**A fixed point approach to unitary $C_0$-groups in Hilbert space**

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## 1. Introduction, main results

Throughout this work $\mathbf{H}$ denotes an arbitrarily fixed infinite dimensional complex Hilbert space with the scalar product $\langle x|y\rangle$ which is linear in $x$ and conjugate linear in $y$, giving rise to the norm $\|x\| = \langle x|x\rangle^{1/2}$. We denote the open unit ball $\{e \in \mathbf{H} : \ \|e\| < 1\}$ with $\mathbf{B}$ and for any vector $a \in \mathbf{H}$ we shall write $a^* := [x \mapsto \langle x|a\rangle]$ for its dual functional.
Recall that the group $\mathrm{Aut}(\mathbf{B})$ of all holomorphic automorphisms of $\mathbf{B}$ consists the biholomorphic maps $\mathbf{B} \leftrightarrow \mathbf{B}$, and the $\mathbf{H}$-unitary operators restricted to $\mathbf{B}$ form the isotropy subgroup of the origin of $\mathrm{Aut}(\mathbf{B})$. Stone's classical theorem on strongly continuous one-parameter groups of unitary operators can be reformulated in terms of $\mathrm{Aut}(\mathbf{B})$ as a statement that the infinitesimal generator of a strongly continuous one-parameter subgroup of $\mathrm{Aut}(\mathbf{B})$ leaving fixed the origin can be identified canonically with the restriction $iA|\mathbf{B}$ where $A$ is a possibly unbounded self-adjoint linear operator with dense domain in $\mathbf{H}$. The first attempt to reach an analogous description for the strongly continuous one-parameter subgroups of $\mathrm{Aut}(\mathbf{B})$ formed by possibly non-linear maps can be found in Vesentini's celebrated paper [15] in 1987 based on a linear model generalizing naturally a well-known analogous concept for finite dimensional Möbius groups. Later on [16] he returned to the theme with the aim of extending the results to strongly continuous one-parameter semigroups holomorphic automorphisms of the unit ball of a Cartan factor of type 1 that is a space of the form $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ with Hilbert spaces $\mathbf{H}_k$ where a linear model is still available. Katshkevich-Reich-Shoiket [9] extended these investigations to general strongly continuous one-parameter semigroups of holomorphic fractional linear transformations. Nevertheless a simple explicit algebraic description for these semigroups seems not yet being appeared in the literature.
Our purpose in this paper will be to develop an alternative shorter approach to the description of vector fields arising as infinitesimal generators of strongly continuous one-parameter subsemigroups of $\mathrm{Aut}(\mathbf{B})$. Though several details presented here seem to be contained implicitly in [15,16,9], our treatment treatment based on the existence of joint fixed points uses essentially different ideas which may be of independent geometric interest concerning the structure of the Banach-Lie group of the surjective isometries of a hyperbolic space. We try give a self-contained presentation staring only from the familiar form (2.3) for the Möbius shifts establish first in [5,Ch.X] in infinite dimensions. At the beginning we also provide some simple general results concerning the existence of joint fixed points and continuity of boundary extensions in the setting of reflexive spaces. We pay particular attention (Section 4) to the characterize the cases where a Kaup type formula ($x \mapsto b - \langle x|b\rangle + iAx$ given first in [10] for the *uniformly* continuous case) is available for the vector fields of the infinitesimal generators. We focus to one-parameter groups, establishing the following main results.

**Theorem 1.1.** *Assume* $\big[\Psi^t:\ t\in\mathbb{R}\big]$ *is a strongly continuous one-parameter group of holomorphic automorphisms of* **B**.[1] *Then there exists a vector* $\overline{x}$ *with* $\|\overline{x}\|\le 1$ *along with a constant* $\lambda\in\mathbb{R}$ *and a densely defined possibly unbounded self-adjoint operator* $A:\mathbf{Z}\to\mathbf{H}$ *with dense domain such that*

$$\mathbf{B}\cap(\overline{x}+\mathbf{Z})=\mathbf{D}\quad \textit{where}\quad \mathbf{D}:=\big\{x\in\mathbf{D}:\ t\mapsto\Psi^t(x)\ \textit{is differentiable on}\ \mathbb{R}\big\},\tag{1.2}$$

$$\frac{d}{dt}\Big|_{t=0}\Psi^t(x)=-\big\langle(iA-\lambda)(x-\overline{x})\big|\overline{x}\big\rangle x+(iA+\lambda)(x-\overline{x})\qquad(x\in\mathbf{D}).\tag{1.3}$$

*Given any tuple* $(A,\overline{x},\lambda)$ *consisting of a densely defined self-adjoint operator* $A:\mathbf{Z}\to\mathbf{H}$, *a vector* $\overline{x}$ *with* $\|\overline{x}\|\le 1$ *and a real number* $\lambda$, *there exists (a necessarily unique) strongly continuous one-parameter group* $[\Psi^t:\ t\in\mathbb{R}]$ *satisfying* (1.2) *and* (1.3) *if and only if one of the following alternatives holds:* (1) $\|\overline{x}\|=1$; (2) $\|\overline{x}\|<1,\ \lambda=0$.

**Corollary 1.4.** *If* $\|\overline{x}\|=1$ *above and* $\overline{x}$ *is an eigenvector of the operator* $A$ *with eigenvalue* $\rho(\in\mathbb{R})$ *then the following alternatives hold: either for some* $0\ne\lambda\in\mathbb{R}$ *we have*

$$\Psi^t(x_0+\xi\overline{x})=\frac{e^{\lambda t}\exp(itA)x_0}{2\lambda-\varphi(\rho,\lambda,t)(\xi-1)}+\Big(1+\frac{2\lambda e^{2\lambda t}(\xi-1)}{2\lambda-\varphi(\rho,\lambda,t)(\xi-1)}\Big)\overline{x}$$

*where* $\lambda\ne 0$ *and* $\varphi(\rho,\lambda,t):=(i\rho-\lambda)e^{2\lambda t}-(\lambda+i\rho)$, *or*

$$\Psi^t(x_0+\xi\overline{x})=\frac{\exp(itA)x_0}{1+i\rho t(\xi-1)}+\Big(1+\frac{\xi-1}{1+i\rho t(\xi-1)}\Big)\overline{x}$$

*for all* $t\in\mathbb{R}$, *and* $x_0+\xi\overline{x}\in\mathbf{B}$ *with* $x_0\perp\overline{x}$.

**Corollary 1.5.** *If* $\overline{x}\ne\overline{x}'\in\partial\mathbf{B}$ *are the only common fixed poits of* $[\overline{\Psi^t}:t\in\mathbb{R}]$, *there exists* $\Theta\in\mathrm{Aut}(\mathbf{B})$ *along with a constant* $0\ne\lambda\in\mathbb{R}$ *such that* $\overline{\Theta}(\overline{x})=\overline{x}$, $\overline{\Theta}(\overline{x}')=-\overline{x}$ *and*

$$\Theta\circ\Psi^t\circ\Theta^{-1}(x+\xi\overline{x})=\frac{e^{\lambda t}\exp(itA)x_0}{2\lambda-\varphi(0,\lambda,t)(\xi-1)}+\Big(1+\frac{2\lambda e^{2\lambda t}(\xi-1)}{2\lambda-\varphi(0,\lambda,t)(\xi-1)}\Big)\overline{x}$$

*for all* $t\in\mathbb{R}$, *and* $x_0+\xi\overline{x}\in\mathbf{B}$ *with* $x_0\perp\overline{x}$.

## 2. Preliminaries: linear model with joint fixed points

**Lemma 2.1.** *Assume* $\mathcal{K}$ *is a compact topological space and let* $[f_t:\ t\in\mathbb{R}_+]$ *be a one-parameter semigroup of continuous maps* $\mathcal{K}\to\mathcal{K}$ *admitting fixed points such that all the functions* $t\mapsto f_t(x)$ *are continuous. Then also* $\bigcap_{t\in\mathbb{R}_+}\mathrm{Fix}(f_t)\ne\emptyset$.

[1] That is $\Psi^{t+s}=\Psi^t\circ\Psi^s\in\mathrm{Aut}(\mathbf{B})$ for all couples $t,s\in\mathbb{R}$ and the functions $[t\mapsto\Psi^t(x)]$ are continuous $\mathbb{R}\to\mathbf{H}$ for any fixed vector $x\in\mathbf{B}$.

**Proof.** Consider any parameter $t > 0$ and a point $x \in \mathrm{Fix}(f_t)$. For $n = 1, 2, \ldots$ recursively we have $f_{nt}(x) = f_t\big(f_{(n-1)t}(x)\big) = x$. Thus $\mathrm{Fix}(f_{nt}) \supset \mathrm{Fix}(f_t) \neq \emptyset$ $(t \in \mathbb{R}_+,\ n = 1, 2, \ldots)$. From the continuity of the maps $f_t$ it follows that $\mathrm{Fix}(f_{1/n!})$ $(n = 1, 2, \ldots)$ is a decreasing sequence of non-empty compact sets with non-empty intersection $\mathcal{X} := \bigcap_n \mathrm{Fix}(f_{1/n!})$. Since any rational number $0 \neq q \in \mathbb{Q}_+$ can be written in the form $q = m/n!$ for suitable integers $m, n > 0$, it follows even that $\bigcap_{q \in \mathbb{Q}_+} \mathrm{Fix}(f_q) = \mathcal{X} \neq \emptyset$. Consider any parameter $t > 0$ and any point $x \in \mathcal{X}$. Given any sequence $q_1, q_2, \ldots \in \mathbb{Q}_+$ converging to $t$, the continuity of the orbit $t \mapsto f_t(x)$ ensures that $x = \lim_n f_{q_n}(x) = f_t(x)$.

**Lemma 2.2.** *Let $\mathbf{K}$ be a domain in a Banach space $\mathbf{E}$ and let $f_t : \mathbf{D}_t \to \mathbf{E}$ $(t \in \mathbb{R}_+]$ be a family of holomorphic maps defined on open neighborhoods of $\overline{\mathbf{K}}$ such that the restrictions $[f_t|\mathbf{K} : t \in \mathbb{R}_+]$ form a strongly continuous one-parameter semigroup. Assume that for every boundary point $x \in \partial\mathbf{K}$ there exists a 1-dimensional complex disc $\Delta_x$ centered in $x$ and intersecting $\mathbf{K}$ such that $\Delta_x \subset \bigcap_{t \in [0,\delta_x]} \mathbf{D}_t$ and $\bigcup_{t \in [0,\delta_x]} f_t(\Delta_x)$ is a bounded set for some $\delta_x > 0$. Then $[f_t|\overline{\mathbf{K}} : t \in \mathbb{R}]$ is also a strongly continuous one-parameter semigroup.*

**Proof.** By assumption $f_s(f_t(x)) = f_{s+t}(x)$ $(x \in \mathbf{K},\ s, t \in \mathbb{R}_+)$. Since the maps $f_t|\overline{\mathbf{K}}$ are all continuous, and it follows $f_s(f_t(x)) = f_{s+t}(x)$ $(x \in \overline{\mathbf{K}},\ s, t \in \mathbb{R}_+)$ that is $[f_t|\overline{\mathbf{K}} : t \in \mathbb{R}_+]$ is a one-parameter semigroup of continuous maps on the closure $\overline{\mathbf{K}}$. Hence, to complete the proof, it suffices to see only that for any $x \in \partial\mathbf{K}$, the function $t \mapsto f_t(x)$ is continuous on some neighborhood the origin, namely on $[0, \delta_x)$. Fix any $x \in \partial\mathbf{K}$ and consider a convergent sequence $t_n \to t$ within $[0, \delta_x]$. We show the convergence $f_{t_n}(x) \to f_t(x)$ as follows. We can write $\Delta_x = \{x + \zeta v : |\zeta| < 1\}$ with a suitable vector $v \in \mathbf{E}$. By assumption, the functions $g_n(\zeta) := f_{t_n}(x + \zeta v)$ $(n = 1, 2, \ldots)$ are uniformly bounded and holomorphic on the unit disc. Furthermore they are assumed to converge pointwise to $g(\zeta) := f_t(x + \zeta v)$ on the non-empty open complex domain $\{\zeta \in \mathbb{C} :\ x + \zeta v \in \mathbf{K}\}$. In Banach space setting, pointwise convergence implies uniform convergence on compact sets for holomorphic maps [12]. In particular, we have uniform convergence for $[g_n : n = 1, 2, \ldots]$ on some compact disc with positive radius. By a theorem of Vigué [17, 8], for a uniformly bounded sequence of holomorphic maps, the uniform convergence on some subdomain entails locally uniform (and hence pointwise) convergence on the whole domain. In particular, $g_n \to g$ pointwise and hence $f_{t_n}(x) = g_n(0) \to g(0) = f_t(x)$.

**Corollary 2.3.** *If $\mathbf{E}$ is a JB*-triple, $\mathbf{K}$ is its open unit ball and $[f_t : t \in \mathbb{R}]$ is a strongly continuous one-parameter subgroup of $\mathrm{Aut}(\mathbf{K})$ then the maps $\overline{f}_t$ obtained with graph closure from the respective $f_t$, form a strongly continuous one-parameter group of maps $\overline{\mathbf{K}} \to \overline{\mathbf{K}}$.*

**Proof.** It is well-known [11] that we can write $f_t = M_{f_t(0)} \circ U_t$ with some invertible linear operator $U_t \in \mathcal{L}(\mathbf{E})$ and a so-called Möbius transformation with the fractional linear form $x \mapsto f_t(0) + B_t[I + L(x, f_t(0)]^{-1}x$ where $B_t, L(x, f_t(0)) \in \mathcal{L}(\mathbf{E})$ and $\|L(x, f_t(0))\| \leq \|f_t(0)\|\|x\|$. In particular each $f_t$ extends holomorphically to the ball of radius $1/\|f_t(0)\|$. Thus the conditions required by the lemma are fulfilled since $\lim\limits_{t \to 0} f_(0) = 0$ by assumption.

Henceforth we focus to the case of the unit ball $\mathbf{B}$ of an infinite dimensional Hilbert space.

Recall [5, Ch. VI] that the group of $\mathrm{Aut}(\mathbf{B})$ all holomorphic automorphisms of $\mathbf{B}$ admits a matrix representation. Namely each element $\Psi$ of $\mathrm{Aut}(\mathbf{B})$ has the fractional linear form

$$\Psi(x) = \frac{Ax+b}{\langle x|c\rangle + d}\,, \qquad A\in\mathcal{L}(\mathbf{H}),\quad b,c\in\mathbf{H},\quad d\in\mathbb{C} \tag{2.4}$$

and we have

$$\Psi_1\circ\Psi_2(x) = \Psi_1\big(\Psi_2(x)\big) = \frac{Ax+b}{\langle x|c\rangle + d} \quad \text{whenever} \quad \begin{bmatrix} A & b\\ c^* & d\end{bmatrix} = \begin{bmatrix} A_1 & b_1\\ c_1^* & d_1\end{bmatrix}\begin{bmatrix} A_2 & b_2\\ c_2^* & d_2\end{bmatrix}.$$

This representation is unique up to a constant, since in (2.4) we necessarily have

$$\begin{bmatrix} A & b\\ c^* & d\end{bmatrix} = d\begin{bmatrix} \mathrm{Q}_a & a\\ a^* & 1\end{bmatrix}\begin{bmatrix} U & 0\\ 0 & 1\end{bmatrix} \qquad \text{where} \quad a:=\Psi(0),\ U=(\beta_a^2\mathrm{P}_a+\beta_a\overline{\mathrm{P}}_a)^{-1}\Psi'(0)$$

in terms of the the Fréchet derivative $\Psi'$ and the standard notations

$$\mathrm{P}_a:=\big[\text{orthogonal projection } \mathbf{H}\to\mathbb{C}a\big],\quad \beta_a=\sqrt{1-\|a\|^2},\quad \mathrm{Q}_a:=\mathrm{P}_a+\beta_a(I-\mathrm{P}_a).$$

We call the matrix

$$\widetilde{\Psi} := \begin{bmatrix} \mathrm{Q}_a & a\\ a^* & 1\end{bmatrix}\begin{bmatrix} U & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix} \mathrm{Q}_aU & a\\ (U^*a)^* & 1\end{bmatrix}$$

corresponding to the case with constant $d=1$ the *canonical representation* of $\Psi$. In the sequel we shall write

$$\mathcal{H}:=\mathbf{H}\oplus\mathbb{C}=\big\{\big[\begin{smallmatrix} x\\ \xi\end{smallmatrix}\big] : x\in\mathbf{H},\ \xi\in\mathbb{C}\big\}$$

and identify the matrix $M := \big[m_{ij}\big]_{i,j=1}^2$ where $m_{11}\in\mathcal{L}(\mathbf{H})$, $m_{12}\in\mathbf{H}$, $m_{2,1}\in\mathbf{H}^*$ and $m_{22}\in\mathbb{C}$ with the linear operator $\big[\begin{smallmatrix} x\\ \xi\end{smallmatrix}\big]\mapsto M\big[\begin{smallmatrix} x\\ \xi\end{smallmatrix}\big]$ on $\mathcal{H}$. Notice that, by (2.4) we have

$$\Psi(x) = \left[\widetilde{\Psi}\begin{bmatrix} x\\ 1\end{bmatrix}\right]_{\mathbb{C}}^{-1}\left[\widetilde{\Psi}\begin{bmatrix} x\\ 1\end{bmatrix}\right]_{\mathbf{H}} \qquad (x\in\mathbf{B}) \tag{2.5}$$

where $[\,\cdot\,]_{\mathbb{C}}$ resp. $[\,\cdot\,]_{\mathbf{H}}$ are the standard notations for the canonical projections $\mathcal{H}\to\mathbb{C}$ resp. $\mathcal{H}\to\mathbf{H}$. It is immediate that any $\Psi\in\mathrm{Aut}(\mathbf{B})$ extends holomorphically to the ball $(1-\|\Psi(0)\|)^{-1}\mathbf{B}$. Hence we can define the group of all automorphisms of the closed unit ball $\overline{\mathbf{B}}:=\{x\in\mathbf{H}:\ \|x\|\le 1\}$ as

$$\mathrm{Aut}(\overline{\mathbf{B}}) := \big\{\overline{\Psi}:\ \Psi\in\mathrm{Aut}\,(\mathbf{B})\big\} \quad \text{where} \quad \overline{\Psi} := \big[\text{continuous extension of } \Psi \text{ to } \overline{\mathbf{B}}\big].$$

It is also well-known [5, Ch.VI] that any mapping $\overline{\Psi}\in\mathrm{Aut}(\overline{\mathbf{B}})$ is weakly continuous and preserves the Grassmann family $\mathrm{Aff}(\overline{\mathbf{B}})$ of all complex affine closed subspaces intersected with $\overline{\mathbf{B}}$.[1] By Schauder's fixed point theorem, $\mathrm{Fix}(\overline{\Psi})\neq\emptyset$ since $\overline{\mathbf{B}}$ is weakly compact. Moreover we have the following alternatives:

[1] If $x=\sum_{k=1}^2\lambda_k x_k$ with $\lambda_1,\lambda_2\in\mathbb{C}$ and $\sum_{k=1}^2\lambda_k=1$ then $\overline{\Psi}(x)=\sum_{k=1}^2\alpha_k\overline{\Psi}(x_k)$ for some $\alpha_1,\alpha_2\in\mathbb{C}$ with $\sum_{k=1}^2\alpha_k=1$ $\big($namely $\alpha_k=\lambda_k[1+\langle x_k|U^*a\rangle]/[1+\langle\lambda_1x_1+\lambda_2x_2|U^*a\rangle]\big)$.

(1) $\mathrm{Fix}(\overline{\Psi}) \in \mathrm{Aff}(\overline{\mathbf{B}})$, (2) $\mathrm{Fix}(\overline{\Psi})$ consists of two boundary points.

In case (2) from the proof of [5, Thm.VI.4.8] we see even that $\overline{\Psi} = \overline{\Phi} \circ \overline{\Theta}_a \circ \overline{\Phi}^{-1}$ with a suitable automorphism $\Phi \in \mathrm{Aut}(\overline{\mathbf{B}})$ and a *Möbius shift*

$$\Theta_a : x \mapsto \frac{\mathrm{Q}_a x + a}{1 + \langle x|a\rangle} \tag{2.6}$$

for some $0 \neq a \in \mathbf{B}$ such that $\mathrm{Fix}(\overline{\Theta}_a) = \{-e, e\}$ where $e := a/\|a\|$.

**Remark 2.7.** In finite dimensions, it is customary to normalize (2.4) by requiring $\det \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} = 1$. Thus, in case of $\dim(\mathbf{H}) = N$, in this manner one can establish a canonical identification of $\mathrm{Aut}(\mathbf{B})$ with a subgroup of the classical matrix group $\mathrm{SL}(N+1)$. Though in infinite dimensions such a normalization is not available, for one-parameter groups with common fixed point there is an alternative way as follows.

**Definition 2.8.** Let $\big([\Psi^t : t \in \mathbb{R}], \overline{x}\big)$ be a couple of a one-parameter subgroup of $\mathrm{Aut}(\mathbf{B})$ with common fixed point $\overline{x}$ for the continuous extensions of its members to $\overline{\mathbf{B}}$: $\overline{x} \in \overline{\mathbf{B}}$, $\overline{\Psi}^t(\overline{x}) = \overline{x}$ $(t \in \mathbb{R})$. In terms of the canonical representations define

$$\widehat{\Psi}^t_{\overline{x}} := \Big[\widetilde{\Psi^t}\begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}\Big]_{\mathbb{C}}^{-1} \widetilde{\Psi^t} = \frac{1}{1 + \langle U_t \overline{x}|a_t\rangle} \begin{bmatrix} Q_t & a_t \\ a_t^* & 1 \end{bmatrix} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix} \qquad (t \in \mathbb{R}).$$

where $a_t = \Psi_t(0) \in \mathbf{B}$, $U_t \in \mathcal{L}(\mathbf{H})$ is a suitable unitary operator and $Q_t := \mathrm{Q}_{a_t} = P_t + \beta_t(I - P_t)$ with $P_t x := \mathrm{P}_{a_t} x = \|a_t\|^{-2}\langle x|a_t\rangle a_t$, $\beta_t = \sqrt{1 - \|a_t\|^2}$.
Later on, conveniently we shall simply write $\widehat{\Psi}^t$ instead of $\widehat{\Psi}^t_{\overline{x}}$ without danger of confusion.

**Remark 2.9.** As we have seen $\Psi^t(x) = (1 + \langle U_t x|a_t\rangle)^{-1}[Q_t U^t x + a_t] = \big[\Theta_{a_t} \circ U_t\big](x)$. Thus, by construction we have

$$\overline{\Psi^t}(x) = \Big[\widehat{\Psi}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbb{C}}^{-1} \Big[\widehat{\Psi}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbf{H}} \quad (x \in \overline{\mathbf{B}}), \qquad \widehat{\Psi}^t \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}.$$

It is worth to notice that the term $\langle U_t \overline{x}|a_t\rangle$ is actually independent of $U_t$ as

$$\langle U_t \overline{x}|a_t\rangle = \frac{\langle \overline{x} - a_t|a_t\rangle}{1 - \langle \overline{x}|a_t\rangle}, \qquad \widehat{\Psi}^t = \frac{1 - \langle \overline{x}|a_t\rangle}{1 - \langle a_t|a_t\rangle} \begin{bmatrix} Q_t & a_t \\ a_t^* & 1 \end{bmatrix} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.10}$$

Proof: In general we have $P_t y = \langle y|a_t\rangle\langle a_t|a_t\rangle^{-2} a_t$ $(0 \neq a_t, y \in \mathbf{H})$. It follows $\langle P_t U_t \overline{x}|a_t\rangle = \langle U_t \overline{x}|a_t\rangle$ with $\langle \overline{P}_t U_t \overline{x}|a_t\rangle = 0$ for any $t \in \mathbb{R}$. Thus multiplying the fixed point equation $\overline{x} = \overline{\Psi}^t(\overline{x}) = (1 + \langle U_t \overline{x}|a_t\rangle)^{-1}(P_t + \beta_t \overline{P}_t) U_t \overline{x}$ with $|a_t\rangle$, we get $(1 + \langle U_t \overline{x}|a_t\rangle)^{-1}\langle U_t \overline{x} + a_t|a_t\rangle = \langle \overline{x}|a_t\rangle$ whence the relations (2.10) are immediate.

The power style indexing of $\widehat{\Psi}^t$ in $t$ is justified by the proposition below.

**Proposition 2.11.** *Given a strongly continuous one-parameter group* $[\Psi^t : t \in \mathbb{R}]$ *in* $\mathrm{Aut}(\mathbf{B})$ *with common fixed point* $\overline{x} \in \overline{\mathbf{B}}$*, the family* $[\widehat{\Psi}^{t}_{\overline{x}} : \ t \in \mathbb{R}]$ *is a strongly continuous one-parameter group of operators in* $\mathcal{H}$*.*

**Proof.** Since $\Psi^t \circ \Psi^s = \Psi^{t+s}$ $(t, s \in \mathbb{R})$, for the representation matrices we have $\widehat{\Psi}^t \widehat{\Psi}^s = d_{t,s} \widehat{\Psi}^{t+s}$ with suitable constants $d_{t,s} \in \mathbb{C}_*$. The fixed point property $\overline{\Psi}^t(\overline{x}) = \overline{x}$ implies

$$\widehat{\Psi}^t \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R}).$$

Hence necessarily $d_{t,s} = 1$ $(t, s \in \mathbb{R})$, thus $\big[\widehat{\Psi}^t : t \in \mathbb{R}\big]$ is a one-parameter matrix group.
By assumption, the function $t \mapsto a_t = \Psi^t(0)$ is norm-continuous $\mathbb{R} \to \mathbf{B}$. Hence we can deduce the strong continuity of the $\mathbf{H}$-unitary operator valued function $t \mapsto U_t$. Namely consider any vector $x \in \mathbf{H}$. To establish the norm-continuity of the function $t \mapsto U_t$, we may assume without loss of generality that $x \in \mathbf{B}$. Then, by the aid of the Möbius shifts (2.3) we can write

$$U_t x = \big[\Theta_{a_t}^{-1} \circ \Psi_t\big](x) = \Theta_{-a_t}\big(\Psi(x)\big) \qquad (t \in \mathbb{R}).$$

Observe that the norm continuity of $t \mapsto a_t$ implies the continuity of $t \mapsto \langle x | a_t \rangle$ and $t \mapsto \beta_t \in [0, 1)$ entailing the norm-continuity of $t \mapsto P_t + \beta_t \overline{P}_t \in \overline{\mathrm{Ball}}\big(\mathcal{L}(\mathbf{H})\big)$. Hence the required norm-continuity of $t \mapsto U_t x = (1 - \langle x | a_t \rangle)^{-1} [(P_t x - a_t + \beta_t \overline{P}_t x]$ is immediate. In general, the product of two bounded strongly continuous linear operator valued functions $\mathbb{R} \to \mathcal{L}(\mathbf{X})$ over a normed space $\mathbf{X}$ is strongly continuous. Hence we conclude that the entries $(1, 1), (1, 2), (2, 1)$ resp. $(2, 2)$ of the matrices $\widehat{\Psi}^t$ are strongly continuous functions $\mathbb{R} \to \mathcal{L}(\mathbf{H})$, $\mathbb{R} \to \mathbf{H}$, $\mathbb{R} \to \mathbf{H}^* \simeq \mathbf{H}$ resp. $\mathbb{R} \to \mathbb{R}$ which completes the proof.

**Corollary 2.12.** *Given a strongly continuous one-parameter group* $[\mathcal{T}^t : \ t \in \mathbb{R}]$ *in* $\mathcal{L}(\mathcal{H})$*, the following statements are equivalent*

(i) *for all* $t \in \mathbb{R}$*, the maps* $x \mapsto \big[\mathcal{T}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]\big]_{\mathbb{C}}^{-1} \big[\mathcal{T}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]\big]_{\mathbf{H}}$ *belong to* $\mathrm{Aut}(\mathbf{B})$*;*

(ii) *we have* $\mathcal{T}^t = e^{\mu t} \widehat{\Psi}^t$ $(t \in \mathbb{R})$ *for some strongly continuous one-parameter subgroup* $[\Psi^t : \ t \in \mathbb{R}]$ *of* $\mathrm{Aut}(\mathbf{B})$ *and a constant* $\mu \in \mathbb{C}$*;*

(iii) *each operator* $\mathcal{T}^t$ *maps the cone* $\mathcal{K} := \big\{ \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] : |\xi|^2 > \|x\|^2 \big\}$ *onto itself;*

(iv) *each operator* $\mathcal{T}^t$ *maps* $\partial\mathcal{K} := \big\{ \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] : |\xi|^2 = \|x\|^2 \big\}$ *onto itself.*

**Proof.** The implication (ii)⇒(i) is trivial by (2.5).

Proof of (i)⇒(ii): By assumption the maps $\Psi^t(x) := \big[\mathcal{T}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]\big]_{\mathbb{C}}^{-1} \big[\mathcal{T}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]\big]_{\mathbf{H}}$ $(t \in \mathbb{R},\ x \in \mathbf{B})$. are well-defined holomorphic automorphisms of the unit ball $\mathbf{B}$. By (2.5) we have $\mathcal{T}^t = d_t \widehat{\Psi}^t$ $(t \in \mathbb{R})$ with suitable constants $d_t \in \mathbb{C}_*$. Fixing any point $x \in \mathbf{B}$, the strong continuity of the group $[\mathcal{T}^t : \ t \in \mathbb{R}]$ implies the continuity of the function $\mathcal{T}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]$ whence we deduce also the continuity of $t \mapsto \Psi^t(x)$ which entails the continuity of $t \mapsto \widehat{\Psi}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] = d_t^{-1} \mathcal{T}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]$ and hence the continuity of $t \mapsto d_t$. By the one-parameter group property, all the relations

$\mathcal{T}^{t+s} = \mathcal{T}^t\mathcal{T}^s$, $\widehat{\Psi}^{t+s} = \widehat{\Psi}^t\widehat{\Psi}^s$ $(t,s \in \mathbb{R})$ hold. Therefore $d_{t+s} = d_t d_s$ $(t,s \in \mathbb{R})$ and the continuity of $t \mapsto d_t$ establishes the existence of a constant $\mu \in \mathbb{C}$ with $d_t = e^{\mu t}$ $(t \in \mathbb{R})$.

Proof of (i)⇔(iii)⇔(iv): Consider the projective Hilbert space $\mathcal{H}_*/\approx$ associated with $\mathcal{H}$ regarded as the set of all nontrivial punctured complex rays $\mathbb{C}_*\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$ with the factor topology.[2] By homogeneity, any injective linear operator $\mathcal{T} \in \mathcal{L}(\mathcal{H})$ acts holomorphically on $\mathcal{H}_*/\approx$ by its factorization $\mathcal{T}_\approx : \mathbb{C}_*\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] \mapsto \mathbb{C}_*\mathcal{T}\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$. In particular, as admitting a continuous inverse, each map $\mathcal{T}^t_\approx$ is a holomorphic automorphism of $\mathcal{H}/\approx$. Hence the equivalences (i)⇔(iii)⇔(iv) are straightforward consequences of the facts that, with the embedding $\Pi : x \mapsto \mathbb{C}\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ and its inverse $\pi\big(\mathbb{C}\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]\big) := x/\xi$ $(\xi \neq 0)$, we have $\Pi\mathbf{B} := \mathcal{K}$, $\pi\mathcal{K} = \mathbf{B}$ and $\big[\mathcal{T}^t\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbb{C}}^{-1}\big[\mathcal{T}^t\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbf{H}} = \pi \circ \mathcal{T}^t_\approx \circ \Pi\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ whenever $\big[\mathcal{T}^t\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbb{C}} \neq 0$.

**Corollary 2.13.** *Given any $\Theta \in \mathrm{Aut}(\overline{\mathbf{B}})$, the $\Theta$-shifted automorphisms $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ form strongly continuous one-parameter group with common fixed point $\overline{y} := \Theta(\overline{x})$ when extended continuously to $\overline{\mathbf{B}}$ and* $\widehat{\Phi}^t_{\overline{y}} = e^{\mu t}\widetilde{\Theta}^{-1}\widehat{\Psi}^t_{\overline{x}}\widetilde{\Theta}$ $(t\in\mathbb{R})$ *for some $\mu\in\mathbb{C}$.*

## 3. Infinitesimal generators

Throughout this section, let $\big([\Psi^t\colon t \in \mathbb{R}], \overline{x}\big)$ be an arbitrarily fixed couple of a strongly continuous one-parameter group in $\mathrm{Aut}(\mathbf{B})$ with a common fixed point for the continuous extensions in $\overline{\mathbf{B}}$. Recalling the Hille–Yosida theorem [13, Kap.10], Proposition 2.11ensures that the differential

$$(3.1) \qquad \mathcal{A} : \mathfrak{h} \mapsto \frac{d}{dt}\widehat{\Psi}^t\mathfrak{h} \quad \text{with} \quad \mathcal{D} := \mathrm{dom}(\mathcal{A}) = \big\{\mathfrak{h} \in \mathcal{H} : t \mapsto \widehat{\Psi}^t\mathfrak{h} \text{ is differentable on } \mathbb{R}\big\}$$

(called the *infinitesimal generator* of the linear model $\big[\widehat{\Psi}^t\colon t\in\mathbb{R}\big]$ where $\widehat{\Psi}^t \equiv \widehat{\Psi}^t_{\overline{x}}$ for short) is a not necessarily bounded linear map with closed graph and $[\widehat{\Psi}^t : t\in\mathbb{R}]$-invariant domain being dense in $\mathcal{H}$. Instead of the differential $\mathcal{A} = \frac{d}{dt}|_{t=0}\widehat{\Psi}^t$ of the representations we are primarily interested in the differential

$$\Omega := \frac{d}{dt}\Big|_{t=0}\Psi^t : \mathbf{D} \to \mathbf{H} \quad \text{where} \quad \mathbf{D} = \mathrm{dom}(\Omega) = \Big\{x \in \mathbf{B} :\ \frac{d}{dt}\Big|_{t=0}\Psi^t(x) \text{ exists}\Big\}.$$

In order that we could regard the vector field $\Omega$ as a non-linear infinitesimal generator for $[\Psi^t :\ t \in \mathbb{R}]$, we should see the density of $\mathbf{D}$ in $\overline{\mathbf{B}}$. In order to establish a non-linear Stone-type theorem, we should determine precise links to self-adjoint linear operators.

**Lemma 3.2.** **D** *is $\big[\overline{\Psi}^t : t\in\mathbb{R}\big]$-invariant. We have $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathcal{D} \iff x\in\mathbf{D}$ whenever $x\in\mathbf{B}$.*

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[2] As usually, $\mathcal{H}_* := \mathcal{H}\setminus\{0\}$ with the equivalence relation $\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right] :\iff \mathbb{C}_*\left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right] = \mathbb{C}_*\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$ where $\mathbb{C}_* := \mathbb{C}\setminus\{0\}$. A subset of $\mathcal{H}/\approx$ is open if the union of it members (rays in $\mathcal{H}_*$) is open in $\mathcal{H}$.

**Proof.** The $\big[\Psi^t : t\in\mathbb{R}\big]$-invariance of $\mathbf{D}$ is clear from the group property $\Psi^{t+s} = \Psi^t \circ \Psi^s$ $(t, s \in\mathbb{R})$. Moreover even $\mathrm{dom}\big(\frac{d}{dt}\big|_{t=0}\Theta\circ\Psi^t\circ\Theta^{-1}\big) = \Theta(\mathbf{D})$ whenever $\Theta$ is any holomorphic automorphism of $\mathbf{B}$. Hence, given any point $x \in \mathbf{B}$, we have $x \in \mathbf{D}$ if and only if $0 = \Theta_{-x}(x)\in\mathrm{dom}\big(\frac{d}{dt}\big|_{t=0}\Phi^t\big)$ with the one-parameter group of the maps $\Phi^t := \Theta_{-x} \circ \Psi^t \circ \Theta_x$ in terms of the Möbius transformations (2.3). That is, without loss of generality, it suffices only to see the equivalence $0 \in \mathrm{dom}\big(\frac{d}{dt}\big|_{t=0}\Phi^t\big) \iff \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \in \mathcal{D}$. According to (2.10), by setting $a_t := \Psi^t(0)$ and $\mathfrak{a}_t := \widehat{\Psi}^t\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ we have $\mathfrak{a}_t = (1 - \langle a_t|\overline{x}\rangle(1 - \|a_t\|^2)^{-1}\left[\begin{smallmatrix}a_t\\1\end{smallmatrix}\right]$ $(t \in \mathbb{R})$. Hence the curves $t \mapsto a_t$ resp. $t \mapsto \mathfrak{a}_t$ are differentiable in the same time, which completes the proof.

For later use we also introduce the notations

$$\mathbf{Z} := \Big\{z \in \mathbf{H} :\ \begin{bmatrix}z\\0\end{bmatrix} \in \mathcal{D}\Big\}; \qquad Bz := \Big[\mathcal{A}\begin{bmatrix}z\\0\end{bmatrix}\Big]_{\mathbf{H}},\ \Lambda z := \Big[\mathcal{A}\begin{bmatrix}z\\0\end{bmatrix}\Big]_{\mathbb{C}} \quad (z \in \mathbf{Z}).$$

**Lemma 3.3.** *$\mathbf{Z}$ is a dense linear submanifold in $\mathbf{H}$ with $\mathbf{D} = \big(\overline{x} + \mathbf{Z}\big) \cap \mathbf{B}$ and $\mathcal{D} = \left[\begin{smallmatrix}\mathbf{Z}\\0\end{smallmatrix}\right] + \mathbb{C}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right]$. The set $\mathbf{D}$ is dense in $\mathbf{B}$ and*

$$\frac{d}{dt}\Big|_{t=0}\overline{\Psi}^t(x) = [\Lambda(\overline{x} - x)]x + B(x - \overline{x}) \qquad (x \in \mathbf{D}).$$

**Proof.** By definition, $\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] \in \mathcal{D}$ with $\mathcal{A}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = 0$ since $\widehat{\Psi}^t\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right]$ $(t \in \mathbb{R})$. Since $\mathcal{D}$ is closed for linear combinations, it follows that $\left[\begin{smallmatrix}\mathbf{Z}\\0\end{smallmatrix}\right] + \mathbb{C}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = \mathcal{D}$ and that $\mathbf{Z}$ is the image of $\mathcal{D}$ by the bounded linear operator $\Pi\left[\begin{smallmatrix}x\\\xi\end{smallmatrix}\right] := x - \xi\overline{x}$. Since $\Pi\mathcal{H} = \mathbf{H}$ and since $\mathcal{D}$ is dense in $\mathcal{H}$, $\mathbf{Z} = \Pi\mathcal{D}$ is also dense in $\mathbf{H} = \Pi\mathcal{H}$. From Lemma 3.2 we know that $\mathbf{D} = \mathbf{B}\cap\big\{x :\ \left[\begin{smallmatrix}x\\1\end{smallmatrix}\right] \in \mathcal{D}\big\}$. Hence the relation $\mathbf{D} = \big(\overline{x}+\mathbf{Z}\big)\cap\mathbf{B}$ along with the density of $\mathbf{D}$ in $\mathbf{B}$ is immediate. Given any $x \in \mathcal{D}$, the relation $\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right] \in \mathcal{D}$ implies $\mathcal{A}\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right] = \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]$. Since $\overline{\Psi}^t(x) = \big\{\widehat{\Psi}^t\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\big\}_{\mathbb{C}}^{-1}\big\{\widehat{\Psi}^t\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\big\}_{\mathbf{H}}$ along with $\widehat{\Psi}^0 = \mathrm{Id}$ and $\mathcal{A}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = 0$, we get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\overline{\Psi}^t(x) &= -\Big[\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbb{C}}^{-2}\Big(\frac{d}{dt}\Big|_{t=0}\Big\{\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big\}_{\mathbb{C}}\Big)\Big[\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbf{H}} + \Big[\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbb{C}}^{-1}\frac{d}{dt}\Big|_{t=0}\Big\{\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big\}_{\mathbf{H}} = \\ &= -\Big[\mathcal{A}\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbb{C}}x + \Big[\mathcal{A}\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbf{H}} = -\Big[\mathcal{A}\begin{bmatrix}x-\overline{x}\\0\end{bmatrix}\Big]_{\mathbb{C}}x + \Big[\mathcal{A}\begin{bmatrix}x-\overline{x}\\0\end{bmatrix}\Big]_{\mathbf{H}}.\end{aligned}$$

**Lemma 3.4.** *Suppose a Hilbert space $\mathbf{W}$ is the orthogonal sum of the subspaces $\mathbf{W}_1, \mathbf{W}_2$ and $\mathcal{C}$ is the infinitesimal generator of a strongly continuous one-parameter subgroup $[T^t : t\in\mathbb{R}]$ of $\mathcal{L}(\mathbf{W})$. Then, for the cone $\mathbf{K} := \big\{w_1\oplus w_2 : \|w_1\| > \|w_2\|\big\}$, we have $T^t\mathbf{K} = \mathbf{K}$ $(t\in\mathbb{R})$ if and only if $\mathcal{C}$ is tangent to the boundary of $\mathbf{K}$ that is if*

$$\text{(3.5)}\qquad \mathrm{Re}\big\langle\mathcal{C}(w_1\oplus w_2)\big|w_1\big\rangle = \mathrm{Re}\big\langle\mathcal{C}(w_1\oplus w_2)\big|w_2\big\rangle \quad \big(w_1\oplus w_2\in\mathrm{dom}(\mathcal{C}),\ \|w_1\| = \|w_2\|\big).$$

**Proof.** It is immediate that $T^t\mathbf{K}\subset \mathbf{K}$ $(t \in \mathbb{R}) \Rightarrow T^t\partial\mathbf{K} \subset \partial\mathbf{K}$ $(t \in \mathbb{R}) \Rightarrow \frac{d}{dt}\big|_{t=0}T^t(w_1 \oplus w_2) \in \mathrm{Tan}_{w_1\oplus w_2}(\mathbf{K})$ for $w_1 \oplus w_2 \in \mathrm{dom}(\mathcal{C}) \Rightarrow$ (3.5). Assume (3.5) and let $P$ denote the canonical projection of $\mathbf{W}$ onto $\mathbf{W}_1$ and define $T^{t,s} := \exp(t\mathcal{C} + sP)$ $(s, t \in \mathbb{R})$. By the theorem of bounded perturbations [4, p.158] the operators $T^{t,s}$ are all well-defined. Moreover, by [4, Corollary 1.7 p. 161] (applied with $B := sP$ and $A := \mathcal{C}$ there) we have $\lim_{s\to 0} T^{t,s}w = T^t w$ $(w \in \mathrm{dom}(\mathcal{C}), t \in \mathbb{R})$. Therefore, to establish that $T^t\mathbf{K} \subset \mathbf{K}$ $(t \in \mathbb{R})$, it suffices to see only that $T^{t,s}w \in \mathbf{K}$ whenever $w \in \mathrm{dom}(\mathcal{C}) \cap \mathbf{K}$ and $t, s > 0$. To proceed to contradiction, let $s, t > 0$ and $w := w_1 \oplus w_2 \in \mathrm{dom}(\mathcal{C})$ with $\|w_1\| > \|w_2\|$ but $\|[T^{t,s}w]_1\| \le \|[T^{t,s}w]_2\|$. The function $\delta(\tau) := \|[T^{\tau,s}w]_1\|^2 - \|[T^{\tau,s}w]_2\|^2$ is differentiable in $\tau$ on the whole $\mathbb{R}$ and $\delta(0) > 0 \ge \delta(t)$. Thus there exist a point $t_* \in (0, t]$ such that $\delta(\tau) > 0 = \delta(t_*)$ $(0 \le \tau < t_*)$. Since $\delta(t_*) = 0$, the vector $w_* := T^{t_*,s}w$ belongs to $\partial\mathbf{K}$ and hence $\mathrm{Re}\langle \mathcal{C}w_* | [w_*]_1\rangle = \mathrm{Re}\langle \mathcal{C}w_* | [w_*]_2\rangle$. We get the contradiction $0 \ge \delta'(t_*) = 2\mathrm{Re}\langle(\mathcal{C} + sP)w_* | [w_*]_1\rangle - 2\mathrm{Re}\langle(\mathcal{C} + sP)w_* | [w_*]_2\rangle = 2s\|[w_*]_1\|^2 > 0$.

**Corollary 3.6.** $\mathrm{Re}\Big( - \Lambda v + \big\langle Bv \big| \overline{x}\big\rangle + \big\langle Bv \big| v\big\rangle\Big) = 0$ *whenever* $v \in \mathbf{Z}$ *with* $\|\overline{x} + v\| = 1$.

**Proof.** By Corollary 2.12, we have $\widehat{\Psi}^t \mathcal{K} = \mathcal{K}$ $(t \in \mathbb{R})$ where $\mathcal{K} := \big\{ \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] : |\xi| > \|x\| \big\} \subset \mathcal{H}$. An application of Lemma 3.4 with $\mathbf{W}_1 := \mathbb{C}$, $\mathbf{W}_2 := \mathbf{H}$, $\mathbf{K} := \mathcal{K}$, $T^t := \widehat{\Psi}^t$, $\mathcal{C} := \mathcal{A}$ establishes that $\mathrm{Re}\,\big[(\Lambda x)\overline{\xi}\big] = \mathrm{Re}\,\big\langle Bx \big| x\big\rangle$ whenever $\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] \in \mathcal{D}$ and $\|x\| = |\xi|$. We obtain the statement with the choice $x := v + \overline{x}$ and $\xi := 1$ if $v \in \mathbf{Z}$ with $\|v + \overline{x}\| = 1$ because then, by Lemma 3.3, we have $\big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$.

**Proposition 3.7.** *For some symmetric linear operator* $A : \mathbf{Z} \to \mathbf{H}$ *and a suitable constant* $\lambda \in \mathbb{R}$ *which is necessarily* $= 0$ *if* $\|\overline{x}\| \ne 1$*, we have* (1.3) *as*

$$B = iA + \lambda I, \qquad \Lambda z = \big\langle (iA - \lambda I)z \big| \overline{x}\big\rangle \quad (z \in \mathbf{Z}).$$

**Proof.** Consider any vector $0 \ne z \in \mathbf{Z}$. Let $\zeta \in \mathbb{C}$ be the (unique) constant such that $\overline{x} + \zeta z \perp z$ and define $\varrho := \sqrt{1 - \|\overline{x} + \zeta z\|^2}$. Actually we have $\zeta = -\langle \overline{x}|z\rangle / \langle z|z\rangle$ and $1 \ge \|\overline{x}\|^2 = \|\overline{x} + \zeta z\|^2 + \| - \zeta z\|^2$ showing that both $\zeta$ and $\varrho$ are well-defined. Consider the unit vectors

$$v_\varphi := \overline{x} + \zeta z + e^{i\varphi}\varrho z \qquad (\varphi \in \mathbb{R}).$$

According to Corollary 3.6,

$$\mathrm{Re}\Big((\zeta + e^{i\varphi}\varrho)\big[ - \Lambda z + \big\langle Bz \,\big|\, \overline{x}\big\rangle\big] + |\zeta + e^{i\varphi}\varrho|^2 \big\langle Bz \,\big|\, z\big\rangle\Big) = 0 \qquad (\varphi \in \mathbb{R}).$$

Thus the identity $\mathrm{Re}\big(\alpha + \beta e^{i\varphi} + \gamma e^{-i\varphi}\big) = 0$ for all $\varphi \in \mathbb{R}$ holds with the constants $\alpha := \zeta\big[ - \Lambda z + \big\langle Bz \,\big|\, \overline{x}\big\rangle\big] + (|\zeta|^2 + \varrho^2)\big\langle Bz \,\big|\, z\big\rangle$, $\beta := \varrho\,\big[ - \Lambda z + \big\langle Bz \,\big|\, \overline{x}\big\rangle + \overline{\zeta}\big\langle Bz \,\big|\, z\big\rangle\big]$ and $\gamma := \varrho\zeta\big\langle Bz \,\big|\, z\big\rangle$. Since $2\,\mathrm{Re}\big(\alpha + \beta e^{i\varphi} + \gamma e^{-i\varphi}\big) = 2\,\mathrm{Re}(\alpha) + (\beta + \overline{\gamma})e^{i\varphi} + (\overline{\beta} + \gamma)e^{-i\varphi}$, we have necessarily $\mathrm{Re}(\alpha) = \beta + \overline{\gamma} = 0$. From the relation $\beta + \overline{\gamma} = 0$, it follows

$$\Lambda z - \langle Bz \,|\, \overline{x}\rangle = 2\,\overline{\zeta}\,\mathrm{Re}\,\langle Bz \,|\, z\rangle = -2\,\langle z \,|\, \overline{x}\rangle \,\frac{\mathrm{Re}\,\langle Bz \,|\, z\rangle}{\langle z \,|\, z\rangle}\ , \tag{3.8}$$

and substituting this into the relation $0 = \mathrm{Re}(\alpha)$, we get

$$0 = (\varrho^2 - |\zeta|^2)\,\mathrm{Re}\,\big\langle Bz \,\big|\, z\big\rangle = \big(1 - \|\overline{x}\|^2\big)\,\mathrm{Re}\,\big\langle Bz \,\big|\, z\big\rangle\ . \tag{3.8$'$}$$

From (3.8) we see also that

$$z \mapsto \frac{\mathrm{Re}\big\langle Bz \,\big|\, z\big\rangle}{\big\langle z \,\big|\, z\big\rangle} = -\frac{1}{2}\,\frac{\Lambda z - \big\langle Bz \,\big|\, \overline{x}\big\rangle}{\big\langle z \,\big|\, \overline{x}\big\rangle}$$

is a real valued Gâteaux holomorphic function on the algebraically open and in $\mathbf{Z}$ algebraically dense domain $\{z \in \mathbf{Z} : z \not\perp \overline{x}\}$ which is possible only if being constant on $\mathbf{Z}$. By writing $\lambda$ for this constant value, from (3.8) and (3.8′) we conclude that

$$\Lambda z = \big\langle (B-2\lambda I)z \,\big|\, \overline{x}\big\rangle, \quad \mathrm{Re}\big\langle \big(B-\lambda I\big)z \,\big|\, z\big\rangle = 0 \qquad (z \in \mathbf{Z}),$$

and, in particular, if $\|\overline{x}\| < 1$ then necessarily $\lambda = 0$ above. By setting $A := -i(B+\lambda I)$, hence the statement including the symmetry of $A$ is immediate.

The following geometric converse of Proposition 3.7 is elementary:

**Remark 3.9.** Given a dense linear submanifold $\mathbf{Z}$ in $\mathbf{H}$, a symmetric linear operator $A : \mathbf{Z} \to \mathbf{H}$, a vector $\overline{x} \in \overline{\mathbf{B}}$, the vector field $\Omega_\lambda(x) := -\langle (iA-\lambda I)(x-\overline{x}|\overline{x}\rangle x + (iA+\lambda I)(x-\overline{x})$ define for $\overline{x} + \mathbf{Z}$ is tangent to the unit sphere $\partial\mathbf{B}$ at the points $x \in \overline{x} + \mathbf{Z}$ with $\|x\| = 1$ whenever either $\|\overline{x}\| = 1$ and $\lambda \in \mathbb{R}$ or $\|\overline{x}\| < 1$ and $\lambda = 0$.

In the sequel we proceed to the problem if the operator $A$ in Proposition 3.7 arising from the differential $\frac{d}{dt}\big|_{t=0}\Psi^t$ of a strongly continuous one-parameter subgroup of $\mathrm{Aut}(\mathbf{B})$ is necessarily self-adjoint and conversely if every self-adjoint operator may appear there.

## 4. The Jordan case

We continue the previous investigations with unchanged notations but under the additional hypothesis that

$$0 \in \mathbf{D} = \mathrm{dom}\Big(\frac{d}{dt}\Big|_{t=0}\Psi^t\Big) \;. \tag{4.1}$$

As we know, $\mathbf{D} = \big\{x \in \mathbf{B} : \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathcal{D}\big\} = \mathbf{B} \cap (\overline{x} + \mathbf{Z})$ where $\mathcal{D} = \mathrm{dom}(\mathcal{A})$ with $\mathcal{A} = \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t_{\overline{x}}$ and $\mathbf{Z} = \big\{x \in \mathbf{H} : \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}\big\}$ is a dense complex linear submanifold in $\mathbf{H}$. Thus, as a consequence of (4.1), for the distinguished common fixed point of the extended automorphisms $\overline{\Psi}^t$ we have $\overline{x} \in \mathbf{Z} = \mathrm{dom}(B) = \mathrm{dom}(\Lambda)$. Therefore also

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{D} = \begin{bmatrix} \mathbf{Z} \\ \mathbb{C} \end{bmatrix}, \quad \mathcal{A}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} B(x - \xi\overline{x}) \\ \Lambda(x - \xi\overline{x}) \end{bmatrix} = \begin{bmatrix} B & -B\overline{x} \\ \Lambda & -\Lambda\overline{x} \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} \quad (x \in \mathbf{Z},\ \xi \in \mathbb{C}).$$

**Remark 4.2.** Recall [10] that the complete holomorphic vector fields on $\mathbf{B}$ are the infinitesimal generators of the uniformly continuous one-parameter subgroups of $\mathrm{Aut}(\mathbf{B})$

and, with suitable $c \in \mathbf{H}$ and a bounded self-adjoint operator $C \in \mathcal{L}(\mathbf{H})$ they can be written in the form $x \mapsto a - \{xc^*x\} + iCx$ by means of the Jordan triple product.

$$\{xy^*z\} := \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x \tag{4.3}$$

In terms of the factorization $\Psi^t = \Theta_{a_t} \circ U_t|\mathbf{B}$ we introduce the following vector resp. not necessarily bounded symmetric linear operator:

$$b := \frac{d}{dt}\Big|_{t=0} \Psi^t(0) = \lim_{t\to 0} \frac{1}{t} a_t, \quad R := -i\frac{d}{dt}\Big|_{t=0} U_t : x \mapsto \lim_{t\to 0} \frac{1}{it}(U_t - I)x.$$

**Proposition 4.4.** *Under hypothesis* (4.1), *we have* $\mathbf{D} = \mathbf{B} \cap \mathbf{Z}$ *along with* $\mathrm{dom}(R) \supset \mathbf{Z}$ *and the vector field* $\Omega := \frac{d}{dt}\big|_{t=0}\Psi^t$ *admits the Jordan form*

$$\Omega(x) = b - \{xb^*x\} + iRx \qquad (x \in \mathbf{D}). \tag{4.5}$$

**Proof.** The relation $\mathbf{D} = \mathbf{B} \cap \mathbf{Z}$ is clear since $\overline{x} \in \mathbf{Z}$. By the definition of the generator $\mathcal{A}$,

$$\mathcal{A}\begin{bmatrix}0\\1\end{bmatrix} = \lim_{t\to 0} \frac{1}{t}\big(\widehat{\Psi}^t - \widehat{\Psi}^0\big)\begin{bmatrix}0\\1\end{bmatrix} = \lim_{t\to 0} \frac{1}{t}\Big\{\frac{1}{1 + \langle U_t\overline{x}|a_t\rangle}\begin{bmatrix}a_t\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\Big\}.$$

Since $\lim_{t\to 0} a_t = 0$ and $\|U_t\overline{x}\| \le 1$ $(t \in \mathbb{R})$, taking (2.10) into account, we see that the limit

$$b := \lim_{t\to 0} \frac{1}{t} a_t = \frac{d}{dt}\Big|_{t=0} \Psi^t(0)$$

is well-defined and

$$B\overline{x} = -\Big[\mathcal{A}\begin{bmatrix}0\\1\end{bmatrix}\Big]_{\mathbf{H}} = -b, \quad \Lambda\overline{x} = -\Big[\mathcal{A}\begin{bmatrix}0\\1\end{bmatrix}\Big]_{\mathbb{C}} = \langle \overline{x} \,|\, b\rangle.$$

As a consequence we also have

$$\begin{aligned}\beta_t &= \sqrt{1 - \|a_t\|^2} = \sqrt{1 - \|tb + o(t)\|^2} = 1 - \frac{1}{2}\|b\|^2 t^2 + o(t^2),\\ Q_t &= P_t + \beta_t(I - P_t) = I + (1 - \beta_t)P_t = I + o(t) \quad \text{(in operator norm)}.\end{aligned}$$

Since also $U_t z = z + o(1)$ in norm, hence we deduce that for any vector $z \in \mathbf{Z}$, and $\zeta \in \mathbb{C}$,

$$\widehat{\Psi}^t\begin{bmatrix}z\\\zeta\end{bmatrix} = \frac{1 - \langle\overline{x}|a_t\rangle}{1 - \|a_t\|^2}\begin{bmatrix}Q_tU_t & a_t\\ a_t^*U_tz & 1\end{bmatrix}\begin{bmatrix}z\\\zeta\end{bmatrix} = \begin{bmatrix}U_tz - t\langle\overline{x}|b\rangle z\\ t\langle U_tz|b\rangle + \zeta\end{bmatrix} + o(t) \quad \text{in norm.} \tag{4.6}$$

By definition, $\big[\begin{smallmatrix}Bz\\ \Lambda z\end{smallmatrix}\big] = \lim_{t\to 0} \frac{1}{t}\big(\widehat{\Psi}^t - I\big)\big[\begin{smallmatrix}z\\0\end{smallmatrix}\big]$. Hence with well-defined limits we conclude that

$$Bz = \lim_{t\to 0} \frac{1}{it}(U_t - I)z - \langle\overline{x}\,|b\rangle z \qquad (z \in \mathbf{Z}). \tag{4.7}$$

The strong limit of $t^{-1}(U_t - I)|\mathbf{Z}$ is necessarily symmetric due to the fact that each $U_t$ is unitary. Thus comparing (4.7) with Proposition 3.7 stating that the operator $B$ has the form $iA + \lambda I$ with some $\lambda \in \mathbb{R}$ and a symmetric operator $A$ with $\mathrm{dom}(A) = \mathbf{Z}$, we get

$$\lambda = -\mathrm{Re}\langle \overline{x} \,|\, b\rangle\,, \quad A = \lim_{t\to 0}^{\text{strong}} \left(U^t - I\right) - \mathrm{Im}\langle \overline{x} \,|\, b\rangle I \Big| \mathbf{Z} \quad \text{in 3.7.} \tag{4.8}$$

We calculate $\Omega$ by substituting (4.7-8) into its form $\Omega(x) = \left[\Lambda(\overline{x}-x)\right]x + B(x-\overline{x})$ applying also the relations $B\overline{x} = -b$, $\Lambda\overline{x} = \langle x|b\rangle$, $B = iA+\lambda I$, $\Lambda x = \langle (iA-\lambda I)x|\overline{x}\rangle = \langle (B-2\lambda I)x|\overline{x}\rangle$. Namely, given any vector $x \in \mathbf{D}$, taking into account the antisymmetry of the operator $iA = B-\lambda I$, we can write

$$\begin{aligned}
\Omega(x) &= \left[\Lambda\overline{x}\right]x - \left[\Lambda x\right]x + Bx - B\overline{x} = \\
&= \langle \overline{x}|b\rangle x - \langle (B-2\lambda I)x|\overline{x}\rangle x + Bx + b = \\
&= b + \left[\langle \overline{x}|b\rangle I + B\right]x - \langle (B-\lambda I)x|\overline{x}\rangle x + \lambda\langle x|\overline{x}\rangle x = \\
&= b + iRx + \langle x|(B-\lambda I)\overline{x}\rangle x + \lambda\langle x|\overline{x}\rangle x = \\
&= b + iRx + \langle x|B\overline{x}\rangle x = b + iRx - \langle x|b\rangle x.
\end{aligned}$$

**Corollary 4.9.** $\mathbf{Z} = \mathrm{dom}(R)$ *that is* $x \in \mathbf{Z}$ *if and only if the limit* $\lim_{t\to 0} t^{-1}(U^t x - x)$ *exists.*

**Proof.** Recall that $\mathbf{Z} = \left\{x \in \mathbf{H} : \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t\left[\begin{smallmatrix} x \\ 0\end{smallmatrix}\right] \text{ is well-defined}\right\}$. From Proposition 4.4 we know that $iRx = \lim_{t\to 0} t^{-1}(U_t - I)x$ is well-defined for every vector $x \in \mathbf{Z}$. Conversely, suppose $u = \lim_{t\to 0} t^{-1}(U_t x - x)$ is well-defined. Then $U_t x = x + tu + o^{\mathrm{norm}}(t)$ and (4.6) establishes that $\widehat{\Psi}^t\left[\begin{smallmatrix} x \\ 0\end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ 0\end{smallmatrix}\right] + t\left[\begin{smallmatrix} u-\langle \overline{x}|a_t\rangle x \\ \langle x|b\rangle\end{smallmatrix}\right] + o^{\mathrm{norm}}(t)$.

**Lemma 4.10.** *We have* $U_{-t} = U_t^{-1} = U_t^*,\ \ a_{-t} = -U_t^* a_t \quad (t \in \mathbb{R})$.

**Proof.** Given any $t \in \mathbb{R}$, we have $\Psi^{-t} = \Psi_t^{-1}$ that is $\Theta_{a_{-t}}U_{-t} = \left[\Theta_{a_t}U_t\right]^{-1} = U_t^{-1}\Theta_{a_t}^{-1} = U_t^{-1}\Theta_{-a_t} = \left[U_t^{-1}\Theta_{-a_t}U_t\right]U_t^{-1} = \Theta_{U_t^{-1}(-a_t)}U_t^{-1}$. By the unambiguous decomposability of holomorphic automorphisms of circular domains into Möbius and unitary parts [2], hence we deduce that $\Theta_{a_{-t}} = \Theta_{-U_t^{-1}a_t}$ and $U_{-t} = U_t^{-1}$.

**Lemma 4.11.** *The operator* $R$ *is self-adjoint with* $\mathrm{dom}(R) = \mathbf{Z}$.

**Proof.** In view of (4.6), and since $U_t x = x + itRx + o^{\mathrm{norm}}(t)$ for any $x \in \mathbf{Z} = \mathrm{dom}(R) = \mathrm{dom}(\frac{d}{dt}\big|_{t=0}U_t)$, we conclude that

$$\frac{d}{dt}\Big|_{t=0}\widehat{\Psi}^t\begin{bmatrix} x \\ \xi\end{bmatrix} = \begin{bmatrix} iR-\langle \overline{x}|b\rangle I & b \\ b^* & -\langle \overline{x}|b\rangle\end{bmatrix}\begin{bmatrix} x \\ \xi\end{bmatrix} \quad \text{for any } \begin{bmatrix} x \\ \xi\end{bmatrix} \in \mathcal{D} = \begin{bmatrix} \mathbf{Z} \\ \mathbb{C}\end{bmatrix}. \tag{4.12}$$

The linear operator in $\mathcal{L}(\mathcal{H})$ with matrix $\left[\begin{smallmatrix} -\langle \overline{x}|b\rangle I & b \\ b^* & -\langle \overline{x}|b\rangle\end{smallmatrix}\right]$ is bounded. Since $\mathcal{A} = \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t$ with domain $\mathcal{D}$ is the generator of a strongly continuous semigroup in $\mathcal{L}(\mathcal{H})$, by the theorem of bounded perturbations [4], also the operator with matrix $\begin{bmatrix} iR & 0 \\ 0 & 0 \end{bmatrix}$ with domain $\mathcal{D}$ is the generator of a strongly continuous one-parameter subgroup of $\mathcal{L}(\mathcal{H})$ entailing that $iR$ is the generator of a strongly continuous group $\big[V_t : t \in \mathbb{R}\big]$ in $\mathcal{L}(\mathbf{H})$. Since $U_{-t} = U_t^{-1} = U_t^*$, the arguments on sun adjoint semigroups in [4, p. 69] show that $\lim_{t\to 0} t^{-1}(U_t^* - I) = -iR$ is the generator of the sun adjoint group $\big[V_t^* : t \in \mathbb{R}\big] = \big[V_{-t} : t \in \mathbb{R}\big]$ and we have $-iR = (iR)^*$ which completes the proof.

**Theorem 4.13.** *Any vector field of the form* (4.5) *where $R$ is a not necessarily bounded self-adjoint operator with dense domain $\mathbf{Z} \subset \mathbf{H}$, is the infinitesimal generator defined on $\mathbf{D} := \mathbf{Z} \cap \mathbf{B}$ of a pointwise continuous one-parameter group $[\Phi^t : \ t \in \mathbb{R}]$ of holomorphic automorphisms of $\mathbf{B}$.*[3]

**Proof.** It suffices to see that there is a strongly (i.e. pointwise) continuous one-parameter group $\big[\mathcal{V}^t : \ t \in \mathbb{R}\big]$ of bounded linear operators of the space $\mathcal{H}$ such that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{V}^t \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \quad (x \in \mathbf{Z},\ \xi \in \mathbb{C}), \qquad \mathcal{V}^t \mathcal{K} \subset \mathcal{K} := \Big\{ \begin{bmatrix} x \\ \xi \end{bmatrix} : \ \|x\|^2 \geq |\xi|^2 \Big\}.$$

Namely, in this case the maps

$$\Phi^t(x) := \Big[\mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbb{C}}^{-1} \Big[\mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbf{H}} \qquad (t \in \mathbb{R},\ x \in \mathbf{D})$$

suit the requirements of the theorem since $x \in \mathbf{D} \Rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathcal{K} \Rightarrow \mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix} \Rightarrow \Phi^t(x) \in \mathbf{D}$ and $x \in \mathbf{D} \Rightarrow \frac{d}{dt}\big|_{t=0} \Phi^t(x) = -\big[\mathcal{V}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\big]_{\mathbb{C}}^{-2} \frac{d}{dt}\big|_{t=0} \big[\mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\big]_{\mathbb{C}} \big[\mathcal{V}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\big]_{\mathbf{H}} + \big\{\mathcal{V}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\big\}_{\mathbb{C}}^{-1} \frac{d}{dt}\big|_{t=0} \big[\mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\big]_{\mathbf{H}}$ $= -\big[\big[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\big] \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]\big]_{\mathbb{C}} x + \big[\big[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\big] \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big]\big]_{\mathbf{H}} = -\langle x|b\rangle x + iRx + b = \Omega(x)$. Notice that, by Corollary 2.12, a strongly continuous one parameter group of linear operator leaves the cone $\mathcal{K}$ invariant if all its members map the boundary $\partial\mathcal{K} = \big\{\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] : \|x\| = |\xi|\big\} = \big\{\big[\begin{smallmatrix} x \\ e^{i\tau}\|x\| \end{smallmatrix}\big] : x \in \mathbf{H},\ \tau \in \mathbb{R}\big\}$ into itself. Therefore it suffices to check that there is a (necessarily unique) strongly continuous one-parameter group in $\mathcal{L}(\mathcal{H})$ with domain $\mathbf{Z} \oplus \mathbb{C} = \big[\begin{smallmatrix} \mathbf{Z} \\ \mathbb{C} \end{smallmatrix}\big]$ such that

$$\frac{d}{dt} \mathcal{V}^t \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix} \mathcal{V}^t \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \Big\| \Big[\mathcal{V}^t \begin{bmatrix} x \\ \|x\| \end{bmatrix}\Big]_{\mathbf{H}} \Big\| = \Big| \Big[\mathcal{V}^t \begin{bmatrix} x \\ \|x\| \end{bmatrix}\Big]_{\mathbb{C}} \Big| \qquad (x \in \mathbf{Z},\ t \in \mathbb{R}).$$

By Stone's theorem, the $\mathcal{H}$-unitary operators $\mathcal{W}^t \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] := \big[\begin{smallmatrix} \exp(itR)x \\ \xi \end{smallmatrix}\big]$ form a strongly continuous one-parameter group whose infinitesimal generator is defined on $\mathrm{dom}(R) \oplus \mathbb{C} = \mathbf{Z} \oplus \mathbb{C}$ with the diagonal matrix $\big[\begin{smallmatrix} iR & 0 \\ 0 & 0 \end{smallmatrix}\big]$. Since the matrix $\big[\begin{smallmatrix} 0 & b \\ b^* & 0 \end{smallmatrix}\big]$ represents a bounded linear operator in $\mathcal{H}$, by the theorem of bounded perturbations [4], there is a strongly continuous one-parameter group $\big[\mathcal{V}^t : \ t \in \mathbb{R}\big]$ whose generator is defined on $\mathbf{Z} \oplus \mathbb{C}$ with the matrix $\big[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\big]$. In particular $\frac{d}{dt} \mathcal{V}^t \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] = \big[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\big] \mathcal{V}^t \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big]$ $(t \in \mathbb{R},\ x \in \mathbf{Z})$. To complete the proof, we show that necessarily

$$\frac{d}{dt} \Big[ \Big\| \Big[\mathcal{V}^t \begin{bmatrix} x \\ \|x\| \end{bmatrix}\Big]_{\mathbf{H}} \Big\|^2 - \Big| \Big[\mathcal{V}^t \begin{bmatrix} x \\ \|x\| \end{bmatrix}\Big]_{\mathbb{C}} \Big|^2 \Big] = 0 \qquad (t \in \mathbb{R},\ x \in \mathbf{Z}).$$

[3] That is, for all $x \in \mathbf{D} := \mathbf{Z} \cap \mathbf{B}$, the functions $t \mapsto \Phi^t(x)$ range in $\mathbf{Z}$, they are differentiable and satisfy the identity $\frac{d}{dt} \Phi^t(x) = \mathcal{V}\big(\Phi^t(x)\big)$ $(t \in \mathbb{R})$.

Consider any vector $x \in \mathbf{Z}$ and write $\left[\begin{smallmatrix} x_t \\ \xi_t \end{smallmatrix}\right] := \mathcal{V}^t \left[\begin{smallmatrix} x \\ \|x\| \end{smallmatrix}\right]$ for all parameters $t \in \mathbb{R}$. Then

$$\begin{aligned} \frac{d}{dt}\Big[\|x_t\|^2 - |\xi_t|^2\Big] &= 2\,\mathrm{Re}\Big[\big\langle dx_t/dt \big| x_t\big\rangle - \big(d\xi_t/dt\big)\overline{\xi_t}\Big] = \\ &= 2\,\mathrm{Re}\left\{\Big\langle \Big[\begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix}\begin{bmatrix} x_t \\ \xi_t \end{bmatrix}\Big]_{\mathbf{H}} \Big| x_t \Big\rangle - \Big[\begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix}\begin{bmatrix} x_t \\ \xi_t \end{bmatrix}\Big]_{\mathbb{C}} \big(d\xi_t/dt\big)\overline{\xi_t}\right\} = \\ &= 2\,\mathrm{Re}\left[\big\langle iRx_t + \xi_t b \big| x_t\big\rangle - \big\langle x_t \big| b\big\rangle \overline{\xi_t}\right] = \\ &= -2\,\mathrm{Im}\big\langle Rx_t \big| x_t\big\rangle + 2\,\mathrm{Im}\big(\big\langle \xi b \big| x_t\big\rangle - \big\langle x_t \big| \xi b\big\rangle\big) = 0. \end{aligned}$$

## 5. Proof of the main results

**Proof of Theorem 1.1.** Consider any pointwise continuous one-parameter subgroup $\big[\Psi^t : t \in \mathbb{R}\big]$ of $\mathrm{Aut}\big(\mathbf{B}\big)$ and let us fix any common fixed point $\overline{x} \in \overline{\mathbf{B}}$ of the continuous extensions of the maps $\Psi^t$ $(t \in \mathbb{R})$ (guaranteed by Lemma 2.4). From Proposition 3.7, we know already that (1.2-3) hold for some dense linear complex submanifold $\mathbf{Z}$ of the underlying Hilbert space with a symmetric linear operator $A$ with $\mathrm{dom}(A) = \mathbf{Z}$. We have to see that $A$ is even self-adjoint in any case and, conversely every self-adjoint operator with domain $\mathbf{Z}$ may appear in (1.3) with any constant $\lambda \in \mathbb{R}$ if $\overline{x} \in \partial\mathbf{B}$ or $\lambda = 0$ if $\overline{x} \in \mathbf{B}$. In order to establish a link to the Jordan case, fix any point $c \in \mathbf{D} = \mathrm{dom}\big(\frac{d\Psi^t}{dt}\big|_{t=0}\big)$ and let

$$\Phi^t := \Theta_{-c}\Psi^t\Theta_c \quad (t \in \mathbb{R}), \quad \overline{y} := \Theta_c(\overline{x})$$

by means of the Möbius transformations (2.3). Clearly $\big[\overline{\Phi}^t : t \in \mathbb{R}\big]$ is a strongly continuous one-parameter subgroup of $\mathrm{Aut}\big(\mathbf{B}\big)$ such that

$$\text{(5.1)} \qquad 0 = \Theta_{-c}(c) \in \Theta_{-c}\left(\mathrm{dom}\Big(\frac{\partial\Psi^t}{dt}\Big|_{t=0}\Big)\right) = \mathrm{dom}\Big(\frac{\partial\Phi^t}{dt}\Big|_{t=0}\Big), \quad \overline{y} \in \bigcap_{t\in\mathbb{R}} \mathrm{Fix}\big(\overline{\Phi}^t\big).$$

Thus we can apply the results of Section 4 in particular Lemma 4.11 to $[\Phi^t : t \in \mathbb{R}]$ to conclude that there is a dense complex linear submanifold $\mathbf{Y} \subset \mathbf{H}$ along with a self-adjoint operator $R$ with $\mathrm{dom}(R) = \mathbf{Y}$ and a vector $b \in \mathbf{H}$ such that

$$\widehat{\Phi}^t_{\overline{y}} = e^{-\langle \overline{x}|b\rangle t} \exp\left(t\begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix}\right) \qquad (t \in \mathbb{R}).$$

Hence Corollary 2.11 establishes the existence of a constant $\nu \in \mathbb{C}$ with

$$\widehat{\Psi}^t_{\overline{x}} = e^{\nu t}\beta^{-2}\widetilde{\Theta_c}\widehat{\Phi}^t_{\overline{y}}\widetilde{\Theta_{-c}} \qquad (t \in \mathbb{R})$$

due to the identity $\widetilde{\Theta_c}\widetilde{\Theta_{-c}} = \beta^2\mathcal{I} = (1 - \|c\|^2)\left[\begin{smallmatrix} I & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for the canonical representations

$$\widetilde{\Theta_{\pm c}} = \begin{bmatrix} Q & \pm c \\ \pm c^* & 1 \end{bmatrix} \quad \text{where} \;\; Q := \beta I + (1-\beta)P, \;\; \beta := \sqrt{1 - \|c\|^2}, \;\; P := \mathrm{P}_c.$$

By passing to infinitesimal generators, with $\mu := \nu - \langle \overline{x}|b\rangle$, we get

$$\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\begin{bmatrix} x\\ \xi\end{bmatrix} = \mu\begin{bmatrix} x\\ \xi\end{bmatrix} + \beta^{-2}\widetilde{\Theta_c}\begin{bmatrix} iR & b\\ b^* & 0\end{bmatrix}\widetilde{\Theta_{-c}}\begin{bmatrix} x\\ \xi\end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x\\ \xi\end{bmatrix} \in \mathrm{dom}\Big(\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\Big). \tag{5.2}$$

From Lemma 3.3 and (5.1) we see that

$$\begin{bmatrix} \mathbf{Z}\\ 0\end{bmatrix} + \mathbb{C}\begin{bmatrix} \overline{x}\\ \xi\end{bmatrix} = \mathrm{dom}\Big(\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\Big) = \widetilde{\Theta_c}\mathrm{dom}\Big(\frac{d\widehat{\Phi}^t_{\overline{y}}}{dt}\Big|_{t=0}\Big) = \widetilde{\Theta_c}\begin{bmatrix} \mathbf{Y}\\ \mathbb{C}\end{bmatrix}.$$

Thus, given any vector $z \in \mathbf{H}$, we have $z \in \mathbf{Z}$ if and only if $\left[\begin{smallmatrix} z\\ 0\end{smallmatrix}\right] = \widetilde{\Theta_c}\left[\begin{smallmatrix} y\\ \eta\end{smallmatrix}\right]$ for some $y \in \mathbf{Y}$ and $\eta \in \mathbb{C}$ that is if $z = Qy - \langle y|c\rangle c = \big[Q - \|c\|^2 P\big]y$ for some $y \in \mathbf{Y}$. It follows

$$\mathbf{Z} = \big[Q - \|c\|^2 P\big]\mathbf{Y} = Q^{-1}\mathbf{Y} \tag{5.3}$$

because the operators $P, Q$ commute, we have $\beta^2 = 1 - \|c\|^2 > 0$ and $Q\big[Q - \|c\|^2 P\big] = \big[\beta I + (1-\beta)P\big]\big[\beta I + (\beta^2 - \beta)P\big] = \beta^2 I$. We are now ready to establish the self-adjointness of the operator $A$ in (1.3). By (5.2) we have

$$(iA + \lambda I)z = \left[\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\begin{bmatrix} z\\ 0\end{bmatrix}\right]_{\mathbf{H}} = \mu z + \beta\big[iQRQz - Qbc^*z + cb^*Qz\big] \quad (z \in \mathbf{Z}).$$

That is, with the bounded self-adjoint operator $S := i\beta[Qbc^* - cb^*Qz] = [iQbc^*] + [iQbc^*]^*$ we have $A = \beta QRQ + S + i(\lambda - \mu)I$. We know the symmetry of $A$ already entailing the relations $\mu = \lambda$ with $A = \beta QRQ + S$. Here the operator $QRQ$ self-adjoint with $\mathrm{dom}(QRQ) = Q^{-1}\mathrm{dom}(R) = Q^{-1}\mathbf{Y} = \mathbf{Z} = \mathrm{dom}(A)$ since $R$ is a self-adjoint operator with $\mathrm{dom}(R) = \mathbf{Y}$ while $Q$ is an invertible bounded self-adjoint operator, Therefore, as being the bounded self-adjoint perturbation, the operator $A$ is necessarily self-adjoint.
To see the converse, we need only to check the reversibility of some of our previous arguments. Assuming $A$ to be self-adjoint in (1.2-3), it is the theorem of bounded perturbations [4] ensures that the operator $\mathcal{A}\left[\begin{smallmatrix} z+\xi\overline{x}\\ \xi\end{smallmatrix}\right] := \left[\begin{smallmatrix} (iA+\lambda I)z\\ \langle (iA-\lambda I)z|\overline{x}\rangle(z+\xi\overline{x})\end{smallmatrix}\right]$ $(z \in \mathbf{Z},\ \xi \in \mathbb{C})$ is the infinitesimal generator of a strongly continuous one-parameter subgroup $[\mathcal{U}^t : t \in \mathbb{R}]$ of $\mathcal{L}(\mathbf{H})$ with graph being tangent to the boundary of cone $\mathcal{K}$ in Corollary 2.12(iii) and we have $\mathcal{K} = \mathcal{U}^t\mathcal{K}$ $(t \in \mathbb{R})$. Hence the holomorphic maps $\Psi^t(x) := \big[\mathcal{U}^t\left[\begin{smallmatrix} x\\ 1\end{smallmatrix}\right]\big]_{\mathbb{C}}^{-1}\big[\mathcal{U}^t\left[\begin{smallmatrix} x\\ 1\end{smallmatrix}\right]\big]_{\mathbf{H}}$ are well-defined on the unit ball $\mathbf{B}$ leaving it invariant, and form a strongly continuous one-parameter subgroup in $\mathrm{Aut}(\mathbf{B})$.

**Proof of Corollary 1.4.**
By assumption $A\overline{x} = \rho\overline{x}$ in particular $\overline{x} \in \mathrm{dom}(A)$. Thus we are in the Jordan case $0 \in \mathrm{dom}(\Omega)$ and hence

$$\begin{aligned}\Omega(x) &= \big\langle x - \overline{x}\big|(iA + \lambda I)\overline{x}\big\rangle x + \big(iA + \lambda I\big)(x - \overline{x} = \\ &= b - \langle x|b\rangle x + iR \qquad \text{with} \ \ b = \Omega(0) = (-\lambda - i\rho)\overline{x}, \ \ iR = \Omega'(0) = iA + i\rho I\ .\end{aligned}$$

Recall that we have $\Psi^t(x) = \big[\widehat{\Psi}^t\big(\begin{smallmatrix}x\\1\end{smallmatrix}\big)\big]_{\mathbb{C}}^{-1}\big[\widehat{\Psi}^t\big(\begin{smallmatrix}x\\1\end{smallmatrix}\big)\big]_{\mathbf{H}}$ $x \in \mathbf{H})$ for the strongly continuous one-parameter group $[\widehat{\Psi}^t : t \in \mathbb{R}]$ which, according to (4.12), has the infinitesimal generator

$$\frac{d}{dt}\Big|_{t=0}\widehat{\Psi}^t = \begin{bmatrix} iR - \langle \overline{x}|b\rangle & b \\ b^* & (-\langle \overline{x}|b\rangle \end{bmatrix} = \begin{bmatrix} iA + \lambda I & (-\lambda - i\rho)\overline{x} \\ (-\lambda + i\rho)\overline{x}^* & \lambda - i\rho \end{bmatrix}.$$

Since the operator $A$ is self-adjoint with eigenvector $\overline{x}$, it is reduced by the subspaces $\mathbb{C}\overline{x}$ and $\mathbf{H}_0 := \overline{x}^\perp (= \{x \in \mathbf{H} : x \perp \overline{x}\})$. Thus in terms of $\mathbf{H}_0 \oplus \mathbb{C}\overline{x} \oplus \mathbb{C}$-matrices,

$$\frac{d}{dt}\Big|_{t=0}\widehat{\Psi}^t = \begin{bmatrix} (iA + \lambda I)\big|\mathbf{H}_0 & 0 & 0 \\ 0 & \lambda + i\rho & -\lambda - i\rho \\ 0 & -\lambda + i\rho & \lambda - i\rho \end{bmatrix}$$

entailing that (a) in case $0 \neq \lambda \in \mathbb{R}$ we have

$$\widehat{\Psi}^t = \begin{bmatrix} e^{i\lambda t}\exp(itA)\big|\mathbf{H}_0 & 0 \\ 0 & \exp t\left[\begin{smallmatrix} \lambda+i\rho & -\lambda-i\rho \\ -\lambda+i\rho & \lambda-i\rho \end{smallmatrix}\right] \end{bmatrix} =$$

$$= \begin{bmatrix} e^{i\lambda t}\exp(itA)\big|\mathbf{H}_0 & 0 & 0 \\ 0 & (\lambda + i\rho)e^{2\lambda t} + (\lambda + i\rho) & (-\lambda - i\rho)e^{2\lambda t} + (\lambda - i\rho) \\ 0 & (-\lambda + i\rho)e^{2\lambda t} + (\lambda + i\rho) & (\lambda - i\rho)e^{2\lambda t} + (\lambda - i\rho) \end{bmatrix},$$

(b) in case of $\lambda = 0$ we have

$$\widehat{\Psi}^t = \begin{bmatrix} e^{i\lambda t}\exp(itA)\big|\mathbf{H}_0 & 0 \\ 0 & \exp it\rho\left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} e^{i\lambda t}\exp(itA)\big|\mathbf{H}_0 & 0 & 0 \\ 0 & 1 + it\rho & -it\rho \\ 0 & it\rho & 1 - it\rho \end{bmatrix}.$$

Hence the statement is straightforward from the relation

$$\Psi^t(x_0 + \xi\overline{x}) = \big[\widehat{\Psi}^t[x_0, \xi, 1]^{\mathrm{T}}\big]_3^{-1}\Big(\big[\widehat{\Psi}^t[x_0, \xi, 1]^{\mathrm{T}}\big]_2\overline{x} + \big[\widehat{\Psi}^t(x_0, \xi, 1]^{\mathrm{T}}\big]_1\Big).$$

**Lemma 5.4.** *The group* $\mathrm{Aut}(\overline{\mathbf{B}})$ *is 2-transitive on* $\partial\mathbf{B}$ *i.e. given any tuple* $(e_1, f_1, e_2, f_2)$ *of unit vectors with* $e_1 \neq e_2$, $f_1 \neq f_2$, *there is a transformation* $\Theta \in \mathrm{Aut}(\mathbf{B})$*such that* $\overline{\Theta}(e_k) = f_k$ $(k = 1, 2)$.

**Proof.** Let $e_1, f_1, e_2, f_2 \in \partial\mathbf{B}$ with $e_1 \neq e_2$, $f_1 \neq f_2$. Let $c$ be the center of the circlar arc $C$ in the 1-dimensional complex disc $D := \mathbf{B} \cap \{(1 - \zeta)e_1 + \zeta e_2 : \zeta \in \mathbb{C}\}$ connecting the points $e_1, e_2$ and being orthogonal in real sense to the boundary circle of $D$. Since holomorphic fractional linear transformations preserve affine lines, the Möbius shift $\Theta_c^{-1} = \Theta_{-c}(x) = (1-\langle x|c\rangle)^{-1}(Q_c x - c)$ maps $D$ onto an affine disc passing through $0 = \Theta_{-c}(c)$ and containg the points $g_k := \overline{\Theta_{-c}}(e_k)$ $(k = 1, 2)$ as boundary points. Since the continuous extension $\overline{\Theta_{-c}}$ preserves the unit sphere $\partial\mathbf{B}$, The image $\Theta_{-c}(D)$ is a disc of the form $\Theta_{-c}(D) = \{\zeta g_k : |\zeta| < 1\}$ $(k = 1, 2)$. Since holomorphic maps are conformal, the image

$\Theta^{-1}(C)$ is an arc passing through the origin and being orhogonal to the boundary circle which is possible only if it is a diameter of the form $\{\tau g : -1 < \tau < 1\}$ with $\|g\| = 1$ that is we have $\overline{\Theta_{-c}}(e_k) = g_k = (-1)^k g$ $(k = 1, 2)$ for some unit vector $g$. Similarly, $\overline{\Theta_{-d}}(f_k) = (-1)^k h$ $(k = 1, 2)$ for some unit vector $h$ and a Möbius shift $\Theta_{-d}$. It is a well-known elementary fact that any the unitary group is transitive on $\partial\mathbf{B}$, actually e.g. we have $Ug = h$ with a twisted reflection $U := \kappa[I - 2\mathrm{P}_{h-\kappa g}]$ where the constant $\kappa \in \mathbb{C}$ is of modulus 1 such that $\zeta g + h \perp \zeta g - h$ i.e. $\kappa\langle g|h\rangle \in \mathbb{R}$. Therefore the transformation $\Theta := \Theta_d \circ U \circ \Theta_{-c}$ suits the requirements of the lemma.

**Proof of Corollary 1.5.**
Due to the 2-transitivity on the boundary, there exists $\Theta \in \mathrm{Aut}(\mathbf{B})$ such that $\Theta(\overline{x}) = \overline{x}$, $\Theta(\overline{x}') = -\overline{x}$. Thus, by passing to the group $[\Theta \circ \Psi^t \circ \Theta^{-1} : t \in \mathbb{R}]$ instead of $[\Psi^t : t \in \mathbb{R}]$, we may assume without loss of generality that $\{\pm\overline{x}\} = \mathrm{Fix}\big(\overline{\Phi}^{t_0}\big)$ with the effect

$$-\big\langle (iA - \lambda I)(x - \overline{x})\big|\overline{x}\big\rangle x + (iA + \lambda I)(x - \overline{x}) = 0 \quad \text{for} \quad x = \pm\overline{x}.$$

Regarding the case $x = -\overline{x}$, since $\langle\overline{x}|\overline{x}\rangle = 1$, it follows $\big\langle A\overline{x}\big|\overline{x}\big\rangle\overline{x} + A\overline{x} = 0$. However, since the operator $A$ is self-adjoint, necessarily also $A\overline{x} = \big\langle A\overline{x}\big|\overline{x}\big\rangle\overline{x}$ implying $A\overline{x} = 0$. Hence the statement is immediate from Corollary 1.4.

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