - a)) = Uz + Ue$$

$$\Phi(e) = e \iff [1 + L(Ue, a)]B(a)^{-1/2}(e - a) = Ue$$

$$[1 + L(Uz + Ue, a)]B(a)^{-1/2}(w + (e - a)) - [1 + L(Ue, a)]B(a)^{-1/2}(e - a) = Uz$$

$$[1 + L(Uz + Ue, a)]B(a)^{-1/2}w + L(Uz, a)B(a)^{-1/2}(e - a) = Uz$$

$$w = B(a)^{1/2}[1 + L(Uz + Ue, a)]^{-1}[Uz - L(Uz, a)B(a)^{-1/2}(e - a)]$$

$\Phi(z+e) - e = w = (A_z + B)^{-1} C z$

$A_z = L(Uz, a) B(a)^{-1/2}, \quad B = [1 + L(Ue, a)] B(a)^{-1/2}, \quad C = U + L(U\bullet, a) B(a)^{-1/2}(a - e)$

$\Lambda z = \mathrm{D}_e \Phi = \frac{d}{dz}|_{z=0} (A_z + B)^{-1} C z = B^{-1} C z$

$\Phi^t(z+e) - e = (A_{t,z} + B_t)^{-1} C_t z$

$z \in \mathbf{Z} \Rightarrow \quad t \mapsto \Lambda^t z = B_t^{-1} C_t z$ diff.

$B_t^{-1} C_t z = B(a_t)^{1/2} [1 + L(U_t e, a_t)]^{-1} [U_t z + L(U_t z, a_t) B(a_t)^{-1/2}(a - e)]$

Question: $t \mapsto B_t^{-1} C_t z \Rightarrow^? t \mapsto U_t z$ diff.

$a \mapsto B(a)^\alpha, \ u \mapsto [1 + L(u, e)]^n$ norm-analytic in $\mathrm{GL}(\mathbf{E})$

$t \mapsto U_t z + L(U_t z, a_t) B(a_t)^{-1/2}(a - e)$ diff. $\quad (z \in \mathbf{Z})$

$L(U_t z, a_t) B(a_t)^{-1/2}(a - e) = \{[U_t z][a_t][B(a_t)^{-1/2}(a - e)]\} =$

$= U_t U_{-t} \{[U_t z][a_t][B(a_t)^{-1/2}(a - e)]\} = U_t \{z [U_{-t} a_t][U_{-t} B(a_t)^{-1/2}(a - e)]\}$ diff. in t

Theorem. If $0 \in \mathrm{dom}(\Phi')$ and $\bigcap_{t \in \mathbf{R}} \mathrm{Fix}(\Phi^t) \neq \emptyset$ then the generator Φ' is of Kaup's type:

$\mathrm{dom}(\Phi')$ subtriple in $\mathbf{E}$, $\quad \Phi'(z) = a - \{zaz\} + iAz$ closed.

Proof. $\mathrm{dom}(\Phi') = \{x : t \mapsto U_t \text{ diff.}\} = \mathrm{dom}(\Lambda')$ dense in $\mathbf{E}$, Λ' closed lin. op.

$\Phi^t(z+e) - e = (A_{t,z} + B_t)^{-1} C_t z$

$\Psi'(z+e) = -(A_{t,z} + B_t)^{-1} \left[\frac{d}{dt}(A_{t,z} + B_t)\right] (A_{t,z} + B_t)^{-1} C_t \big|_{t=0} + (A_{t,z} + B_t)^{-1} \frac{d}{dt} C_t z \big|_{t=0}$

$\Lambda'(z) = -B_t^{-1} \left[\frac{d}{dt} B_t\right] B_t^{-1} C_t \big|_{t=0} + B_t^{-1} \left[\frac{d}{dt} B_t\right] \big|_{t=0}$

Let $x_n \to x$, $\Psi'(x_n) \to y$.

$z_n := x_n - e$,

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

Let $x \in \mathrm{dom}(\Psi')$, $\||| = 1$, $\varphi \in \mathbf{E}^*$, $\langle \varphi, x \rangle = \|\varphi\| = 1$

Φ' is a TANGENT vector field to ∂bfB

$0 = \mathrm{Re}\langle \varphi \circ \overline{\kappa}, \Phi'(\kappa x) \rangle \qquad \Leftarrow |\kappa| = 1$

$\zeta \mapsto \langle \varphi, \Phi'(\zeta x) \rangle = \sum_{n=0}^{\infty} \alpha_n \zeta^n$ holomorphic

$\mathrm{Re}\big(\overline{\kappa} \sum_{n=0}^{\infty} \alpha_n \kappa^n\big) = 0$

$\sum_{n=0}^{\infty} (\alpha_n \kappa^{n-1} + \overline{\alpha_n} \kappa^{1-n}) = 0 \quad (|\kappa| = 1)$

$$\sum_{n=-\infty}^{\infty} \beta_n \kappa^n = 0 \qquad \beta_n = \alpha_{n+1} \ (n \geq 2), \quad \beta_n = \overline{\alpha_{1-n}} \ (n \leq -2),$$

$$\beta_1 = \alpha_2 + \overline{\alpha_0}, \quad \beta_{-1} = \alpha_0 + \overline{\alpha_2}, \quad \beta_0 = \alpha_1 + \overline{\alpha_1}$$

$\alpha_n = 0 \ (|n| \geq 2), \quad \alpha_1 + \overline{\alpha_1} = 0, \quad \alpha_2 = -\overline{\alpha_0}$

CONSIDER $\Omega() := \Phi'(x) - \{xbx\}$ INSTEAD OF Φ', $\quad b := \Psi'(0) = \frac{d}{dt} a(t)\big|_{t=0}$

This is also tangent to ∂bfB with $\Omega(0) = 0$

$\Omega(\zeta x) = \zeta \Omega(x)$ HOMOGENITY

SPIN FACTOR

$\big(\mathbf{E}, \langle \cdot | \cdot \rangle\big)$ Hilbert space, $\quad x \mapsto \overline{x}$ conjugation, $\quad \langle x|y \rangle^- = \langle \overline{x} | \overline{y} \rangle$

$$\{xay\} = \langle x|a \rangle y + \langle y|a \rangle x - \underbrace{\langle x|\overline{y} \rangle}_{\langle y|\overline{x} \rangle} \overline{a}$$

$$[\ \text{TRIPOTENTS}\,] = \Big\{ \lambda e : \ e \in \mathrm{Re}(\mathbf{E}), \ \lambda \in \mathbf{T}, \ \langle e|e \rangle = 1 \Big\} \cup \cup \Big\{ u + iv : \ u, v \in \mathrm{Re}(\mathbf{E}), \ \langle u|u \rangle = \langle v|v \rangle = 1/2, \ \langle u|v \rangle = 0 \Big\}$$

$U_t = \kappa_t V_t$: $\quad V_t \quad$ real $\langle \cdot | \cdot \rangle$-unitary, $\mathrm{Re}(\mathbf{E}) \to \mathrm{Re}(\mathbf{E})$, $\kappa_t \in \mathbf{T}$.

Problem. $z \in \mathbf{D} \Rightarrow^{?} \overline{z} \in \mathbf{D}$ $\quad$ $\big($i.e. $t \mapsto U_t z$ diff. $\Rightarrow^{?}$ $t \mapsto U_t \overline{z}$ diff.$\big)$ $\quad$ [YES]

Lemma. $\exists\, x \quad t \mapsto U_t x, U_t \overline{x}$ diff. $\Longrightarrow \quad \exists\, t \mapsto \varepsilon_t \in \{\pm 1\} \quad t \mapsto \varepsilon_t \kappa_t$ diff.

Proof. $t \mapsto \overline{U_t \overline{x}} = \overline{\kappa_t V_t \overline{x}} = \overline{\kappa_t} V_t x$ diff.

$t \mapsto \langle \kappa_t V_t x | \overline{\kappa_t} V_t x \rangle = \kappa_t^2$ diff.

$\forall\, h \in \mathbf{R} \quad \exists\, I_h$ open intv. around h, $\qquad \mathrm{Re}(\kappa_t^2 / \kappa_h^2) > 0$ $(t \in I_h)$

$\ldots, J_{-2}, J_{-1}, J_0, J_1, J_2, \ldots$ chain of intervals $\quad J_k \subset I_{h_k}$ $(k = 0, \pm 1, \ldots)$

$\exists\, k \mapsto \nu_k \in \{\pm 1\} \qquad \varepsilon_t := \nu_k \mathrm{sgn}(\kappa_t / \kappa_h)$ $(t \in J_k)$ well-def. and suits

Corollary. $\mathbf{F} \cap \mathrm{conj}(\mathbf{F}) \neq 0 \Rightarrow \mathbf{F} = \mathrm{conj}(\mathbf{F})$

Proof. $0 \neq x \in \mathbf{F} \cap \mathrm{conj}(\mathbf{F}) \Rightarrow t \mapsto U_t x, U_t \overline{x}$ diff. $\Rightarrow t \varepsilon_t \kappa_t, \varepsilon_t \kappa_t^{-1}$ diff.

$z \in \mathbf{F} \Rightarrow t \mapsto \varepsilon_t V_t z = \varepsilon_t \kappa_t^{-1} U_t z$ diff. $\Rightarrow t \mapsto \mathrm{conj}\big(\varepsilon_t \kappa_t^{-1} V_t z\big) = U_t \overline{z}$ diff.

Proposition. $\mathbf{F}$ is closed under conjugation in any case.

Proof. The only case of a JB*-subtriple $\mathbf{H}$ such that $\mathbf{H} \cap \mathrm{conj}(\mathbf{H}) = 0$ is if $\mathbf{H}$ is a Hilbert space spanned by a collinear grid $\{2^{-1/2}(u_k + i v_k) : k \in \mathcal{K}\}$ where $\{a_k, b_k : k \in |\mathcal{K}\}$ is $\langle \cdot | \cdot \rangle$-orthononormed. Also $\mathrm{TRIP}(\mathbf{H}) = \{w + iT(w) : w \in \mathbf{G}, \langle w | w \rangle = 1/2\}$ with some subspace $\mathbf{G} \subset \mathrm{Re}(\mathbf{E})$ and an isometry $T : \mathrm{Sphere}(\mathbf{G}) \to \mathrm{Re}(\mathbf{E})$. The case $\mathbf{F} = \mathbf{H}$ is impossible: then $t \mapsto a_t = w_t + iT(w_t)$ diff. $\Rightarrow t \mapsto \overline{a_t} = w_t - iT(w_t)$ diff. $\Rightarrow \{a_t, \overline{a_t} : t \in \mathbf{R}\} \subset \mathbf{F}$.

Assumption without loss of gen.: $U_t = \kappa_t V_t, \quad t \mapsto \kappa_t$ diff.

Notation: $\mathbf{F}^\perp := \{x \in \mathbf{E} : \langle x | \mathbf{F} \rangle = 0\}$. $\quad (\neq \mathbf{F}^{\perp \mathrm{Jordan}})$

Proposition. $\mathbf{E} = \mathbf{F}$ $\;$ (i.e. $\mathbf{F}^\perp = 0$).

Proof. $\mathbf{F} = \mathrm{conj}(\mathbf{F}) \Rightarrow \mathbf{F}^\perp = \mathrm{conj}(\mathbf{F}^\perp)$ spin factor. $\dim(\mathbf{F}^\perp) > 0 \Rightarrow \exists\, y \in \mathbf{F}^\perp \quad 0 \neq y = \overline{y}$

Calculate $t \mapsto \Phi^t(y) = M_{a(t)} \circ U_t y$.

$M_a(x) = a + B(a)^{1/2}[1 + L(x,a)]^{-1}x, \quad B(a) = 1 - 2L(a) + Q_a^2 : z \mapsto z - 2\{aaz\} + \{a\{aza\}a\}$

$y \in \mathbf{F}^\perp,\ a \in \mathbf{F} \Rightarrow \qquad \langle y|f\rangle = \langle y|\overline{f}\rangle = 0 \ (f \in \mathbf{F})$

$\{fgy\} = \langle f|g\rangle y + \langle y|g\rangle f - \langle y|\overline{f}\rangle \overline{g} = \langle f|g\rangle y, \ \ \{fyg\} = \langle f|y\rangle g + \langle g|y\rangle f - \langle g|\overline{f}\rangle \overline{y} = -\langle g|\overline{f}\rangle \overline{y}$

$x_1 + y_1 = (1 + L(y,a))^{-1} y$

$y = (1 + L(y,a))(x_1 + y_1) = x_1 + y_1 + \{yax_1\} + \{yay_1\}$

$0 = x_1 - \langle y|\overline{y_1}\rangle \overline{a} \quad$ ($\mathbf{F}$-component), $\qquad y = y_1 + \langle x_1|a\rangle y \quad$ ($\mathbf{F}^\perp$-component)

$\gamma = \gamma(y, y_1) := \langle y|\overline{y_1}\rangle = \langle y_1|\overline{y}\rangle$

$x_1 = \langle y|\overline{y_1}\rangle \overline{a} = \gamma \overline{a}, \qquad y_1 = (1 - \langle x_1|a\rangle) y = (1 - \gamma\langle \overline{a}|a\rangle) y$

$$\gamma = \langle y_1|y\rangle = (1 - \gamma\langle \overline{a}|a\rangle)\langle y|\overline{y}\rangle, \quad \Longrightarrow \quad \gamma = \frac{\langle y|\overline{y}\rangle}{1 + \langle \overline{a}|a\rangle\langle y|\overline{y}\rangle}$$

$$[1 + L(y,a)]^{-1} y = x_1 + y_1 = \gamma \overline{a} + (1 - \gamma\langle a|\overline{a}\rangle) y = \frac{\langle y|\overline{y}\rangle \overline{a} + y}{1 + \langle \overline{a}|a\rangle\langle y|\overline{y}\rangle}$$

$z \perp \mathbf{F} \Rightarrow \quad B(a)z = z - 2\{aaz\} + \{a\{aza\}a\} = z - 2\langle a|a\rangle z + |\langle a|\overline{a}\rangle|^2 z$

$B(a)^{1/2} z = \beta(a) z \quad \beta(a) := \sqrt{1 - 2\langle a|a\rangle + |\langle a|\overline{a}\rangle|^2}$

$U_t y = \kappa_t V_t y, \quad t \mapsto \langle U_t y|\overline{U_t y}\rangle = \kappa_t^2 \langle y|\overline{y}\rangle$ diff.

$$\begin{aligned} t \mapsto \Phi^t(y) = M_{a(t)} \circ U_t y &= a(t) + B(a(t))^{1/2}[1 + L(U_t y, a(t))]^{-1} U_t y = \\ &= a(t) + \beta(a(t)) \frac{\langle y|\overline{y}\rangle \overline{a(t)} + U_t y}{1 + \langle \overline{a(t)}|a(t)\rangle\langle y|\overline{y}\rangle} \end{aligned}$$

IF $\dim(\mathbf{F}^\perp = 1$ THEN $V_t y = y$ and $T_t y = \kappa_t y \Longrightarrow \dim(\mathbf{F}^\perp) = 1$ impossible

CASE $\dim(\mathbf{F}^\perp) > 1$

We can find $y \in \mathbf{F}^\perp$ with $0 \neq y \perp \overline{y}$

Calculate $t \mapsto \Phi^t(x+y) = M_{a(t)} \circ U_t(x+y)$.

$M_a(x+y) = a + B(a)^{1/2}[1 + L(x+y,a)]^{-1}(x+y), \quad B(a) = 1 - 2L(a) + Q_a^2 : z \mapsto$

$z - 2\{aaz\} + \{a\{aza\}a\}$

$y \in \mathbf{F}^\perp,\ a \in \mathbf{F} \Rightarrow \qquad \langle y|f\rangle = \langle y|\overline{f}\rangle = 0\ (f \in \mathbf{F})$

$\{fgy\} = \langle f|g\rangle y + \langle y|g\rangle f - \langle y|\overline{f}\rangle \overline{g} = \langle f|g\rangle y,\ \ \{fyg\} = \langle f|y\rangle g + \langle g|y\rangle f - \langle g|\overline{f}\rangle \overline{y} = -\langle g|\overline{f}\rangle \overline{y}$

$x_1 + y_1 = (1 + L(x+y,a))^{-1}(x+y)$

$x + y = (1 + L(x+y,a))(x_1+y_1) = x_1 + y_1 + \{xax_1\} + \{xay_1\} + \{yax_1\} + \{yay_1\}$

$x = x_1 + \{xax_1\} - \langle y|\overline{y_1}\rangle \overline{a}$ ($\mathbf{F}$-component), $\qquad y = y_1 + \langle x|a\rangle y_1 + \langle x_1|a\rangle y$ ($\mathbf{F}^\perp$-component)

$\gamma_0 = \gamma_0(x_1,a) := \big(1 - \langle x_1|a\rangle\big)/(1 + \langle x|a\rangle\big)$

$y_1 = \gamma_0 y$

Consider vectors y with $0 \neq y \perp \overline{y}$: $\quad x = x_1 + \{xax_1\} - \langle y\big|\overline{\gamma_0\, y}\rangle \overline{a} = x_1 + \{xax_1\}$

$$x_1 = [1 + L(x,a)]^{-1}x, \qquad y_1 = \frac{1 - \langle [1 + L(x,a)]^{-1}x|a\rangle}{1 + \langle x|a\rangle} = \gamma(x,a)y$$

$x_2 + y_2 = B(a)^{1/2}(x_1 + y_1)$

$$M_a(x+y) = a + B(a)^{1/2}(x_1+y_1) = a + B(a)^{1/2}\big([1 + L(x,a)]^{-1}x + \gamma(x,a)y\big] = \\ = M_a(x) + \gamma(x,a)B(a)^{1/2}y \qquad \text{if } y \perp \overline{y} \in \mathbf{F}^\perp$$

$z \perp \mathbf{F} \Rightarrow \quad B(a)z = z - 2\{aaz\} + \{a\{aza\}a\} = z - 2\langle a|a\rangle z + |\langle a|\overline{a}\rangle|^2 z$

$B(a)^{1/2}z = \beta(a)z \quad \beta(a) := \sqrt{1 - 2\langle a|a\rangle + |\langle a|\overline{a}\rangle|^2}$

If $y \perp \overline{y} \in \mathbf{F}^\perp$ then $U_t y \in \mathbf{F}^\perp$, $\langle U_t y|\overline{U_t y}\rangle = \langle \kappa_t V_t|\overline{\kappa_t V_t y}\rangle = \kappa_t^2 \langle y|\overline{y}\rangle = 0$,
$\Phi^t(x+y) = M_a(U_t x + U_t y) = M_{a(t)}(U_t x) + \beta(a(t))\gamma(U_t x, a(t))U_t y =$

$$= \Phi^t(x) + \beta(a(t))\gamma(U_t x, a(t))U_t y$$

$\gamma(0,a) \equiv 0$, $t \mapsto a(t)$ diff. $\Rightarrow$

$t \mapsto \Phi^t(y) = \underbrace{\Phi^t(0)}_{a(t)} + \beta(a(t))y$ diff. whenever $y \perp \overline{y} \in \mathrm{Ball}(\mathbf{F}^\perp)$

Thus $0 \neq y \in \mathbf{F}^\perp = 0$ contradiction if we assume $\dim(\mathbf{F}^\perp) > 1$

Proof. $\mathbf{F} = \mathrm{conj}(\mathbf{F}) \Rightarrow \mathbf{F}^\perp = \mathrm{conj}(\mathbf{F}^\perp)$ spin factor. $\dim(\mathbf{F}^\perp) > 1 \Rightarrow \exists\, y \in \mathbf{F}^\perp \quad 0 \neq y \perp \overline{y}$

Calculate the effect of $\Phi^t = M_{a(t)} \circ U_t$ on $\mathbf{F}^\perp$.

$M_a(x) = a + B(a)^{1/2}[1 + L(x,a)]^{-1}x, \quad B(a) = 1 - 2L(a) + Q_a^2 : z \mapsto z - 2\{aaz\} + \{a\{aza\}a\}$

$y \in \mathbf{F}^\perp \Rightarrow \qquad \langle y|f\rangle = \langle y|\overline{f}\rangle = 0 \ (f \in \mathbf{F})$

$\{fgy\} = \langle f|g\rangle y + \langle y|g\rangle f - \langle y|\overline{f}\rangle \overline{g} = \langle f|g\rangle y, \ \ \{fyg\} = \langle f|y\rangle g + \langle g|y\rangle f - \langle g|\overline{f}\rangle \overline{y} = -\langle g|\overline{f}\rangle \overline{y}$

$x_1 + y_1 = (1 + L(x+y,a))^{-1}(x+y)$

$x + y = (1 + L(x+y,a))(x_1+y_1) = x_1 + y_1 + \{xax_1\} + \{xay_1\} + \{yax_1\} + \{yay_1\}$

$x = x_1 + \{xax_1\} - \langle y|\overline{y_1}\rangle \overline{a}, \qquad y = y_1 + \langle x|a\rangle y_1 + \langle x_1|a\rangle y$

$y_1 = \dfrac{1 - \langle x_1|a\rangle}{1 + \langle x|a\rangle} y = \gamma_0(x_1,a)y$

Consider vectors y with $y \perp \overline{y}$: $\quad x = x_1 + \{xax_1\} - \langle y|\overline{\gamma_0 y}\rangle \overline{a} = x_1 + \{xax_1\}$

$x_1 = [1 + L(x,a)]^{-1}x, \quad y_1 = \dfrac{1 - \langle [1 + L(x,a)]^{-1}x|a\rangle}{1 + \langle x|a\rangle} y \ \ \underline{= \gamma(x,a)y} \quad (y \perp \overline{y})$

$x_2 + y_2 = B(a)^{1/2}(x_1 + y_1)$

$$\begin{aligned} M_a(x+y) &= a + B(a)^{1/2}(x_1+y_1) = a + B(a)^{1/2}\big([1 + L(x,a)]^{-1}x + \gamma(x,a)y\big] = \\ &= M_a(x) + \gamma(x,a)B(a)^{1/2}y \qquad \text{if } y \perp \overline{y} \in \mathbf{F}^\perp \end{aligned}$$

$z \perp \mathbf{F} \Rightarrow \quad B(a)z = z - 2\{aaz\} + \{a\{aza\}a\} = z - 2\langle a|a\rangle z + |\langle a|\overline{a}\rangle|^2 z$

$B(a)^{1/2}z = \beta(a)z \quad \beta(a) := \sqrt{1 - 2\langle a|a\rangle + |\langle a|\overline{a}\rangle|^2}$

If $y \perp \overline{y} \in \mathbf{F}^\perp$ then $U_t y \in \mathbf{F}^\perp$, $\langle U_t y | \overline{U_t y}\rangle = \langle \kappa_t V_t | \overline{\kappa_t V_t y}\rangle = \kappa_t^2 \langle y|\overline{y}\rangle = 0$,

$$\begin{aligned}\Phi^t(x+y) &= M_a(U_t x + U_t y) = M_{a(t)}(U_t x) + \beta(a(t))\gamma(U_t x, a(t))U_t y = \\ &= \Phi^t(x) + \beta(a(t))\gamma(U_t x, a(t))U_t y\end{aligned}$$

$\gamma(0,a) \equiv 0$, $t \mapsto a(t)$ diff. $\Rightarrow$

$t \mapsto \Phi^t(y) = \underbrace{\Phi^t(0)}_{a(t)} + \beta(a(t))y$ diff. whenever $y \perp \overline{y} \in \mathrm{Ball}(\mathbf{F}^\perp)$

Thus $0 \neq y \in \mathbf{F}^\perp = 0$ contradiction if we assume $\dim(\mathbf{F}^\perp) > 1$

$x_1 := \Phi^t(x)$, $y_1 := \beta(a(t))\gamma(U_t x, a(t))U_t y \qquad \langle y_1 | \overline{y_1}\rangle = 0$

$$\begin{aligned}\Phi^{t+h}(x+y) &= \Phi^h\big(\Phi^t(x+y)\big) = \Phi^h(x_1+y_1) = \Phi^h(x_1) + \beta(a(h))\gamma(U_h x_1, a(h))U_h y_1 = \\ &= \Phi^{t+h}(x) + \beta(a(h))\gamma\big((U_h\Phi^t(x), a(h)\big)\beta(a(t))\gamma\big((U_t x, a(t)\big)U_h U_t y\end{aligned}$$

$$\Phi^{t+h}(x+y) = \Phi^{t+h}(x) + \beta(a(t+h))\gamma\big(U_{t+h}x, a(t+h)\big)U_{t+h}y$$

$$U_h U_t y = \frac{\beta(a(h))\gamma\big(U_h\Phi^t(x), a(h)\big)\beta(a(t))\gamma\big(U_t x, a(t)\big)}{\beta(a(t+h))\gamma\big(U_{t+h}x, a(t+h)\big)} U_{t+h} \quad \big(\mathrm{Span}\{\text{admissible } y\} = \mathbf{F}^\perp\big)$$

$x := 0 \Rightarrow \quad x_1 = \Phi^t(x) = a(t)$, $\Phi^h(x_1) = a(t+h)$, $\gamma(0,a) = 1$

$$U_h U_t = \lambda(h,t)U_{t+h}, \qquad \lambda(h,t) := \frac{\beta(a(h))\gamma\big(U_h a(t), a(h)\big)\beta(a(t))}{\beta(a(t+h))}$$

Fractional linear forms

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2),$$

$$\mathcal{F}(\mathcal{A}) : X \mapsto (AX+B)(CX+D)^{-1} = \big[\mathcal{A}(X\ 1)^{\mathrm{T}}\big]_1 \big[\mathcal{A}(X\ 1)^{\mathrm{T}}\big]_2^{-1}$$

$$\mathcal{F}(\mathcal{A}\mathcal{B}) = \mathcal{F}(\mathcal{A}) \circ \mathcal{F}(\mathcal{B})$$

$$M_a = \mathcal{F}(\mathcal{M}_a, \quad \mathcal{M}_a = \mathrm{diag}\begin{pmatrix} (1-aa^*)^{-1/2} \\ (1-a^*a)^{-1/2}\end{pmatrix}\begin{bmatrix} 1 & a \\ a^* & 1\end{bmatrix}$$

Surj. lin. isom: $X \mapsto UXV^*$, unitary $U \in \mathcal{L}(\mathbf{H}_1)$, $V \in \mathcal{L}(\mathbf{H}_2)$

$\Phi^t := \mathcal{F}(\mathcal{A}_t), \qquad \big[\phi^t : t \in \mathbf{R}\big]$ str.cont,1prg.

$\mathcal{A}_t = \mathcal{M}_{a(t)}\mathrm{diag}(U_t, V_t)$

Attention: $U \otimes V^* = \mathcal{F}\big(\mathrm{diag}(U,V)\big) = \mathcal{F}\big(\kappa\mathrm{diag}(U,V)\big)$ with any $\kappa \in \mathbf{T}$

Adjusted str.cont.: [Stachó JMAA 2010, Cor. 2.6]

$\exists\, t \mapsto \kappa(t) \in \mathbf{T} \quad t \mapsto \kappa(t)U_t, t \mapsto \kappa(t)V_t$ str.cont.

Case of $\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ with $r := \dim(\mathbf{H}_2) < \infty$

Assumptions without loss of gen. *up to Möbius equ.*:

(1) $\mathcal{A}_t\mathcal{A}_h = \lambda(t,h)\mathcal{A}_{t+h}, \qquad \lambda(t,h) \in \mathbf{T} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$

(2) $\mathcal{A}_t = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \qquad t \mapsto A_t, B_t, C_t, D_t$ str.cont.

(3)* $\exists$ common fixed point (by reflexivity): $\mathcal{F}(\mathcal{A}_t)E = E$ $(t \in \mathbf{R})$.
Interesting only: $\|E\| = 1$, $\quad E$ TRIP

$\lambda(t,h) = \mathcal{A}_{-(t+h)}\mathcal{A}_t\mathcal{A}_h$ cont. in t,h $\quad$ (prod. of unif.bded. str.cont. lin. maps)

$S_t := \big[\mathcal{A}_t(E\ 1)^{\mathrm{T}}\big]_2$.

$\mathcal{A}_t(E\ 1)^{\mathrm{T}} = (E\ 1)^{\mathrm{T}} S_t,$

$S_t S_h = \lambda(t,h) S_h S_t$

Proposition. $[S_t : t \in \mathbf{R}]$ Abelian family, $\quad \lambda(t,h) \equiv \lambda(h,t)$.

$\mathrm{trace}\, AB = \mathrm{trace}\, BA$.

$\mathrm{trace}\, S_t S_h = \lambda(t,h)\mathrm{trace}\, S_h S_t = \lambda(t,h)\mathrm{trace}\, S_t S_h$.

$[\lambda(t,h) - 1]\mathrm{trace}\, S_t S_h = 0$

$\mathrm{trace}\, S_t S_h \to \mathrm{trace}\, 1 = \dim(\mathbf{H}_2)$ $(t \to 0)$.

$\exists\, \varepsilon > 0 \quad \lambda(t,h) = 1$ $(|t|, |h| < \varepsilon)$.

$S_t \smile S_h$ for $|t|, |h| < \varepsilon$.

$u, v \in \mathbf{R}$, $u/m, v/m \in (-\varepsilon, \varepsilon)$,

$S_u = \widetilde{\lambda} S^m_{u/m}$, $S_v = \widetilde{\mu} S^m_{v/m}$ $\quad \exists\, \widetilde{\lambda}, \widetilde{\mu} \in \mathbf{T}, \Longrightarrow S_u \smile S_v$ Q.e.d.

Remark: In infinite dimensions, $AB = \lambda BA \neq 0 \not\Rightarrow A \smile B$ even if $\lambda \in \mathbf{T}$.

Example: $A : e_n \mapsto e_{n+1}$ $(n = 0, \pm 1, \ldots)$ bilateral shift, $B : e_n \mapsto \lambda^n e_n$.

$S_t = \sigma_1(t) e_1 \otimes e_1^* + \cdots + \sigma_r(t) e_r \otimes e_r^* + N$ Jordan normal form

$\sigma_k(t) \in \mathbf{T}$, $r = \dim(\mathbf{H}_2)$, N nilpotent, $N \smile \sigma_1(t) e_1 \otimes e_1^* + \cdots + \sigma_r(t) e_r \otimes e_r^*$.

$S_t S_h = \lambda(t, h) S_{t+h}$

$\sigma_k(t)\sigma_k(h) = \lambda(t, h)\sigma_k(t + h) \quad (k = 1, \ldots, r)$,

$\lambda(t, h) = \sigma_k(t)\sigma_k(h)\sigma_k(t + h)^{-1}$.

$\kappa(t) := \sigma_1(t)^{-1}$, $\quad \widetilde{S}_t := \kappa(t) S_t$.

$\lambda(t, h) = \kappa_k(t)^{-1}\kappa_k(h)^{-1}\kappa_k(t + h)$.

$\widetilde{S}_t = e_1 \otimes e_1^* + \sum_{k=2}^r \widetilde{\sigma}_k(t) e_k \otimes e_k^* + \widetilde{N}$ Jordan form, $\qquad \widetilde{\sigma}_k(t) = \kappa(t)\sigma_k(t)$.

$$\widetilde{\sigma}_k(t)\widetilde{\sigma}_k(h)\widetilde{\sigma}_k(t + h)^{-1} = \sigma_k(t)\sigma_k(t)\sigma_k(t + h)^{-1}\kappa_k(t)\kappa_k(t)\kappa_k(t + h)^{-1} =$$

$$= \sigma_k(t)\sigma_k(t)\sigma_k(t + h)^{-1}\lambda(t, h)^{-1} = 1.$$

$\widetilde{\mathcal{A}}_t := \kappa(t)\mathcal{A}_t$.

$\mathcal{A}_t\mathcal{A}_h = \lambda(t, h)\mathcal{A}_{t+h} \Longrightarrow \widetilde{\mathcal{A}}_t\widetilde{\mathcal{A}}_h = \widetilde{\mathcal{A}}_{t+h}$.

Proposition. $[\kappa(t) S_t : t \in \mathbf{R}]$, $[\kappa(t)\mathcal{A}_t : t \in \mathbf{R}]$ str.cont.1prg.

Proof. Proved: they are 1prg's. Strong cont.: $t \mapsto \kappa(t) = \sigma_1(t)^{-1} = \langle S_t e_1 | e_1\rangle^{-1}$ cont.

Assumption without loss of gen.:

$\mathbf{(1)}^*$ $[\mathcal{A}_t : t \in \mathbf{R}]$, $[S_t : t \in \mathbf{R}]$ str.cont.1prg. $S_t := [\mathcal{A}_t(E\ 1)^{\mathrm{T}}]_2^{-1} = (C_t E + D_t)^{-1}$

$$\mathcal{A}' := \frac{d}{dt}\Big|_{t=0} \mathcal{A}_t = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{d}{dt}\Big|_{t=0} \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\}$$

$\mathbf{D} := \mathrm{dom}(\mathcal{A}')$ dense lin. submanifold in $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$

Observation: $t \mapsto \Phi^t(X)$ diff. whenever $\begin{bmatrix} Xy \\ y \end{bmatrix} \in \mathbf{D}\ \ \forall y \in \mathbf{H}_2$.

Proof: $X \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2) \implies$ since $\dim(\mathbf{H}_2) < \infty$,

$t \mapsto \mathcal{A}_t \begin{bmatrix} X \\ 1 \end{bmatrix}$ diff. $\iff$ $t \mapsto \mathcal{A}_t \begin{bmatrix} X \\ 1 \end{bmatrix} y = \mathcal{A}_t \begin{bmatrix} Xy \\ y \end{bmatrix}$ diff. $\forall y \in \mathbf{H}_2$.

$\mathcal{A}'$ is of $\mathbf{H}_1 \oplus \mathbf{H}_2$-split matrix form if $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$

$r := \dim(\mathbf{H}_2) < \infty$, $\{e_1, \ldots, e_r\}$ lin.indep. basis in $\mathbf{H}_2$

$\mathcal{A}'$ is $\mathbf{H}_1 \oplus \mathbf{H}_2$-split $\iff$ $0 \oplus e_k = \begin{bmatrix} 0 \\ e_k \end{bmatrix} \in \mathbf{D}$ $(k = 1, \ldots, r)$.

Lemma. $\exists\, F \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ $\|F\| < 1$, $\mathcal{M}_F \mathcal{A}' \mathcal{M}_F^{-1}$ is $\mathbf{H}_1 \oplus \mathbf{H}_2$-split.

Proof. $E = \sum_{k=1}^{r} g_k \otimes e_k^*$, $\{g_1, \ldots, g_r\}$, $\{e_1, \ldots, e_r\}$ ortn. (E TRIP fixed point)

$$g_k \oplus e_k = \begin{bmatrix} Ee_k \\ e_k \end{bmatrix} \in \mathbf{D}, \qquad \mathbf{D} = \mathbf{D}_1^0 \oplus 0 + \sum_{k=1}^{r} \mathbf{C} g_k \oplus e_k, \quad \mathbf{D}_1^0 := \{x : x \oplus 0 \in \mathbf{D}\}$$

$$\exists ?\ \ F = \sum_{k=1}^{r} f_j \otimes e_j^*, \quad \exists ?\ \ h_k \in \mathbf{D} \qquad \mathcal{M}_F(h_k \oplus e_k) = 0 \oplus e_k$$

$$\mathcal{M}_F := \mathrm{diag} \begin{bmatrix} (1 - FF^*)^{-1/2} \\ (1 - F^*F)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & F \\ F^* & 1 \end{bmatrix}, \quad \mathcal{M}_F^{-1} = \mathcal{M}_{-F}$$

$$\mathrm{dom}\big(\mathcal{M}_F \mathcal{A}' \mathcal{M}_F^{-1}\big) = \mathcal{M}_F \mathrm{dom}(\mathcal{A}') = \begin{bmatrix} 1 & F \\ F^* & 1 \end{bmatrix} \mathbf{D} \quad \text{if } \|F\| < 1$$

$$\begin{bmatrix} 1 & F \\ F^* & 1 \end{bmatrix} h_k \oplus e_k = \begin{bmatrix} 1 & F \\ F^* & 1 \end{bmatrix} \begin{bmatrix} h_k \\ e_k \end{bmatrix} = \begin{bmatrix} h_k + Fe_k \\ F^*h_k + e_k \end{bmatrix}$$

Choice for f_k, h_k: $\|F\| < 1$, $F^*h_k + e_k = 0$, $h_k = d_k + g_k$ with $d_k \in \mathbf{D}_1^0$

$$H := \sum_{k=1}^{r} d_k \oplus e_k^*, \quad H = E + \Delta \text{ with } \Delta := \sum_{k=1}^{r} d_k \oplus e_k^*,$$

$$(F^*H + 1)e_k = F^*h_k + e_k, \qquad \begin{bmatrix} 1 & F \\ F^* & 1 \end{bmatrix} \begin{bmatrix} H \\ 1 \end{bmatrix} = \begin{bmatrix} E + F \\ F^*H + 1 \end{bmatrix}$$

Requirement: $F^*H = -1(= -\mathrm{id}_{\mathbf{H}_2}), \quad \|F\| < 1.$

Choice for F: $F := H\Theta, \ \Theta \in \mathcal{L}(\mathbf{H}_2).$

$F^*H = \Theta^* H^* H, \quad \Theta = \Theta^* = -(H^*H)^{-1}$

$$F^*F = \Theta^* H^* H \Theta = (H^*H)^{-1} H^* H (H^*H)^{-1} = (H^*H)^{-1} =$$

$$= \big[(E+\Delta)^*(E+\Delta)\big]^{-1}$$

$\Rightarrow$ the requirement can be fulfilled.

Special choice for Δ to $M_F(E)$ TRIP

$\{d_1, \ldots, d_r\} = \lambda \cdot [\mathrm{ORTN}] = \lambda \Delta_0$ with $\langle d_k | e_\ell \rangle = 0,$

Possible since $\mathbf{D}_0^1$ is of *finite codim.*

$H^*H = (E+\Delta)^*(E+\Delta) = 1 + |\lambda|^2$ in $\mathcal{L}(\mathbf{H}_2)$

E, Δ_0 collinear TRIPs, $\quad F \in \mathbf{R}E + \mathbf{R}\Delta_0, \Rightarrow M_F(E)$ TRIP.

By means of Möbius equivalence, we may assume:

(4) $\mathcal{A}' := \frac{d}{dt}\Big|_{t=0} \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \mathbf{H}_1 \oplus \mathbf{H}_2$-split,

$A : [\mathbf{H}_1\text{−dense}] \to \mathbf{H}_1, \ C \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2), \ D \in \mathcal{L}(\mathbf{H}_2, \mathbf{H}_2).$

$$\mathcal{A}_t = \mathcal{M}_{a(t)} \mathrm{diag}(U_t, V_t) = \mathrm{diag}\begin{pmatrix} [1 - a(t)a(t)^*]^{-1/2} \\ [1 - a(t)^*a(t)]^{-1/2} \end{pmatrix} \begin{bmatrix} 1 & a(t) \\ a(t)^* & 1 \end{bmatrix} \mathrm{diag}(U_t, V_t).$$

$A_t = [1 - a(t)a(t)^*]^{-1/2} U_t, \quad B_t = [1 - a(t)a(t)^*]^{-1/2} a(t) V_t,$

$C_t = [1 - a(t)a(t)^*]^{-1/2} a(t)^* U_t, \quad D_t = [1 - a(t)a(t)^*]^{-1/2} V_t$

$\mathrm{dom}(\mathcal{A}') = \mathbf{D}_1 \oplus \mathbf{H}_2, \qquad \mathbf{D}_1 = \mathrm{dom}(A) = \mathrm{dom}\big(\frac{d}{dt}\big|_{t=0} U_t\big).$

$t \mapsto a(t) = B_t D_t^{-1}$ is differentiable, $\quad a(t) = tb + o(t)$ at $t = 0$

$$\mathcal{A}' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} U' & b \\ b^* & V' \end{bmatrix}, \quad U' := \tfrac{d}{dt}\big|_{t=0} U_t,\ V' := \tfrac{d}{dt}\big|_{t=0} V_t.$$

$$\mathcal{A}_t \begin{bmatrix} E \\ 1 \end{bmatrix} = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} E \\ 1 \end{bmatrix} = \begin{bmatrix} ES_t \\ S_t \end{bmatrix} = \begin{bmatrix} E \\ 1 \end{bmatrix} S_t$$

$[S_t : t \in \mathbf{R}]$ str.cont.1prg, $\quad S' := \tfrac{d}{dt}\big|_{t=0} S_t = \mathrm{gen}[S_t : t \in \mathbf{R}]$

$y \in \mathbf{H}_2 \ \Rightarrow\ t \mapsto \mathcal{A}_t \left[\begin{smallmatrix} Ey \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_t y$ diff., $\quad \left[\begin{smallmatrix} Ey \\ y \end{smallmatrix}\right] \in \mathrm{dom}(\mathcal{A}'), \quad Ey \in \mathbf{D}_1.$

Projective translation: $\mathcal{T} := \begin{bmatrix} 1 & E \\ & 1 \end{bmatrix}, \quad \mathcal{T}^{-1} := \begin{bmatrix} 1 & -E \\ & 1 \end{bmatrix}$

$\mathcal{B}_t := \mathcal{T}^{-1}\mathcal{A}_t\mathcal{T}, \quad \mathcal{B}' := \mathcal{T}^{-1}\mathcal{A}\mathcal{T}$

$\mathcal{A}' = \mathrm{gen}\big[\mathcal{A}_t : t \in \mathbf{R}\big], \quad \mathcal{B}' = \mathrm{gen}\big[\mathcal{B}_t : t \in \mathbf{R}\big], \quad \mathrm{dom}(\mathcal{B}') = \mathcal{T}^{-1}(\mathbf{D}_1 \oplus \mathbf{H}_2).$

$\mathrm{dom}(\mathcal{B}') = \big\{[d - Ey] \oplus y : d \in \mathbf{D}_1,\ y \in \mathbf{H}_2)\big\} = \mathbf{D}_1 \oplus \mathbf{H}_2 (= \mathrm{dom}(\mathcal{A}')).$

$$\mathcal{T}^{-1} \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \mathcal{T} = \begin{bmatrix} 1 & -E \\ & 1 \end{bmatrix} \begin{bmatrix} A_t & A_tE + B_t \\ C_t & C_tE + D_t \end{bmatrix} = \begin{bmatrix} 1 & -E \\ & 1 \end{bmatrix} \begin{bmatrix} A_t & ES_t \\ C_t & S_t \end{bmatrix} =$$
$$= \begin{bmatrix} A_t - EC_t & \mathbf{0} \\ C_t & S_t \end{bmatrix}.$$

$$\mathcal{B}' = \mathcal{T}^{-1}\mathcal{A}'\mathcal{T} = \begin{bmatrix} A' - EC' & 0 \\ C' & S' \end{bmatrix} = \begin{bmatrix} U' - Eb^* & 0 \\ b^* & b^*E + V' \end{bmatrix}$$

$W_t := \big[\mathcal{B}_t\big]_{11}$ str.cont.1prg. $\quad W' = \mathrm{gen}[W_t : t \in \mathbf{R}] = A' - EC' = U' - Eb^*$

$S_t := \big[\mathcal{B}_t\big]_{22}$ str.cont.1prg. $\quad S' = \mathrm{gen}[S_t : t \in \mathbf{R}] = C'E + D' = b^*E + V'$

Triangular lemma [Stachó JMAA 2016, Lemma 3.8] $\Rightarrow$

$$\mathcal{B}' = \mathrm{gen}\Bigg[\underbrace{\begin{bmatrix} W_t & 0 \\ \int_0^t S_{t-h}C'W_h\,dh & S_t \end{bmatrix}}_{\mathcal{B}_t} : t \in \mathbf{R}\Bigg]$$

$\Psi^t = \mathcal{F}(\mathcal{B}_t) :\ X \mapsto W_tX\big[\int_0^t S_{t-h}C'W_hX\,dh + S_t\big]^{-1},$

$\mathcal{A}' = \mathcal{T}\mathcal{B}'\mathcal{T}^{-1} = \mathrm{gen}\big[\mathcal{A}_t : t \in \mathbf{R}\big], \quad T := \mathcal{F}(\mathcal{T}) : X \mapsto X + E$

$\Phi^t = \mathcal{F}\big(\mathcal{A}_t\big) = \mathcal{F}\big(\mathcal{T}\mathcal{B}_t\mathcal{T}^{-1}\big) = T \circ \Psi_t \circ T^{-1}$

Closed integrated form: For all $X \in \mathrm{Ball}\big(\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)\big)$,

$$\Phi^t(X) = E + W_t(X-E)\Big[\int_0^t S_{t-h} \underbrace{C'}_{b^*} W_h(X-E)\, dh + S_t\Big]^{-1}.$$

Vector fields

$\Phi^t(X) \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ $\quad \big[\mathbf{H}_1 \to \mathbf{H}_2$ operators$\big]$

$t \mapsto \Phi^t(X)$ diff. $\iff$ $t \mapsto \Phi^t(X)y$ diff. $\forall\, y$ $\quad (\Leftarrow \dim(\mathbf{H}_2) < \infty.)$

If $\mathrm{ran}(X) \subset \mathbf{D}_1\big(= \mathrm{dom}([\mathcal{A}']_{11})\big)$ then

$t \mapsto \Phi^t(X)y = [A_t X + B_t][C_t X + D_t]^{-1} y$ $\quad$ diff. $\forall\, y$

$\Phi' := \frac{d}{dt}\big|_{t=0} \Phi^t, \qquad \mathrm{dom}(\Phi') = \{X : \mathrm{ran}(X) \subset \mathbf{D}_1\}$

Kaup type formula up to Möbius equ.:

$$\Phi'(X)y = \tfrac{d}{dt}\big|_{t=0}[A_t X + B_t][C_t X + D_t]^{-1} y = [A'X + B']y - X[C'X + D']y =$$

$$= \big[b - Xb^* X + U'X - XV'\big]y$$

$\Phi^t(E) = E \quad (t \in \mathbf{R}) \quad \Rightarrow \quad \Phi'(E) = 0$

$0 = b - Eb^*E + U'E - EV'$

$\Phi'(E) = 0 \iff E^*\Phi'(E) = 0 \iff V' = E^*U'E + E^*b - b^*E$ $\quad$ since $E^*E = 1$

$$\mathcal{B}' = \begin{bmatrix} U' - Eb^* & 0 \\ b^* & b^*E + V' \end{bmatrix} = \begin{bmatrix} U' - Eb^* & 0 \\ b^* & E^*U'E + E^*b \end{bmatrix}$$