On the structure of C_0-semigroups of holomorphic Carathéodory isometries in Hilbert space

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Abstract

We establish closed formulas for all strongly continuous one-parameter semigroups of holomorphic Carathéodory isometries of the unit ball of a Hilbert space in terms of spectral resolutions of skew self-adjoint dilations related to the Reich-Shoikhet nonlinear infinitesimal generator.

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1. Introduction

Throughout this work $\mathbf{H}$ denotes a fixed complex Hilbert space with scalar product $\langle x|\, y\rangle$ which is linear (i.e. $\mathbb{C}$-linear) in x and antilinear in y and we shall write $\|x\| := \langle x|x\rangle^{1/2}$ for the canonical norm. We use the notations $\mathbf{B} := \{x \in \mathbf{H} : \|x\| < 1\}$, $a^* := [x \mapsto \langle x|a\rangle]$ for the open unit ball, and the adjoint representation of bounded linear functionals, respectively. We regard the elements h, h^* ($h \in \mathbf{H}$) as column resp. row matrices and, given a linear map $A : \mathbf{S} \to \mathbf{H}$ on some linear submanifold of $\mathbf{H}$, we apply the canonical $\mathbf{H} \oplus \mathbb{C}$-split matrix identifications $x \oplus \xi \equiv \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$ resp. $\left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] \equiv \big[x \oplus \xi \mapsto (Ax+b) \oplus (c^*x+d)\big]$ with $x \in \mathbf{S}$, $b, c \in \mathbf{H}$ and $\xi, d \in \mathbb{C}$. This gives rise to the familiar linear representation of fractional linear maps on $\mathbf{H}$:

$$\mathfrak{F}\Big(\begin{bmatrix} A & b \\ c^* & d \end{bmatrix}\Big) := \big[x \mapsto (c^*x+d)^{-1}(Ax+b)\big].$$

Our object of chief interest will be the semigroup $\mathrm{Iso}(d_{\mathbf{B}})$ of all holomorphic isometries of $\mathbf{B}$ with respect to the Carathéodory metric $d_{\mathbf{B}}$. Recall [3, 4] that all its elements are fractional linear maps, namely they are compositions of *Möbius transformations*[1] with linear isometries of $\mathbf{H}$ (restricted to $\mathbf{B}$). In 1987, in his pioneering work [12], Vesentini established that the correspondence

$$\mathfrak{F}^{\#} : \big[\mathcal{U}^t : t\in\mathbb{R}_+\big] \mapsto \big[\mathfrak{F}(\mathcal{U}^t)|\mathbf{B} : \ t\in\mathbb{R}_+\big]$$

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[1] Fractional linear transformations mapping $\mathbf{B}$ injectively onto itself.

maps the family $\mathcal{C}_0\mathcal{S}\big(\mathrm{Iso}(\mathbf{H})\big)$ of all strongly continuous one-parameter semigroups of linear isometries of the indefinite norm $\|x\|^2-|\xi|^2$ on $\mathbf{H}\oplus\mathbb{C}$ into the family $\mathcal{C}_0\mathcal{S}\big(\mathrm{Iso}(d_{\mathbf{B}})\big)$ of all strongly continuous one-parameter semigroups $[\Psi^t : t\in\mathbb{R}_+]\subset\mathrm{Iso}(d_{\mathbf{B}})$.[2] According to [12, Th.VII], given $[\mathcal{U}^t : t\in\mathbb{R}_+]\in\mathfrak{S}$ with the infinitesimal generator $\mathcal{A}=\frac{d}{dt}\big|_{t=0+}\mathcal{U}^t$, for the corresponding non-linear objects $\Psi^t := \mathfrak{F}(\mathcal{U}^t)\big|\mathbf{B}$ we have $\big\{p\in\mathbf{B} : t\mapsto\Psi^t(p)$ is differentiable$\big\} = \big\{x\in\mathbf{B} :\ x\oplus 1\in\mathrm{dom}(\mathcal{A})\big\}$, and the latter set is dense in the ball $\mathbf{B}$. It is well known [12, 5] that here we can identify the linear operator $\mathcal{A}$ (which is densely defined in $\mathbf{H}\oplus\mathbb{C}$) with an $\mathbf{H}\oplus\mathbb{C}$-split matrix if and only if the orbit $t\mapsto\Psi^t(0)$ is differentiable. This happens if and only if the generator $\mathcal{A}$ has the form

$$\mathcal{A}=\begin{bmatrix} iA+\nu & b\\ b^* & \nu\end{bmatrix},\qquad \nu\in\mathbb{C},\ b\in\mathbf{H},\ A\in\mathrm{Her}_{\mathrm{s}}(\mathbf{H}) \tag{1.1}$$

with $\mathrm{dom}(\mathcal{A})=\mathrm{dom}(A)\oplus\mathbb{C}$ where $\mathrm{Her}_{\mathrm{s}}(\mathbf{H})$ stands for the family of all unbounded $\mathbf{H}$-*hermitian* operators (maximal symmetric closed linear operators with dense domain in $\mathbf{H}$). Even the cases with non-differentiable 0-orbit can be treated by passing to a semigroup $[\Phi^t : t\in\mathbb{R}_+]$ of the form $\Theta^{-1}\circ\Psi^t\circ\Theta$ with any Möbius transformation Θ such that $\Theta(0)\in\mathrm{dom}(\Gamma)$. Since the Möbius group is transitive on $\mathbf{B}$, hence any strongly continuous one-parameter semigroup $[\Psi^t : t\in\mathbb{R}_+]\in\mathcal{C}_0\big(\mathrm{Iso}(d_{\mathbf{B}})\big)$ is equivalent up to a Möbius transformation (*Möbius equivalent* for short in the sequel) to a semigroup $[\Phi^t : t\in\mathbb{R}_+]\in\mathcal{C}_0\big(\mathrm{Iso}(d_{\mathbf{B}})\big)$ whose infinitesimal generator [8, 9] has the form

$$\Gamma(x)=\frac{d}{dt}\Big|_{t=0+}\Phi^t \;=b-\langle x|b\rangle x+iAx,\quad x\in\mathrm{dom}(R)\cap\mathbf{B} \tag{1.2}$$

with some maximal symmetric operator A defined densely on $\mathbf{H}$ and some vector $b\in\mathbf{H}$. Also conversely, if iA is the infinitesimal generator for some strongly continuous one-parameter subsemigroup of $\mathcal{L}(\mathbf{H})$ then, for any $b\in\mathbf{H}$, the vector field (1.2) is the infinitesimal generator of a strongly continuous one-parameter subsemigroup of $\mathrm{Iso}(d_{\mathbf{B}})$. It is worth noticing that Kaup [6, 7] achieved a far-reaching Jordan-theoretical analog of (1.2) describing the complete holomorphic vector fields of the unit ball of JB*-triple and integrated them for the case $A=0$ resulting in a fractional linear type formula for generalized Möbius transformations. However, strong continuity destroys such an elegant setting. In [13, 5] these considerations were extended to semigroups of fractional linear transformations arising from a strongly continuous one-parameter semigroup applied to the solutions of Ricatti type equations $\dot{x}=\Gamma(x)$ with vector fields analogous to (1.2) in reflexive Hilbert C^*-modules, but without providing explicit algebraic formulas.

2. Results

Henceforth, for short, C_0-*semigroup* [resp. C_0-*group*] will mean *strongly continuous one-parameter semigroup* [-*group*]. We shall write $\mathrm{gen}[U^t : t\in\mathbb{R}_+]$ or $\mathrm{gen}[\widetilde{U}^t : t\in\mathbb{R}]$ for

[2] It seems that so far no argument appeared in the literature concerning the plausible surjectivity of the map $\mathfrak{F}^\#$. The question is rather harmless in our setting: in the case of the unit ball of a Hilbert space an argument with joint fixed points (Lemma 3.1) furnishes a positive answer. However, e.g. in the case of the unit ball of $\mathcal{L}(\mathbf{H})$, the surjectivity of the respective $\mathfrak{F}^\#$ seems to be open and highly non-trivial.

the infinitesimal generator of the C_0-semigroup $[U^t : t \in \mathbb{R}_+]$ or C_0-group $[\widetilde{U}^t : t \in \mathbb{R}]$, respectively. Given a closed subspace $\mathbf{K}$ in the Hilbert spaces $\mathbf{H}$ or $\mathbb{C}\overline{e} \oplus \mathbf{H}$, let $\mathrm{P}_{\mathbf{K}}$ be the orthogonal projection onto $\mathbf{K}$ (without danger of confusion).

In this paper we develop a triangularization method leading to explicit algebraic formulas for a C_0-semigroup generated by a vector field (1.2). This will be done in terms of fixed points of Γ and quadratures of a C_0-semigroup formed by complex linear isometries of a suitable 1-codimensional subspace of $\mathbf{H}$. As a consequence we conclude that any C_0-semigroup of holomorphic Carathéodory isometries of $\mathbf{B}$ admits a *dilation* to a C_0-group of surjective holomorphic Carathéodory isometries of the unit ball of some covering Hilbert space. Our fixed-point approach seems to be new even in finite dimensions (with uniformly continuous one-parameter groups).

Recall [3] that any Carathéodory isometry $\Psi \in \mathrm{Iso}(d_{\mathbf{B}})$ admits a continuous extension $\overline{\Psi}$ to the closed unit ball $\overline{\mathbf{B}}$. Given a C_0-semigroup $\mathbf{\Psi} = [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$, the extensions $\overline{\mathbf{\Psi}} := [\overline{\Psi^t i} : t \in \mathbb{R}_+]$ form also a C_0-semigroup (see [10] in a more general setting). According to [12, Section 7] $\overline{\mathbf{\Psi}}$ admits common fixed points whose family $\mathrm{Fix}(\overline{\mathbf{\Psi}})$ consists of one or two boundary points or it is the intersection of $\overline{\mathbf{B}}$ with some closed complex-affine submanifold containing points from $\mathbf{B}$. In the latter case $\mathbf{\Psi}$ is simply Möbius equivalent to a C_0-semigroup of linear isometries of $\mathbf{H}$ restricted to $\mathbf{B}$.

Our main goal is the following classification of the remaining cases with explicit formulas up to Möbius equivalence.

Theorem 2.1. *Suppose the vector field* (1.2) *is the infinitesimal generator of a C_0-semigroup $\mathbf{\Phi} := [\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ having a common boundary fixed point $\overline{e} \in \mathrm{Fix}(\overline{\mathbf{\Phi}}) \cap \partial\mathbf{B}$. Then for all points $x_0 + \xi\overline{e} \in \mathbf{B}$ with $x_0 \perp \overline{e}$ we have*

$$
\begin{aligned}
\mathrm{P}_{\mathbb{C}\overline{e}}\, \Phi^t(\xi\overline{e}+x_0) &= \Big[1 - (1-\xi)e^{-2\lambda t}\big/\varphi_{\lambda,\mu}(t,x_0,\xi)\Big]\overline{e}\,, \\
\mathrm{P}_{\mathbf{H}_0}\Phi^t(\xi\overline{e}+x_0) &= \Big[(1-\xi)e^{-2\lambda t}\big(\textstyle\int_0^t e^{\lambda s}V_0^s\, ds\big)b_0 + e^{-\lambda t}V_0^t x_0\Big]\Big/\varphi_{\lambda,\mu}(t,x_0,\xi)
\end{aligned}
\tag{2.2}
$$

where $\mathbf{H}_0 := \mathbf{H} \ominus \mathbb{C}\overline{e}$, $\lambda := \mathrm{Re}\langle\overline{e}|b\rangle$, $\mu := \mathrm{Im}\langle\overline{e}|b\rangle$, $b_0 := \mathrm{P}_{\mathbf{H}_0}b$ and $[V_0^t : t \in \mathbb{R}_+]$ is the C_0-semigroup of all linear $\mathbf{H}_0$-isometries generated by the skew-$\mathbf{H}_0$-hermitian operator $iP_{\mathbf{H}_0}(A-\mu)\big|\mathbf{H}_0$ and

$$
\begin{aligned}
\varphi_{\lambda,\mu}(t,x_0,\xi) := 1 &+ (1-\xi)\Big\langle \big(\textstyle\int_0^t e^{-2\lambda s}\int_0^s e^{\lambda r}V_0^r\, dr\, ds\big)b_0 \Big|\ b_0 \Big\rangle - \\
&- (1-\xi)(\lambda + i\mu)\textstyle\int_0^t e^{-2\lambda s} ds + \Big\langle \big(\textstyle\int_0^t e^{-\lambda s}V_0^s\, ds\big)x_0 \Big|\ b_0 \Big\rangle.
\end{aligned}
\tag{2.3}
$$

Remark 2.4. The following converse can be recovered from the proofs later on (see Remark 3.13). Given any couple of vectors $\overline{e}, b_0 \in \mathbf{H}$ such that $\|\overline{e}\| = 1$ and $b_0 \perp \overline{e}$ along with any C_0-semigroup $[V_0^t : t \in \mathbb{R}_+]$ of linear isometries of $\mathbf{H}_0 = \mathbf{H} \ominus (\mathbb{C}\overline{e})$ and two real constants λ, μ, the maps (2.2) form a C_0-semigroup in $\mathrm{Iso}(d_{\mathbf{B}})$.

Remark 2.5. In case of $\lambda \neq 0$, one can express the integrated operators in (2.2) in terms of the resolvent $R(\pm\lambda, iS_0)$ of the $\mathbf{H}_0$-hermitian operator $S_0 := i^{-1}\mathrm{gen}[V_0^t : t \in \mathbb{R}_+]$. Namely we have $\int_0^t e^{-\lambda\tau}V_0^\tau\, d\tau = \big(1 - e^{-\lambda t}V_0^t\big)R(\lambda, iS_0)$, $\int_0^t e^{-2\lambda\tau}\int_0^\tau e^{\lambda\sigma}V_0^\sigma d\sigma\, d\tau = \frac{1}{2\lambda}(1-e^{-2\lambda t})R(-\lambda, iS_0) - \big(1-e^{-\lambda t}V_0^t\big)R(\lambda, iS_0)R(-\lambda, iS_0)$.

Theorem 2.6. *Let* $\boldsymbol{\Psi} := [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ *be a* C_0*-semigroup with* $\overline{e} \in \mathrm{Fix}(\overline{\boldsymbol{\Psi}}) \subset \partial\mathbf{B}$*. Then, with the notations of Theorem 2.1, we have the following alternatives:*

(i) $\mathrm{Fix}(\overline{\boldsymbol{\Psi}})$ *consists of two points and* $\boldsymbol{\Psi}$ *is Möbius equivalent to some* C_0*-semigroup* $[\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ *of the form*

$$\Phi^t(\xi\overline{e} + x_0) = \frac{\xi + \tanh(\lambda t)}{1 + \xi\tanh(\lambda t)}\,\overline{e} + \frac{e^{-\lambda t}}{\cosh(\lambda t) + \xi\sinh(\lambda t)} V_0^t x_0\,; \tag{2.7}$$

(ii) $\{\overline{e}\} = \mathrm{Fix}(\overline{\boldsymbol{\Psi}})$*, there is a* $\boldsymbol{\Psi}$*-invariant disc of the form* $\emptyset \neq (\overline{e} + \mathbb{C}v) \cap \mathbf{B}$ *and* $\boldsymbol{\Psi}$ *is Möbius equivalent to a* C_0*-semigroup* $[\Phi^t : t \in \mathbb{R}_+]$ *of the form*

$$\Phi^t(\xi\overline{e} + x_0) = \frac{1 + i\mu t}{1 - i\mu t}\,\frac{\xi - i\mu t/(1 + i\mu t)}{1 + i\mu t\xi/(1 - i\mu t)}\,\overline{e} + \frac{1}{1 - i\mu t(1 - \xi)}\, V_0^t x_0\,; \tag{2.8}$$

(iii) *there is no* $\boldsymbol{\Psi}$*-invariant disc of the form* $\emptyset \neq (\overline{e}+\mathbb{C}v)\cap\mathbf{B}$ *and* $\boldsymbol{\Psi}$ *is Möbius equivalent to a* C_0*-semigroup* $[\Phi^t : t \in \mathbb{R}_+]$ *of the form* (2.2) *with* $\lambda = 0$.

Remark 2.9. In the setting of Theorem 2.1, a non-empty disc $(\overline{e} + \mathbb{C}v) \cap \mathbf{B}$ is $[\Phi^t : t \in \mathbb{R}_+]$-invariant if and only if $\overline{e} \not\perp v \in \mathrm{dom}(A)$ and $(iA + \langle\overline{e}|\, b\rangle)v \in \mathbb{C}\overline{e}$ as established in Lemma 4.1. Hence, only cases (i) or (ii) may appear in finite dimensions. Example 4.2, with possible independent interest for physics or stochastic processes, shows that case (iii) is not void.

Recall that, as an implicit simple special case[3] of [1, Main Theorem], every C_0-semigroup $[U^t : t\in\mathbb{R}_+]$ of $\mathbf{H}$-isometries admits a *unitary group dilation* in the following sense: there exists a Hilbert space $\widehat{\mathbf{H}}$ containing $\mathbf{H}$ as a subspace along with a C_0-group $[\widehat{U}^t : t\in\mathbb{R}]$ of unitary operators of $\widehat{\mathbf{H}}$ such that $U^t = \widehat{U}^t\big|\mathbf{H}$ $(t\in\mathbb{R}_+)$. Applying a unitary dilation $[\widehat{V}_0^t : r\in\mathbb{R}]$ of the isometry semigroup $[V_0^t : t\in\mathbb{R}_+]$ in (2.2), we readily obtain the following result with non-linear dilations.

Corollary 2.10. *Given any* C_0*-semigroup* $[\Psi^t : t \in \mathbb{R}_+]$ *of holomorphic Carathéodory isometries of* $\mathbf{B}$*, there is a* C_0*-group* $[\widehat{\Psi}^t : t \in \mathbb{R}_+]$ *of surjective holomorphic Carathéodory isometries of the unit ball* $\widehat{\mathbf{B}}$ *of some Hilbert space* $\widehat{\mathbf{H}}$ *containing* $\mathbf{H}$ *as a subspace such that* $\Psi^t = \widehat{\Psi}^t\big|\mathbf{B}$ $(t \in \mathbb{R}_+)$.

By means of the functional calculus of the skew self-adjoint generator $i\widehat{S}_0$ of the dilation group $[\widehat{V}_0^t : t \in \mathbb{R}]$ of the C_0-semigroup $[V_0^t : t \in \mathbb{R}_+]$, we get the following conclusion in the setting of Theorem 2.1.

Corollary 2.11. *In* (2.2) *we can write*

$$\varphi_{\lambda,\mu}(t,x_0,\xi) = \Big\langle x_0 \Big| f_1(t,\lambda,\widehat{S}_0)b_0\Big\rangle + (1-\xi)\Big[\Big\langle f_2(t,\lambda,\widehat{S}_0)b_0\Big| b_0\Big\rangle - (\lambda + i\mu)\textstyle\int_0^t e^{-2\lambda s}ds\Big] + 1,$$

$$P_{\mathbf{H}_0}\Phi^t(x) = \varphi_{\lambda,\mu}(t,x_0,\xi)^{-1}\Big[e^{-\lambda t}\exp(it\widehat{S}_0)x_0 + (1-\xi)e^{-2\lambda t}f_1(t,\lambda,\widehat{S}_0)b_0\Big]$$

[3]We begin Section 4 with an elementary proof in a Banach space setting.

with the bounded analytic functions $f_j(t,\lambda,\cdot):\mathbb{R}\to\mathbb{C}$ $(j=1,2;\ \lambda,t\in\mathbb{R})$ *defined by*

$$f_1(t,\lambda,\sigma):=\frac{1-e^{-t(\lambda+i\sigma)}}{\lambda+i\sigma}=\sum_{n=1}^{\infty}(-1)^{n-1}(\lambda+i\sigma)^{n-1}\frac{t^n}{n!},\qquad f_2(t,\lambda,\sigma):=$$

$$:=\frac{e^{-2\lambda t}}{2\lambda(\lambda+i\sigma)}+\frac{1}{2\lambda(\lambda-i\sigma)}-\frac{e^{-t(\lambda-i\sigma)}}{\lambda^2+\sigma^2}=\sum_{n=2}^{\infty}\Big[\frac{(-2\lambda)^n}{2\lambda(\lambda+i\sigma)}-\frac{(-\lambda+i\sigma)^n}{\lambda^2+\sigma^2}\Big]\frac{t^n}{n!}.$$

3. Triangularization with boundary fixed points

Lemma 3.1. *Assume* $\mathbf{\Psi}=[\Psi^t:t\in\mathbb{R}_+]\subset\mathrm{Iso}(d_{\mathbf{B}})$ *is a* C_0*-semigroup where* $\Psi^t=\mathfrak{F}(\mathcal{U}_t)\big|\mathbf{B}$ *with* $\mathcal{U}_t\in\mathcal{L}(\mathbf{H}\oplus\mathbb{C})$ $(t\in\mathbb{R}_+)$*. Then there is a family* $[\mu_t:t\in\mathbb{R}_+]\subset\mathbb{C}\setminus\{0\}$ *such that* $\big[\mu_t\mathcal{U}_t:t\in\mathbb{R}_+\big]$ *is a* C_0*-semigroup in* $\mathcal{L}(\mathbf{H}\oplus\mathbb{C})$*.*

Proof. Let o be a common fixed point of the transformations $\overline{\Psi^t}=\mathfrak{F}(\mathcal{U}_t)\big|\overline{\mathbf{B}}$ $(t\in\mathbb{R}_+)$. We are going to show that the choice $\mu_t:=\big[\mathcal{U}_t(o\oplus 1)\big]_{\mathbb{C}}^{-1}$ entailing $\mu_t\mathcal{U}_t(o\oplus 1)=o\oplus 1$ suits our requirements. Consider the matrices $\mathcal{V}^t:=\mu_t\mathcal{U}_t$. Clearly $\mathfrak{F}(\mathcal{V}^t)=\mathfrak{F}(\mathcal{U}_t)$ $(t\in\mathbb{R}_+)$. The map $\mathcal{U}\mapsto\mathfrak{F}(\mathcal{U})\big|\mathbf{B}$ is a homomorphism with respect to compositions, and its preimages are unique up to non-zero factors. Therefore we have $\mathfrak{F}(\mathcal{V}^{t+s})\big|\mathbf{B}=\Psi^{t+s}=\Psi^t\circ\Psi^s=\mathfrak{F}(\mathcal{V}^t\mathcal{V}^s)\big|\mathbf{B}$ and hence $\mathcal{V}^{t+s}=d_{t,s}\mathcal{V}^t\mathcal{V}^s$ $(t,s\in\mathbb{R})$ with suitable constants $d_{t,s}\neq 0$. The fixed point property

$$\mathcal{V}^t(\overline{e}\oplus 1)=\overline{e}\oplus 1\quad(t\in\mathbb{R}_+)\tag{3.2}$$

ensures that $d_{t,s}\equiv 1$, that is, the family $[\mathcal{V}^t:t\in\mathbb{R}_+]$ is a one-parameter matrix semigroup. To see its strong continuity, recall [3, Ch. VI] that the *Möbius shifts*

$$\Theta_a:=\mathfrak{F}\mathcal{M}_a,\ \ \mathcal{M}_a:=\begin{bmatrix}Q_a & a\\ a^* & 1\end{bmatrix},\ \ Q_a:=\mathrm{P}_{\mathbb{C}a}+\sqrt{1-\|a\|^2}(1-\mathrm{P}_{\mathbb{C}a})\quad(a\in\mathbf{B})\tag{3.3}$$

act transitively on $\mathbf{B}$. Thus, since every element of $\mathrm{Iso}(d_{\mathbf{B}})$ keeping the origin fixed is a restriction of a linear isometry of $\mathbf{H}$, we can write $\Psi^t=\Theta_{a_t}\circ U_t$ where $a_t:=\Psi^t(0)$ and U_t is a suitable linear isometry of $\mathbf{H}$. Since $U_t=\mathfrak{F}\big[\begin{smallmatrix}U_t & 0\\ 0 & 1\end{smallmatrix}\big]$, with suitable constants $\delta_t\neq 0$, we have

$$\mathcal{V}_t:=\delta_t\mathcal{M}_{a_t}\begin{bmatrix}U_t & 0\\ 0 & 1\end{bmatrix}=\delta_t\begin{bmatrix}Q_{a_t}U_t & a_t\\ [U_t^*a_t]^* & 1\end{bmatrix}\qquad(t\in\mathbb{R}_+).$$

The value of δ_t is determined unambiguously by (3.2): $\delta_t=\big[1+\langle U_t\overline{e}|a_t\rangle\big]^{-1}$. Thus to complete the proof, it suffices to see the continuity of the functions $t\mapsto a_t$, $t\mapsto[U_tx,Q_{a_t}x]$ $(x\in\mathbf{H})$. It is an immediate consequence of [2, Appendix A6] that the product $t\mapsto A_tB_t$ is strongly continuous for any couple of uniformly bounded strongly continuous operator-valued functions $t\mapsto A_t\in\mathcal{L}(\mathbf{X}_1,\mathbf{X}_2)$, $t\mapsto B_t\in\mathcal{L}(\mathbf{X}_2,\mathbf{X}_3)$ in case of normed spaces $\mathbf{X}_k$. By assumption, the orbit $t\mapsto a_t=\Psi^t(0)$ is a norm-continuous map $\mathbb{R}_+\to\mathbf{B}$ implying the norm continuity of the function $t\mapsto Q_{a_t}$. We deduce the strong continuity of the $\mathbf{H}$-isometry-valued function $t\mapsto U_t$ as follows. Consider any vector $x\in\mathbf{H}$. We may assume $x\in\mathbf{B}$ without loss of generality. Then, by the aid of the Möbius shifts (3.3), we can write

$$U_tx=\big[\Theta_{a_t}^{-1}\circ\Psi_t\big](x)=\Theta_{-a_t}\big(\Psi(x)\big)\qquad(t\in\mathbb{R}_+)$$

whence the continuity of $t\mapsto U_tx=(1-\langle x|a_t\rangle)^{-1}[Q_{a_t}x-a_t]$ is immediate.

3.4. Standard notations, assumptions. Henceforth, for the proofs for Section 2, we assume without loss of generality the following facts.

(i) $\mathbf{\Psi} := [\Psi^t \colon t\in\mathbb{R}_+]$ is an arbitrarily given C_0-semigroup of holomorphic Carathéodory isometries of $\mathbf{B}$ without common fixed point inside $\mathbf{B}$;

(ii) $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ $(t \in \mathbb{R}_+)$ with a suitable Möbius transformation Θ;

(iii) the orbit $t \mapsto \Phi^t(0)$ is differentiable and $\Phi^t = \mathfrak{F}^\# \mathcal{U}^t \big| \mathbf{B}$ with some C_0-semigroup $[U^t : t\in\mathbb{R}_+]$ of linear $\mathbf{H}$-isometries,

$$\mathcal{A} := \mathrm{gen}[\mathcal{U}^t : t\in\mathbb{R}_+] = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix}, \qquad b\in\mathbf{H},\ iA = \mathrm{gen}[U^t : t \in \mathbb{R}_+];$$

(iv) $\overline{e} \in \partial\mathbf{B}$ is a joint boundary fixed point of the maps $\overline{\Phi^t}$, and we write

$$\mathbf{H}_0 := \mathbf{H} \ominus \mathbb{C}\overline{e}, \quad P := \mathrm{P}_{\mathbb{C}\overline{e}}, \quad P_0 := \mathrm{P}_{\mathbf{H}_0} = 1-P, \quad T\colon x \mapsto x+\overline{e}, \quad \mathcal{T} := \begin{bmatrix} \mathrm{id}_{\mathbf{H}} & \overline{e} \\ 0 & 1 \end{bmatrix}.$$

Proposition 3.5. *We have* $\overline{e}\in\mathrm{dom}(A)$ *with* $\mathcal{A}(\overline{e}\oplus 1) = \nu(\overline{e}\oplus 1)$ *and* $b = (\nu - iA)\overline{e}$ *for some* $\nu\in\mathbb{C}$. *The possibly unbounded operator* $A_0 := P_0 A \big| \mathbf{H}_0 \cap \mathrm{dom}(A)$ *is* $\mathbf{H}_0$*-hermitian and, in terms of* $(\mathbb{C}\overline{e} \oplus \mathbf{H}_0 \oplus \mathbb{C})$*-matrices, we have*

$$\mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{bmatrix} -\overline{\nu} & 0 & 0 \\ -b_0 & iA_0 & 0 \\ \nu & b_0^* & \nu \end{bmatrix} \quad \text{where} \quad b_0 := P_0 b, \quad \nu = \langle \overline{e} | \, b \rangle. \tag{3.6}$$

Proof. By assumption 3.4(ii), $\overline{e} \oplus 1$ is a joint eigenvector of the linear operators $\mathcal{U}^t$. Hence $\mathcal{U}^t(\overline{e}\oplus 1) = \zeta_t(\overline{e}\oplus 1)$ $(t\in\mathbb{R}_+)$ with a continuous solution $[t \mapsto \zeta_t]$ of the Cauchy equation $\zeta_{s+t} = \zeta_s\zeta_t$. Thus for some $\nu \in \mathbb{C}$, $\zeta_t = e^{\nu t}$ and we have

$$\overline{e}\oplus 1 \in \mathrm{dom}(\mathcal{A}) = \{\mathfrak{z} : t \mapsto \mathcal{U}^t\mathfrak{z} \text{ is differentiable}\}, \quad \mathcal{A}(\overline{e}\oplus 1) = \nu(\overline{e}\oplus 1).$$

As a consequence, $\overline{e} \in \mathrm{dom}(A) = \mathrm{P}_{\mathbf{H}}\mathrm{dom}(\mathcal{A})$ and the operator

$$\widetilde{A}_0 := A - PA - AP + PAP = (1-P)A(1-P) = P_0 A P_0$$

is a bounded perturbation ranging in $\mathbf{H}_0$ of $A \in \mathrm{Her}_s(\mathbf{H})$ with a self-adjoint operator of finite rank. Hence its restriction A_0 to $\mathbf{H}_0$ is a well-defined unbounded $\mathbf{H}_0$-hermitian operator. Since A is a $(\mathbb{C}\overline{e} \oplus \mathbf{H}_0)$-matrix operator, we can write

$$\mathcal{A} = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} i\alpha & ia_0^* & \beta \\ ia_0 & iA_0 & b_0 \\ \overline{\beta} & b_0^* & 0 \end{bmatrix}, \quad b_0 := P_0 b,\ \beta := \langle b | \overline{e} \rangle,\ a_0 := P_0 A\overline{e},\ \alpha v := \langle A\overline{e} | \overline{e} \rangle$$

in terms of $(\mathbb{C}\overline{e} \oplus \mathbf{H}_0 \oplus \mathbb{C})$-matrices. The eigenvector equation $\mathcal{A}(\overline{e}\oplus 1) = \nu(\overline{e}\oplus 1)$ means that we have $iA\overline{e} + b = \nu\overline{e}$ with $\langle \overline{e} | \, b \rangle = \nu$ implying $i\alpha + \beta = \nu$, $ia_0 + b_0 = 0$, $\overline{\beta} = \nu$. Since

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{T}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

in $(\mathbb{C}\overline{e}\oplus\mathbf{H}_0\oplus\mathbb{C})$-matrix form, hence (3.6) is immediate.

Notation 3.7. Henceforth $[U_0^t : t \in \mathbb{R}_+]$ denotes the C_0-semigroup of $\mathbf{H}_0$-isometries generated by the operator $iA_0 := P_0 A\big|\mathbf{H}_0 \cap \mathrm{dom}(A)$.

Lemma 3.8. *Let $\mathbf{E}_1, \mathbf{E}_2$ be Banach spaces, $\mathcal{G} := \left[\begin{smallmatrix} G_1 & 0 \\ H & G_2 \end{smallmatrix}\right]$ with $H \in \mathcal{L}(\mathbf{E}_1, \mathbf{E}_2)$ and $G_k = \mathrm{gen}[W_k^t : t \in \mathbb{R}_+]$ for some C_0-semigroup $[W_k^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{E}_k)$. Then the family*

$$\mathcal{S}^t := \begin{bmatrix} W_1^t & 0 \\ \int_0^t W_2^{t-s} H W_1^s ds & W_2^t \end{bmatrix} \qquad (t \in \mathbb{R}_+)$$

is a C_0-semigroup in $\mathcal{L}(\mathbf{E}_1 \oplus \mathbf{E}_2)$ such that $\mathrm{gen}[\mathcal{S}^t : t \in \mathbb{R}_+] = \mathcal{G}$.

Proof. The family $\mathbf{W} := [\mathcal{W}^t : t \in \mathbb{R}_+]$ where $\mathcal{W}^t := W_1^t \oplus W_2^t$ is a C_0-semigroup in $\mathbf{E}_1 \oplus \mathbf{E}_2$ and $\mathcal{G}$ is a bounded perturbation of $\mathrm{gen}(\mathbf{W}) = G_1 \oplus G_2 (\equiv \left[\begin{smallmatrix} G_1 & 0 \\ 0 & G_2 \end{smallmatrix}\right])$ with $\mathrm{dom}(\mathcal{G}) = \mathrm{dom}(G_1) \oplus \mathrm{dom}(G_2)$ by the operator $\mathcal{H} := \left[\begin{smallmatrix} 0 & 0 \\ H & 0 \end{smallmatrix}\right]$. According to [2, Thm. III.1.10], for every fixed $\mathfrak{z} = x \oplus y \in \mathrm{dom}(\mathcal{G})$ we have

$$\mathcal{S}^t \mathfrak{z} = \sum_{n=0}^{\infty} S_n(t) \quad \text{where} \quad S_0(t) := \mathcal{W}^t \mathfrak{z}, \quad S_{n+1}(t) = \int_0^t \mathcal{W}^{t-s} \mathcal{H} S_n^{(k)}(s)\, ds.$$

Since $\mathcal{H}$ is an off-diagonal 2×2 triangular operator matrix, $S_n(t) = 0$ for $n > 1$.

3.9. Proof of Theorem 2.1

Since $\mathcal{T}$ is a bounded invertible $\mathbf{H} \oplus \mathbb{C}$-operator and $\mathcal{A} = \mathrm{gen}[\mathcal{U}^t : t \in \mathbb{R}_+]$, we have

$$\mathcal{T}^{-1} \mathcal{A} \mathcal{T} = \mathrm{gen}\big[\mathcal{V}^t : t \in \mathbb{R}_+\big] \quad \text{for} \quad \mathcal{V}^t := \mathcal{T}^{-1} \mathcal{U}^t \mathcal{T}.$$

Since $\Phi^t = \mathfrak{F}(\mathcal{U}^t)\big|\mathbf{B}$ $(t \in \mathbb{R}_+)$, in terms of the translation $Tx := x + \overline{e}$, we can regard the C_0-semigroup $[\mathcal{V}^t : t \in \mathbb{R}^+]$ as the linear representation by means of $\mathfrak{F}$ of the semigroup $[T^{-1} \circ \Phi^t \circ T : t \in \mathbb{R}_+]$ which consists of holomorphic isometries of the shifted ball $\mathbf{B} - \overline{e}$ whose continuous extensions leave the origin fixed. Due to the projective identities $\mathfrak{F}\big(\mathcal{T}^{-1} \mathcal{V} \mathcal{T}\big) = T^{-1} \circ \mathfrak{F}(\mathcal{V}) \circ T \ \ \big(\mathcal{V} \in \mathcal{L}(\mathbf{H} \oplus \mathbb{C})\big)$, for the points $x \in T^{-1}\mathbf{B} = \mathbf{B} - \overline{e}$ we have

$$\mathfrak{F}(\mathcal{V}^t)(x) = \big[\mathfrak{F}(\mathcal{T}^{-1} \mathcal{U}^t \mathcal{T})\big](x) = [T^{-1} \circ \Phi \circ T](x) = \Phi(x + \overline{e}) - \overline{e}.$$

Therefore

$$\Phi^t(x) = \mathfrak{F}(\mathcal{V}^t)(x - \overline{e}) + \overline{e} \quad (x \in \mathbf{B}).$$

By the aid of Lemma 3.8 and (3.6) we calculate a quadrature form for $\mathcal{V}^t$ as follows. Regarding the top left 2×2-corner of the matrix $\mathcal{T}^{-1} \mathcal{A} \mathcal{T}$ we get

$$\begin{bmatrix} -\overline{\nu} & 0 \\ -b_0 & iA_0 \end{bmatrix} = \mathrm{gen}\big[V^t : t \in \mathbb{R}_+\big], \quad V^t = \begin{bmatrix} e^{-\overline{\nu} t} & 0 \\ \int_0^t U_0^{t-s} e^{-\overline{\nu} s}(-b_0) ds & U_0^t \end{bmatrix}. \tag{3.10}$$

Another application of Lemma 3.8 to $\mathcal{T}^{-1} \mathcal{A} \mathcal{T}$ yields

$$\mathcal{V}^t = \begin{bmatrix} V^t & 0 \\ \int_0^t e^{\nu(t-s)} b^* V^s ds & e^{\nu t} \end{bmatrix} \qquad (t \in \mathbb{R}_+). \tag{3.11}$$

As a consequence of (3.11), since $\mathcal{V}^t(x\oplus 1)=\big[V^t x\big]\oplus e^{\nu t}\big[\int_0^t\langle e^{-\nu s}V^\tau x|\,b\rangle d\tau+1\big]$, we get

$$\Phi^t(x)=\frac{e^{-\nu t}V^t(x-\overline{e})}{\big\langle\int_0^t e^{-\nu s}V^s(x-\overline{e})\,ds\big|\,b\big\rangle+1}+\overline{e}\qquad(x\in\mathbf{B},\ t\in\mathbb{R}_+).\tag{3.12}$$

We substitute (3.10) into (3.12) in terms of the new parametrization

$$\lambda=\operatorname{Re}\nu,\quad \mu:=\operatorname{Im}\nu,\quad V_0^t:=e^{-i\mu t}U_0^t.$$

Given any vector $z=z_0+\zeta\overline{e}, z_0\in\mathbf{H}_0$, and recalling the commutativity of convolutions,

$$e^{-\nu t}V^t z=\zeta e^{-2\lambda t}\Big[\overline{e}-\int_0^t e^{\lambda s}V_0^s b_0\,ds\Big]+e^{-\lambda t}V_0^t z_0,\qquad \int_0^t\big\langle e^{-\nu s}V^s z\big|b\big\rangle\,ds=$$
$$=\zeta(\lambda+i\mu)\frac{1-e^{-2\lambda t}}{2\lambda}-\zeta\int_0^t e^{-2\lambda s}\int_0^s e^{\lambda r}\big\langle V_0^r b_0\big|b_0\big\rangle\,dr+\int_0^t e^{-\lambda s}\big\langle V_0^s z_0\big|b_0\big\rangle\,ds\,ds.$$

The statement of Theorem 2.1 is immediate from (3.12) with $z\coloneqq x-\overline{e}=x_0+(\xi-1)\overline{e}$.

Remark 3.13. It is discovered from the above proof that any tuple

$$\mathfrak{a}:=\big(\mathbf{H},\overline{e},[V_0^t:t\in\mathbb{R}_+],b_0,\lambda,\mu\big)$$

with a Hilbert space $\mathbf{H}$, a unit vector $\overline{e}\in\mathbf{H}$, a C_0-semigroup $[V_0^t:t\in\mathbb{R}_+]$ of $\mathbf{H}_0(:=\mathbf{H}\ominus\mathbb{C}\overline{e})$-isometries, a vector $b_0\in\mathbf{H}_0$ and two real constants gives rise to a C_0-semigroup $[\Phi_{\mathfrak{a}}^t:t\in\mathbb{R}_+]$ of holomorphic Carathéodory isometries of the open unit ball $\mathbf{B}$ of $\mathbf{H}\equiv\mathbb{C}\overline{e}\oplus\mathbf{H}_0$ whose generator $\Gamma(x)=\frac{d}{dt}\big|_{t=0+}\Phi_{\mathfrak{a}}^t(x)=\frac{d}{dt}\big|_{t=0+}\mathfrak{F}(\mathcal{T}\mathcal{V}^t\mathcal{T}^{-1})x$ has the form (1.2) with

$$b=\begin{bmatrix}\lambda-i\mu\\ b_0\end{bmatrix},\quad A=\begin{bmatrix}2\mu & -ib_0^*\\ ib_0 & A_0\end{bmatrix},\quad iA_0=\operatorname{gen}[V_0^t:t\in\mathbb{R}_+].\tag{3.14}$$

In particular we can extend $[\Phi_{\mathfrak{a}}^t:t\in\mathbb{R}_+]$ to a C_0-group $[\Phi_{\mathfrak{a}}^t:t\in\mathbb{R}]\subset\operatorname{Iso}(d_{\mathbf{B}})$ if and only if $[V_0^t:t\in\mathbb{R}_+]$ consists of $\mathbf{H}_0$-unitary operators (cf. [12, Theorem II]). Furthermore, given any tuple

$$\mathfrak{b}:=(\mathbf{H},A,\overline{e},\lambda)$$

with a densely defined maximal symmetric linear $\mathbf{H}$-operator A, there is a unique C_0-semigroup $[\Psi_{\mathfrak{b}}^t:t\in\mathbb{R}_+]\subset\operatorname{Iso}(d_{\mathbf{B}})$ whose infinitesimal generator is of the form (1.2) with $b:=(\nu-iA)e$ where $\nu:=\lambda+i\mu$ and $\mu=\langle A\overline{e}|\,\overline{e}\rangle$.

4. Invariant discs

Lemma 4.1. *The C_0-semigroup $[\Phi^t:t\in\mathbb{R}_+]\subset\operatorname{Iso}(d_{\mathbf{B}})$ with generator (1.2) and joint boundary fixed point $\overline{e}\in\partial\mathbf{B}$ admits no invariant disc of the form $\mathbf{B}\cap(\overline{e}+\mathbb{C}\overline{e})\neq\emptyset$ if and only if the operator $iA+\langle\overline{e}|b\rangle$ is not injective or $\overline{e}\in\operatorname{range}(iA-\langle\overline{e}|b\rangle)$.*

Proof. Consider any vector $v\in\mathbf{H}$ such that $\overline{e}+v\in\mathbf{B}$. The disc $\Delta_{\overline{e}}^v:=\mathbf{B}\cap(\overline{e}+\mathbb{C}v)$ is $[\Phi^t:t\in\mathbb{R}_+]$-invariant if and only if the vector field (1.2) is tangent to it, that is, if $b-\langle\overline{e}+\tau v|b\rangle(\overline{e}+\tau v)+iA(\overline{e}+\tau v)\in\mathbb{C}v$ whenever $\overline{e}+\tau v\in\mathbf{B}$. This happens if and only if $-\langle v|b\rangle\overline{e}+iAv=\zeta v$ for some $\zeta\in\mathbb{C}$, because we have $\overline{e}\in\operatorname{dom}(\Gamma)=\operatorname{dom}(A)$

and $\Gamma(\overline{e}) = b - \langle \overline{e}|b\rangle + iA\overline{e} = 0$ (due to the fact that the point $\overline{e}$ is $[\overline{\Phi^t} : t \in \mathbb{R}_+]$-invariant). According to Proposition 3.5, here we have $b = (\nu - iA)\overline{e}$ where $\nu = \langle \overline{e}|b\rangle$. Therefore $\zeta v = -\langle v|(\nu - iA)\overline{e}\rangle\overline{e} + iA\overline{e} = \langle(-\overline{\nu} - iA)v|\overline{e}\rangle\overline{e} + iA\overline{e}$. Notice that, in general, $\mathrm{P}_{\mathbb{C}\overline{e}}x = \langle x|\overline{e}\rangle\overline{e} = x - \mathrm{P}_{\mathbf{H}_0}x$ $(x \in \mathbf{H})$. Thus the disc $\Delta^v_{\overline{e}}$ is $[\Phi^t : t \in \mathbb{R}_+]$-invariant if and only if $-\overline{\nu}\mathrm{P}_{\mathbb{C}\overline{e}}v + \mathrm{P}_{\mathbf{H}_0}(iAv) - \zeta v = 0$ i.e. $\mathrm{P}_{\mathbf{H}_0}(iAv - \zeta v) = 0$ and $\mathrm{P}_{\mathbb{C}\overline{e}}(-\overline{\nu} - \zeta)\overline{e} = 0$ for some $\zeta \in \mathbb{C}$. By assumption $\Delta^v_{\overline{e}} \neq \emptyset$ which is possible if and only if $\mathrm{P}_{\mathbb{C}\overline{e}}v \neq 0$ implying $\zeta = -\overline{\nu}$. Hence we conclude that the $[\Phi^t : t \in \mathbb{R}_+]$-invariance of $\Delta^v_{\overline{e}}$ is equivalent to the relation $\mathrm{P}_{\mathbf{H}_0}(iAv + \overline{\nu})v = 0$ i.e. to $(iA + \overline{\nu})v \in \mathbb{C}\overline{e}$ which completes the proof.

Example 4.2. The C_0-semigroup of the type $\mathbf{\Psi}_{\mathfrak{b}} = [\Psi^t_{\mathfrak{b}} : t \in \mathbb{R}_+]$ in Remark 3.13 with $\mathbf{H} := L^2(\mathbb{R})$, $Af := [x \mapsto xf(x)$ $\big(\mathrm{dom}(A) := \big\{f : \int_{-\infty}^{\infty} |xf(x)|^2\, dx < \infty\big\}\big)$, $\lambda := 0$ and $\overline{e} := (2\pi)^{-1/2}\exp\big(-(x-1)^2/2\big)$ admits no invariant 1-dimensional disc. Indeed, we have $\langle A\overline{e}|\,\overline{e}\rangle = (2\pi)^{-1}\int_{-\infty}^{\infty} x\exp(-(x-1)^2)\, dx = (2\pi)^{1/2} \neq 0$. Thus, according to the construction of the C_0-semigroup $\mathbf{\Psi}_{\mathfrak{b}}$, $\nu = \langle \overline{e}|\, b\rangle = i\mu = i\langle A\overline{e}|\,\overline{e}\rangle/2 \in i\mathbb{R}\setminus\{0\}$. The relation $(iA + \overline{\nu})v = \zeta\overline{e}$ would imply $v = -i\zeta\exp(-(x-1)^2/2)/(x-\mu) \in L^2(\mathbb{R})$ which is possible only if $v = 0$.

4.3. Proof of Theorem 2.6

Recall [3] that the 1-dimensional complex affine discs of the form $\Delta_{p,q} := \big(p + \mathbb{C}(q-p)\big) \cap \mathbf{B}$ $(q \neq p, q \in \partial\mathbf{B})$ are the ranges of complex geodesics for the Carathéodory distance $d_{\mathbf{B}}$, and $d_{\mathbf{B}}$-isometries preserve their family. In particular, in the case when $p \neq q \in \partial\mathbf{B}$ are joint fixed points of the continuous extensions $\overline{\Psi^t}$, the disc $\Delta_{p,q}$ is automatically $[\Psi^t : t \in \mathbb{R}_+]$-invariant. Suppose $\Psi^t(\Delta_{p,q}) = \Delta_{p,q}$ $(t \in \mathbb{R}_+)$. Then the restricted maps $\psi^t_{p,q} := \Psi^t\big|D_{p,q}$ form a C_0-semigroup of holomorphic automorphisms of a 1-dimensional Hilbert ball, thus their continuous extensions $\overline{\psi^t_{p,q}}$ to $\overline{D_{p,q}}$ admit at least one fixed point which is necessarily a joint fixed point for the maps $\overline{\Psi^t}$. A 1-dimensional application of Theorem 2.1 shows that all the orbits $t \mapsto \psi^t_{p,q}(x) = \Psi^t(x)$ $(x \in \Delta_{p,q})$ are automatically real analytic. Hence, given any Möbius transformation Θ, the C_0-semigroup $[\Phi^t : t \in \mathbb{R}_+]$ with $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ leaves the $d_{\mathbf{B}}$-geodesic $D_{\Theta(p),\Theta(q)}$ invariant, and the orbit $t \mapsto \Phi^t(0)$ is differentiable. Conversely, if $[\Phi^t : t \in \mathbb{R}_+]$ is a C_0-semigroup leaving the disc $\Delta_{\overline{e},-\overline{e}} (= \{\zeta\overline{e} : |\zeta| < 1\})$ invariant and $\Phi^t(\overline{e}) = \overline{e}$, $\Psi^t = \Theta^{-1} \circ \Phi^t \circ \Theta$ $(t \in \mathbb{R}_+)$, then the image $\Theta(\overline{\Delta_{\overline{e},-\overline{e}}})$ is a $[\overline{\Psi^t} : t \in \mathbb{R}_+]$-invariant 1-dimensional affine section of $\overline{\mathbf{B}}$ containing a joint fixed point of $[\overline{\Psi^t} : t \in \mathbb{R}_+]$ (namely the point $\Theta^{-1}(\overline{e})$).

Proof of (i), (ii). It remains only to verify the possibility of the simplified representations (2.7), (2.8) by means of an appropriate choice for the coordinatizing Möbius transformation Θ in 3.4. By setting $x_0 := 0$ in (2.2), it is straightforward to check that a C_0-semigroup $[\Phi^t : t \in \mathbb{R}_+]$ of the form (2.2) leaves the disc $\Delta_{\overline{e},-\overline{e}}$ invariant if and only if $b_0 = 0$ and $\Phi^t(\xi\overline{e}) = \omega_{\lambda,\mu}(t,\xi)\overline{e}$ $(|\xi| < 1)$ with the function

$$\omega_{\lambda,\mu}(t,\xi) := 1 - \frac{2\lambda(1-\xi)e^{-2\lambda t}}{2\lambda - (1-\xi)(\lambda + i\mu)(1 - e^{-\lambda t})}.$$

It is also easy to see that the constant 1 is a joint fixed point of all functions $\omega_{\lambda,\mu}(t,\cdot)$. Observe that, for fixed $\lambda, \mu \in \mathbb{R}$, the family $\omega_{\lambda,\mu}(t,\cdot)$ $t \in \mathbb{R}_+)$ admits another fixed point, namely the constant $\xi_{\lambda,\mu} := \frac{i\mu-\lambda}{i\mu+\lambda}$ with modulus 1, if and only if we have $\mu = 0$. Due to folklore 2-transitivity properties of the Möbius group (for a direct proof see [10]), given any two couples $(e_1, e_2), (f_1, f_2) \in [\partial\mathbf{B}]^2$ of distinct boundary points, there exists a Möbius transformation $\Theta^{(e_1,f_1,e_2,f_2)}$ with the effect $e_k \mapsto f_k$ $(k=1,2)$. Thus in case if $[\overline{\Psi^t} : t \in \mathbb{R}_+]$ has only a unique fixed point $p \in \partial\mathbf{B}$ but the disc $\Delta_{p,q}$ is $[\Psi^t : t \in \mathbb{R}_+]$-invariant, we obtain (2.8) with any coordinatization $\Theta := \Theta^{(p,\overline{e},q,\kappa\overline{e})}$ where $|\kappa| = 1$, by substituting $b_0 = 0$ and $\mu = 0$ in (2.2).

If $[\overline{\Psi^t} : t \in \mathbb{R}_+]$ admits two distinct fixed points $p, q \in \partial\mathbf{B}$, then, as we have shown, the disc $\Delta_{p,q}$ is automatically $[\Psi^t : t \in \mathbb{R}_+]$-invariant, and with the choice $\Theta := \Theta^{(p,\overline{e},q,-\overline{e})}$ we get a formula for Φ^t by substituting $b_0 = 0$ and $\mu = 0$ in (2.2) establishing (2.7).

Proof of (iii). Suppose indirectly that $0 \neq \lambda = \mathrm{Re}\langle e|b\rangle$. Then the skew symmetry of iA implies that $\mathrm{range}(iA - \langle e|b\rangle) = \mathbf{H}$. By Lemma 4.1, we have a non-trivial Γ-invariant disc and we are in the setting of (i) or (iii). By assumption, (i) is not the case. However, in the case of (iii) we have $\langle e|b\rangle = i\mu \in i\mathbb{R}$ automatically.

4.4. Proof for Remark 2.5

The operator S_0 is closed with dense domain in $\mathbf{H}_0$. Since S_0 is also symmetric, both $\pm iS_0$ are dissipative (namely $\mathrm{Re}\langle \pm iSx_0|x_0\rangle = 0$ for $x_0 \in \mathrm{dom}(S)$) with the properties that both $\mathrm{range}(\pm iS_0+\delta)$ are dense in $\mathbf{H}$ for any $\delta > 0$ and that the operators $(iS+\delta)^{-1} : \mathrm{range}(S) \to \mathbf{H}_0$ $(0 \neq \delta \in \mathbb{R})$ are all bounded and densely defined.[4] Given $\delta \in \mathbb{R} \setminus \{0\}$, by [2, II. Lemma 1.3], for any $x_0 \in \mathrm{range}(iS_0 - \delta)$ and $t > 0$, we have $\int_0^t e^{-\delta\tau} V_0^\tau x_0 \, d\tau = \int_0^t e^{-\delta\tau} V_0^\tau (iS_0 - \delta)[(iS_0-\delta)^{-1}x_0]\, d\tau = \big(e^{-\delta t}V_0^t - 1\big)(iS_0-\delta)^{-1}x_0$. The boundedness of both the operators V^t and the resolvent $R(\delta, iS) = \mathrm{closure}\big((\delta - iS_0)^{-1}\big)$ establishes 2.5 for $t \in \mathbb{R}_+$ and $0 \neq \lambda \in \mathbb{R}$ with integrals of strongly continuous bounded operator-valued functions.

5. Dilation

Lemma 5.1. *Let $[U^t : t \in \mathbb{R}_+]$ be a C_0-semigroup of linear isometries of a Banach space $\mathbf{E}$. Suppose $\mathbf{E}$ is a subspace of another Banach space $\mathbf{F}$ and there is a surjective isometry $V \in \mathcal{L}(\mathbf{F})$ such that $U^1 = V\big|\mathbf{E}$. Then there is a subspace $\mathbf{E} \subset \widehat{\mathbf{E}} \subset \mathbf{F}$ along with a C_0-group $[\widehat{U}^t : t \in \mathbb{R}]$ of surjective linear isometries of $\widehat{\mathbf{E}}$ such that $U^t = \widehat{U}^t\big|\mathbf{E}$ $(t \in \mathbb{R}_+)$ with $\mathrm{dom}\big(\mathrm{gen}[\widehat{U}^t : t \in \mathbb{R}]\big) \supset \mathrm{dom}\big(\mathrm{gen}[U^t : t \in \mathbb{R}]\big)$.*

Proof. Let $\widehat{\mathbf{E}} := \mathrm{closure}\big(\mathbf{E}_\infty\big)$ in $\mathbf{F}$ where $\mathbf{E}_\infty := \bigcup_{n=0}^\infty$ with $\mathbf{E}_n := V^{-n}\mathbf{E}$. By assumption $V\mathbf{E} = U^1\mathbf{E} \subset \mathbf{E}$. Hence, by induction, we conclude that the subspaces $\mathbf{E}_n$ $(n \in \mathbb{Z}_+)$ form an increasing sequence. Therefore all the operators $U_n^t := V^{-2n}U^{t+n}V^n\big|\mathbf{E}_n$ $(t \geq -n,\ n \in \mathbb{Z}_+)$ are well-defined isometries $\mathbf{E}_n \to \mathbf{E}_{\lceil n-t\rceil}$. We have $U_n^t = U_{n+1}^t\big|\mathbf{E}_n$ for all indices $n \in \mathbb{Z}_+$. Indeed, if $\widehat{x} \in \mathbf{E}_n$ and $t \geq -n$, then

$$\begin{aligned} U_{n+1}^t\widehat{x} &= V^{-2n-2}U^{t+n+1}V^{n+1}\widehat{x} = V^{-2n-2}U^{t+n+1}U^1V^n\widehat{x} = \\ &= V^{-2n-2}U^{t+n+2}V^n\widehat{x} = V^{-2n-2}V^2U^{t+n}V^n\widehat{x} = V^{-2n}U^{t+n}V^n\widehat{x} = U_n^t\widehat{x} \end{aligned}$$

[4] Indeed, $y_0 \perp \mathrm{range}(\pm iS_0 + \delta)$ means $0 = \langle \pm iS_0x_0 - \delta x_0|y_0\rangle$ that is $0 = \langle x_0| \mp iS_0y_0 - \delta y_0\rangle$ for $(x_0 \in \mathrm{dom}(S))$ entailing $\mp iS_0y_0 + \delta y_0 = 0$ with $\delta\|y_0\|^2 = \pm i\langle S_0y_0|y_0\rangle \in i\mathbb{R}$ which is possible only if $y = 0$. Thus by the Lumer-Phillips theorem [2, II. Theorem 3.15], also the operator $-iS_0$ generates a strongly continuous contraction (actually isometry) semigroup and all the values $0 \neq \delta \in \mathbb{R}$ belong to the resolvent set of iS.

since V extends U^1 and we have $V^{n+1}\widehat{x} \in \mathbf{E}$ implying $V^{n+1}\widehat{x} = U^1 V^n \widehat{x}$. Hence

$$U^t_\infty \widehat{x} := \lim_{n\to\infty} U^t_n \widehat{x} = \big[U^t_n \widehat{x} :\ n \in \mathbb{Z}_+,\ n \geq t\big] \qquad (\widehat{x} \in \mathbf{E}_\infty)$$

is a well-defined linear isometry of the linear manifold $\mathbf{E}_\infty$ for any $t \in \mathbb{R}$. Since $\mathrm{range}(U^t_n) \supset V^{-2n}U^{\lceil t\rceil+n}\mathbf{E} = V^{\lceil t\rceil-n}\mathbf{E}$ for $t \geq -n$, we have $\mathrm{range}(U^t_\infty) = \mathbf{E}_\infty$ $(t \in \mathbb{R})$. Thus the operators $\widehat{U}^t := \mathrm{closure}(U^t_\infty)$ $(t \in \mathbb{R})$ are well-defined surjective linear $\widehat{\mathbf{E}}$-isometries, each of which extending the respective U^t. We check that they form a C_0-group as follows. Since $[\widehat{U}^t : t \in \mathbb{R}]$ is an equilipschitzian family, it suffices to see that its restriction $[\widehat{U}^t : t \in \mathbb{R}]$ to the dense submanifold $\mathbf{E}_\infty$ of $\widehat{\mathbf{E}}$ is a C_0-group. Given $s, t \in \mathbb{R}$ and $\widehat{x} \in \widehat{E}_\ell$, we have $\widehat{U}^t\widehat{x} = V^{-2n}U^{t+n}V^n\widehat{x} \in \widehat{\mathbf{E}}_{2n}$ whenever $n \geq -t\ell$ and $\widehat{U}^s(\widehat{U}^t\widehat{x}) = V^{-2m}U^{t+m}V^m\widehat{U}^t\widehat{x}$ whenever $m \geq \max\{-s, 2n\}$. It follows that $\widehat{U}^s\widehat{U}^t\widehat{x} = \widehat{U}^{s+t}\widehat{x}$ because, with $k \geq 2(|s| + |t| + \ell)$, we have

$$\begin{aligned}\widehat{U}^s(\widehat{U}^t\widehat{x}) &= V^{-4k}U^{s+2k}V^{2k}V^{-2k}U^{t+k}V^k\widehat{x} = \\ &= V^{-4k}U^{s+t+3k}V^k\widehat{x} = V^{-4k}U^{s+t+2k}V^{2k}\widehat{x}V^k = \widehat{U}^{s+t}\widehat{x}.\end{aligned}$$

To show the strong continuity, consider any vector $\widehat{x} \in \widehat{E}_\ell$. Then for any integer $n \geq \ell$ the orbit $(-n, \infty) \ni t \mapsto \widehat{U}^t\widehat{x} = V^{-2n}U^{t+n}(V^n\widehat{x})$ is continuous since V^{-2n} is an isometry and $(V^n\widehat{x}) \in \mathbf{E}$. Hence we can see also the required generator domain inclusion property: with $\widehat{x} := x \in \mathrm{dom}\big(\mathrm{gen}[U^t : t \in \mathbb{R}_+]\big)$ we have $V^n x = U^n x \in \mathrm{dom}\big(\mathrm{gen}[U^t : t \in \mathbb{R}_+]\big)$ entailing even the differentiability of the orbits $(-n, \infty) \ni t \mapsto \widehat{U}^t x$.

In particular, since every linear isometry of a Hilbert space admits a unitary dilation [11], in our setting of interest we conclude the following.

Corollary 5.2. *If $[U^t : t \in \mathbb{R}_+]$ is a C_0-semigroup of linear $\mathbf{H}$-isometries, there exists a Hilbert space $\widehat{\mathbf{H}}$ containing $\mathbf{H}$ as a subspace along with a C_0-group $[\widehat{U}^t : t \in \mathbb{R}]$ of $\widehat{\mathbf{H}}$-unitary operators such that $U^t = \widehat{U}^t\big|\mathbf{H}$ $(t \in \mathbb{R}_+)$ whose generator is an extension of* $\mathrm{gen}[U^t : t \in \mathbb{R}_+]$.

5.3. Proof of Corollaries 2.10-11

Given any Hilbert space $\widehat{\mathbf{H}}$ containing $\mathbf{H}$ as a subspace, every Möbius transformation of $\mathbf{H}$ extends to a Möbius transformation of $\widehat{\mathbf{H}}$. Hence it suffices to show only that any C_0-semigroup of the form of Theorem 2.1 admits a dilation of the same algebraic form in a larger Hilbert space. Let $[\Phi^t : t \in \mathbb{R}_+]$ be given as in Theorem 2.1. According to Corollary 3.13, for some tuple $\mathfrak{a} := \big(\mathbf{H}, \overline{e}, [V^t_0 : t \in \mathbb{R}_+], b_0, \lambda, \mu\big)$ we have $\Phi^t = \Phi^t_{\mathfrak{a}} = \mathfrak{F}(\mathcal{U}^t)$ $(t \in \mathbb{R}_+)$ with

$$\mathrm{gen}[\mathcal{U}^t : t \in \mathbb{R}_+] = \begin{bmatrix} iR & b \\ b^* & 0\end{bmatrix} = \begin{bmatrix} i(S_0 + \mu) & -b_0 & b_0 \\ b_0^* & 2i\mu & \lambda - i\mu \\ b_0^* & \lambda + i\mu & 0\end{bmatrix} \tag{5.4}$$

in terms of $\big[\mathbf{H}_0 \oplus (\mathbb{C}\overline{e}) \oplus \mathbb{C}\big]$-matrices. Let $[\widehat{V}^t_0 : t \in \mathbb{R}]$ be the dilation C_0-group of $[V^t_0 : t \in \mathbb{R}_+]$ consisting of unitary operators of a covering Hilbert space $\widehat{\mathbf{H}}_0$ of $\mathbf{H}_0$ with the skew self-adjoint extension $i\widehat{S}_0 = \mathrm{gen}[\widehat{V}^t_0 : t \in \mathbb{R}]$ of iS_0 guaranteed by Corollary 5.2.

Also, by Remark 3.13, the tuple $\widehat{\mathfrak{a}} := \big(\widehat{\mathbf{H}}, \overline{e}, [\widehat{V}_0^t : t \in \mathbb{R}_+], b_0, \lambda, \mu\big)$ where $\widehat{\mathbf{H}} := \widehat{\mathbf{H}}_0 \oplus (\mathbb{C}\overline{e})$ gives rise to a C_0-group $\Phi^t_{\widehat{\mathfrak{a}}} : t \in \mathbb{R}]$ such that $\Phi^t_{\widehat{a}} = \mathfrak{F}(\widehat{\mathcal{U}}^t)$ $(t \in \mathbb{R})$ whose infinitesimal generator can be written in the form of the right-hand side of (5.4) when the entry S_0 is replaced by $\widehat{S}_0$. Hence, by Theorem 2.1, the transformations $\Phi^t_{\mathfrak{a}}$ can be written in the form (2.2) with $\widehat{S}_0$ in place of S_0 and $\widehat{V}_0^t$ in place of V_0^t. Since $\widehat{V}_0^t \big| \mathbf{H}_0 = V_0^t$ $(t \in \mathbb{R}_+)$, it readily follows that $\Phi^t_{\widehat{\mathfrak{a}}} \big| \mathbf{H} = \Phi^t_{\mathfrak{a}}$ $(t \in \mathbb{R}_+)$ which completes the proof of Corollary 2.10.

To prove Corollary 2.11, consider any C_0-semigroup $[\Phi^t_a : t \in \mathbb{R}_+]$ with its dilation group $[\widehat{\Phi}^t_{\mathfrak{a}} : t \in \mathbb{R}_+]$ as above. By construction, the dilation C_0-group $[\widehat{V}_0^t : t \in \mathbb{R}]$ consists of $\widehat{\mathbf{H}}_0$-unitary operators. Thus, in view of Stone's classical theorem, we can apply the functional calculus [11] with its skew self-adjoint generator $i\widehat{S}_0$ when evaluating the transformations $\Phi^t_{\mathfrak{a}}$ by means of (3.12). Actually, for any $t \in \mathbb{R}$ we have

$$\int_0^t e^{-\lambda\tau} \widehat{V}_0^\tau \, d\tau = g_{1,t}(\widehat{S}_0), \qquad \int_0^t e^{-2\lambda\tau} \int_0^\tau e^{\lambda\sigma} \widehat{V}_0^\sigma \, d\sigma \, d\tau = g_{2,t}(\widehat{S}_0)$$

with the functions $\mathbf{s} \mapsto \int_0^t e^{-\lambda\tau} e^{i\tau\mathbf{s}} d\tau$ resp. $\mathbf{s} \mapsto \int_0^t e^{-2\lambda\tau} \int_0^\tau e^{\lambda\sigma} e^{i\sigma\mathbf{s}} d\sigma \, d\tau$ which are real analytic $\mathbb{R} \to \mathbb{C}$. Straightforward calculation establishes their algebraic form and the Taylor series appearing in Corollary 2.11.

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