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Semigroups of Holomorphic Isometries

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Let B be the open unit ball of a complex Hilbert space $\mathscr{H}$. Let $\mathscr{H} \oplus \mathbb{C}$ be the Hilbert space direct sum of $\mathscr{H}$ and $\mathbb{C}$, with inner product (,), and let $\mathfrak{a}$ be the continuous hermitian sesquilinear form defined by $\mathfrak{a}(p, q) = (Jp, q)$, where $p, q \in \mathscr{H} \oplus \mathbb{C}$, J is the operator $J = I \oplus (-1)$ and I is the identity on $\mathscr{H}$.

The group Aut B of all holomorphic automorphisms of B has a faithful representation as a quotient of the group G_0 of all continuous invertible linear operators in $\mathscr{H} \oplus \mathbb{C}$ leaving the sesquilinear form $\mathfrak{a}$ invariant; i.e., there is a homomorphism ϕ_0 of G_0 onto Aut B whose kernel is the center of G_0 (cf. [2, Chap VI] also for bibliographical references).

If $\mathscr{H}$ has infinite dimension, ϕ_0 extends naturally to a homomorphism ϕ of the semigroup G of all continuous linear operators in $\mathscr{H} \oplus \mathbb{C}$ leaving the form $\mathfrak{a}$ invariant onto the semigroup Iso B of all holomorphic maps $B \to B$ which are isometries for the hyperbolic differential metric of B [2].

The homomorphism ϕ_0 and the composition rule in G_0 define in Aut B a Lie group structure whose underlying topology—in accordance with a general result of Vigué [10]—is that of local uniform convergence in B.

The continuous one-parameter groups in the Lie group Aut B correspond (Theorem III) to one-parameter linear uniformly continuous groups in G_0, i.e., to homomorphisms $\mathbb{R} \to G_0$ which are continuous for the norm-topology in the Banach space $\mathscr{L}(\mathscr{H} \oplus \mathbb{C})$ of all bounded linear operators in $\mathscr{H} \oplus \mathbb{C}$.

In Sections 2 and 5 the strongly continuous linear semigroups $\mathbb{R}_+ \to G$ are characterized in terms of their infinitesimal generators. The image by ϕ of such a semigroup T defines a semigroup $\tilde{T}: \mathbb{R}_+ \to \text{Iso } B$ of holomorphic isometries of B. Some results on fixed points of holomorphic isometries of B (established by Hayden and Suffridge in [3] for Aut B and extended to Iso B in [2]) are instrumental in describing in Sections 6–8 the structure of the spectrum of the infinitesimal generator X of T. These results yield a characterization of the case in which T is the restriction to $\mathbb{R}_+$ of a strongly continuous grop $\mathbb{R} \to G_0$, thus providing an extension to the Minkowski form $\mathfrak{a}$ of the classical theorem of M. H. Stone on one-parameter unitary groups in a complex Hilbert space.

If $n = \dim_{\mathbb{C}} \mathscr{H} < \infty$, the group structure of $U(n, 1)$ is not the underlying real structure of a complex Lie group. This classical result holds in general for G_0 and G in the infinite-dimensional case, as follows from properties of holomorphic families of bounded linear operators in $\mathscr{H} \oplus \mathbb{C}$. According to these results—which are established in Section 4— no non-trivial strongly continuous semigroup in $\mathscr{H} \oplus \mathbb{C}$ leaving the form $\mathfrak{a}$ invariant can be extended holomorphically to an open neighborhood of the positive real axis. Similar questions for families of holomorphic isometries in B have been investigated in [9].

The Cauchy problem associated with the infinitesimal generator X of the semigroup $T: \mathbb{R}_+ \to G$ defines, via the homomorphism ϕ, a Riccati-type equation in B. Uniqueness of the solution provided by the semigroup $\tilde{T}$ is discussed in Section 9.

1

If D and D' are open sets in complex Banach spaces, Hol(D, D') will be the set of all holomorphic maps of D into D'; Aut D will be the group—contained in the semigroup Hol(D, D)—of all biholomorphic automorphisms of D.

Let $\mathscr{H}$ be a complex Hilbert space with inner product (|) and norm $\| \ \|$, and let B be the open unit ball of $\mathscr{H}$.

For any $x \in B$, let $\{ \ , \ \}_x$ be the continuous positive-definite inner product on $\mathscr{H}$ defined for v_1, v_2 in $\mathscr{H}$ by

$$\{v_1, v_2\}_x = \frac{1}{(1 - \|x\|^2)^2} ((v_1 \mid x)(x \mid v_2) + (1 - \|x\|^2)(v_1 \mid v_2)). \tag{1.1}$$

The corresponding norm $| \ |_x$ is equivalent to $\| \ \|$ and the map $x \mapsto | \ |_x$ is a differential metric which is contracted by all holomorphic maps of B into B, in the sense that, for every $f \in \text{Hol}(B, B)$ and all $x \in B$, $v \in \mathscr{H}$,

$$|df(x)\, v|_{f(x)} \leqslant |v|_x.$$

In particular, if $f \in \text{Aut } B$ then

$$|df(x)\, v|_{f(x)} = |v|_x$$

for all $x \in B$, $v \in \mathscr{H}$.

The differential metric $x \mapsto | \ |_x$ coincides with the Carathéodory and the Kobayashi metrics on B [2, pp. 153–154] and is called the *hyperbolic metric* on B. In fact, (1.1) shows that if $\mathscr{H} = \mathbb{C}$ and if $B = \Delta$ the open unit disc in $\mathbb{C}$, then $x \mapsto | \ |_x$ is the Poincaré metric on Δ.

Let Iso B be the semigroup of all holomorphic isometries for the hyperbolic metric,

$$\mathrm{Iso}\, B = \{f \in \mathrm{Hol}(B, B) : |df(x)\, v|_{f(x)} = |v|_x \text{ for all } x \in B, \quad v \in \mathcal{H}\}.$$

A faithful representation of Iso B will now be described.

Let $\mathcal{H} \oplus \mathbb{C}$ be the Hilbert space direct sum of $\mathcal{H}$ and $\mathbb{C}$, with inner product $(p_1, p_2) = (x_1 \mid x_2) + \tau_1 \bar{\tau}_2$, where $p_j = (x_j, \tau_j)$, $x_j \in \mathcal{H}$, $\tau_j \in \mathbb{C}$, $j = 1, 2$.

Let J be the self-adjoint unitary operator on $\mathcal{H} \oplus \mathbb{C}$ defined by the matrix

$$J = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

where $I = I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. Let $\mathfrak{a}$ be the continuous hermitian sesquilinear form defined by

$$\mathfrak{a}(p_1, p_2) = (Jp_1, p_2). \tag{1.3}$$

Let G be the semigroup of all linear maps $S: \mathcal{H} \oplus \mathbb{C} \to \mathcal{H} \oplus \mathbb{C}$ which leave $\mathfrak{a}$ invariant,

$$\mathfrak{a}(Sp_1, Sp_2) = \mathfrak{a}(p_1, p_2) \qquad \text{for all } p_1, p_2 \text{ in } \mathcal{H} \oplus \mathbb{C}.$$

It turns out that every $S \in G$ is continuous [2, Theorem VI.3.3, p. 169]. Hence, denoting by S^* the adjoint operator of S, $S \in G$ if, and only if,

$$S^* J S = J.$$

Equivalently, a continuous linear operator S on $\mathcal{H} \oplus \mathbb{C}$ is contained in G if, and only if, S has a matrix representation

$$S = \begin{pmatrix} A & \xi \\ \left(\cdot \,\middle|\, \frac{1}{a} A^* \xi\right) & a \end{pmatrix}, \tag{1.4}$$

whose elements $A \in \mathcal{L}(\mathcal{H})$, $\xi \in \mathcal{H}$, $a \in \mathbb{C}$ satisfy the conditions

$$A^* A = I + \frac{1}{|a|^2} (\cdot \mid A^* \xi)\, A^* \xi, \tag{1.5}$$

$$|a|^2 - \|\xi\|^2 = 1 \tag{1.6}$$

[2, Lemma VI.3.1, p. 166].

Condition (1.5) reads $\|Ax\|^2 = \|x\|^2 + (1/|a|^2)\, |(Ax \mid \xi)|^2$ for all $x \in \mathcal{H}$, so that, in view of (1.6) and of the Schwarz inequality,

$$(1 + \|\xi\|^2)(\|Ax\|^2 - \|x\|^2) = |(Ax \mid \xi)|^2 \leqslant \|Ax\|^2\, \|\xi\|^2,$$

whence

$$\|A\| \leqslant (1 + \|\xi\|^2)^{1/2},$$

or, equivalently in view of (1.6),

$$\left\| \frac{1}{a} A \right\| \leqslant 1. \tag{1.7}$$

Let G_0 be the group of all invertible elements of G. Then $S \in G_0$ if, and only if, $S \in G$ is bijective, or, equivalently, if and only if, A maps $\mathcal{H}$ bijectively onto $\mathcal{H}$ [2, Theorem VI.3.3, p. 169].

Conditions (1.5) and (1.6) imply [2, pp. 176–177] that, if $S \in G$, there exists a neighbourhood U of the closure $\bar{B}$ of B such that

$$\left(x \,\middle|\, \frac{1}{a} A^* \xi\right) + a \neq 0 \qquad \text{for all } x \in U.$$

Let $\tilde{S} \in \mathrm{Hol}(B, \mathcal{H})$ be defined by

$$\tilde{S}(x) = \frac{1}{(x \mid (1/a)\, A^* \xi) + a} (Ax + \xi) \qquad (x \in B).$$

Conditions (1.5) and (1.6) imply [2, pp. 171–172] that $S(B) \subset B$, and the following theorem holds [2, Theorem VI, 4.1, pp. 174–175].

THEOREM. *The function* $S \mapsto \tilde{S}$ *is a surjective homomorphism of* G *onto* Iso B, *mapping* G_0 *onto* Aut B, *whose kernel is the center of* G.

The group Aut B (sometimes called the *Möbius group* of B) acts transitively on B (cf., e.g., [2, Proposition VI.1.5, pp. 148–149]).

Let $\hat{S}$ be the continuous extension of $\tilde{S}$ to $\bar{B}$, for $S \in G$. Then $\hat{S}$ is continuous for the weak topology on $\bar{B}$ [2, Theorem VI.4.5, p. 178]. Thus by the Banach–Alaoglu and the Schauder–Tychonoff theorems $\hat{S}$ has at least one fixed point in $\bar{B}$, i.e.,

$$\mathrm{Fix}\, \hat{S} = \{z \in \bar{B} : \hat{S} z = z\} \neq \varnothing. \tag{1.8}$$

Let $\mathrm{Fix}\, \tilde{S} = \{z \in B : \tilde{S} z = z\}$. If $z \in \bar{B}$ is a fixed point of $\hat{S}$, the point $p = (z, 1)$ is an eigenvector of S with eigenvalue $\mu \neq 0$. Note that $(Jp, p) = \|z\|^2 - 1 \leqslant 0$, and $(Jp, p) < 0$ if, and only if, $z \in B$, i.e., $z \in \mathrm{Fix}\, \tilde{S}$.

Vice versa, if $p=(x,\tau)\neq 0$ is an eigenvector of S with eigenvalue $\mu\neq 0$, and if $(Jp,p)\leqslant 0$, then $z=(1/\tau)x\in \operatorname{Fix}\hat{S}$, and $z\in\operatorname{Fix}\tilde{S}$ if, and only if, $(Jp,p)<0$.

Let $\operatorname{Fix}\tilde{S}\neq\varnothing$. Since $\operatorname{Aut} B$ acts transitively on B, there exists $S_0\in G_0$ such that, setting $S'=S_0\circ S\circ S_0^{-1}$, then $S'(0)=0$, i.e., S' is represented by the matrix

$$S'=\begin{pmatrix} A' & 0\\ 0 & a'\end{pmatrix},$$

where A' is a linear isometry of $\mathscr{H}$ and $a'\in\mathbb{C}$ with $|a|=1$. Since

$$\operatorname{Fix}\hat{S}'=\{x\in\bar{B}: A'x=a'x\}=\{x\in\bar{B}: S'^t(x,1)=a'^t(x,1)\},$$

then

$$\operatorname{Fix}\hat{S}=\{y\in\bar{B}: S_0{}^t(y,1)\in\operatorname{Ker}(a'I_{\mathscr{H}\oplus\mathbb{C}}-S')\}.$$

This proves

PROPOSITION 1.1. *If* $\operatorname{Fix}\tilde{S}\neq\varnothing$ *there exists an eigenvalue* μ *of* S, *with* $|\mu|=1$, *such that, denoting by* $\mathscr{F}\subset\mathscr{H}\oplus\mathbb{C}$ *the corresponding eigenspace, then* $\mathscr{F}\not\subset\mathscr{H}\oplus\{0\}$ *and*

$$\operatorname{Fix}\hat{S}=\{x\in\bar{B}: (x,1)\in\mathscr{F}\}.$$

COROLLARY 1.2. *If* $\operatorname{Fix}\tilde{S}\neq\varnothing$ *there exists a closed affine subspace* $\mathscr{G}\subset\mathscr{H}$ *such that* $\operatorname{Fix}\hat{S}=\bar{B}\cap\mathscr{G}$.

Now let $\operatorname{Fix}\tilde{S}=\varnothing$. In [3] Hayden and Suffridge have shown that, if $\tilde{S}\in\operatorname{Aut} B$, then $\operatorname{Fix}\hat{S}$ contains at most two points. Their proof was shown in [2, pp. 179–181] to hold if $S\in\operatorname{Iso} B$ also. Here is a slightly different argument yielding some supplementary information which will be useful later on.

Let $\operatorname{Fix}\tilde{S}=\varnothing$. If x and y are two distinct points of $\operatorname{Fix}\hat{S}$, then $x\in\partial B$, $y\in\partial B$. Furthermore the eigenvalues ζ and σ of S corresponding to the eigenvectors $p=(x,1)$ and $q=(y,1)$ are distinct, because otherwise every point in the (non-empty) intersection of B with the affine line joining x and y would be contained in $\operatorname{Fix}\tilde{S}$.

If $(Jp,p)=(Jq,q)=0$, then

$$(J(p+q),p+q)=2\operatorname{Re}(Jp,q).$$

Thus, if $(Jp,q)=0$, then $(J(p+q),p+q)=0$, i.e., $\|x+y\|^2=4$, or $\operatorname{Re}(x\mid y)=1$. Hence, by the Schwarz inequality, $\operatorname{Re}(x\mid y)=\|x\|\,\|y\|$, implying immediately that $x=y$. Thus $(Jp,q)\neq 0$, and equality

$$\zeta\bar{\sigma}(Jp,q)=(JSp,Sq)=(Jp,q)$$

implies

$$\zeta\bar{\sigma}=1. \tag{1.9}$$

If $z\in\partial B$ is a fixed point of $\hat{S}$, different from x, and if μ is the eigenvalue of S corresponding to the eigenvector $(z,1)$, then $\zeta\bar{\mu}=1$, and (1.9) yields $\mu=\sigma$. That proves

PROPOSITION 1.3. *If* $\operatorname{Fix}\tilde{S}=\varnothing$, *then* $\operatorname{Fix}\hat{S}$ *consists of two points of* ∂B *at most.*

Furthermore (1.9) yields

LEMMA 1.4. *If* $\operatorname{Fix}\tilde{S}=\varnothing$ *and if* $\operatorname{Fix}\hat{S}$ *contains two points, the corresponding eigenvalues* ζ *and* σ *of* S, *which are related by* (1.9), *are not contained in the unit circle.*

2

Let L be a bounded operator on a complex Hilbert space $\mathscr{K}$. Let $T: t\mapsto T(t)$ $(t\geqslant 0)$ be a linear strongly continuous (i.e., C_0) semigroup on $\mathscr{K}$. The infinitesimal generator of T is a closed linear operator X whose domain $\mathscr{D}(X)$ is dense in $\mathscr{K}$.

THEOREM I. *The semigroup* T *satisfies condition*

$$T(t)^*LT(t)=L \qquad \text{for all } t\geqslant 0 \tag{2.1}$$

if, and only if,

$$L\mathscr{D}(X)\subset\mathscr{D}(X^*) \tag{2.2}$$

and

$$X^*L+LX=0 \qquad \text{on} \quad \mathscr{D}(X). \tag{2.3}$$

Proof. If (2.1) holds, then for p, q in $\mathscr{K}$

$$(T(t)^*LT(t)p,q)=(LT(t)p,T(t)q)=(L(T(t)-I)p,(T(t)-I)q)$$
$$+(L(T(t)-I)p,q)+(Lp,(T(t)-I)q)+(Lp,q),$$

whence

$$(L(T(t)-I)p, (T(t)-I)q)+(L(T(t)-I)p, q)+(Lp, (T(t)-I)q)=0.$$

Thus, for p and q in $\mathscr{D}(X)$, since

$$\lim_{t\downarrow 0}\left(\frac{1}{t}(T(t)-I)p\right)=Xp, \qquad \lim_{t\downarrow 0}\left(\frac{1}{t}(T(t)-I)q\right)=Xq,$$

then

$$(LXp, q)+(Lp, Xq)=0.$$

Hence the linear form $q\mapsto(Xq, Lp)$ is continuous on $\mathscr{D}(X)$. Therefore $Lp\in\mathscr{D}(X^*)$ for all $p\in\mathscr{D}(X)$, and

$$((LX+X^*L)p, q)=0$$

for all $q\in\mathscr{D}(X)$. Thus $(LX+X^*L)p=0$ for all $p\in\mathscr{D}(X)$.

Vice versa, if (2.2) and (2.3) hold, since $T(t)\mathscr{D}(X)\subset\mathscr{D}(X)$ for all $t\geqslant 0$, and since

$$\frac{d}{dt}T(t)p=T(t)Xp=XT(t)p$$

for all $p\in\mathscr{D}(X)$ and all $t\geqslant 0$, then, for p and q in $\mathscr{D}(X)$ and $t\geqslant 0$,

$$\begin{aligned}\frac{d}{dt}(T(t)^*LT(t)p, q)=\frac{d}{dt}(LT(t)p, T(t)q)&=(LXT(t)p, T(t)q)\\&\quad+(LT(t)p, XT(t)q)\\&=((LX+X^*L)T(t)p, T(t)q)=0.\end{aligned}$$

Hence $(d/dt)(T(t)^*LT(t)p)=0$ for $p\in\mathscr{D}(X)$ and for all $t\geqslant 0$, i.e., $T(t)^*LT(t)p$ is independent of $t\geqslant 0$. Thus $T(t)^*LT(t)p=T(0)^*LT(0)p=Lp$ for all $t\geqslant 0$ and all $p\in\mathscr{D}(X)$. Hence $T(t)^*LT(t)=L$ for $t\geqslant 0$ on $\mathscr{D}(X)$, and therefore on $\mathscr{K}$ also. Q.E.D.

For a linear operator X, the resolvent set, the spectrum, the point spectrum and the residual spectrum of X will be denoted by $r(X)$, $\sigma(X)$, $p\sigma(X)$ and $r\sigma(X)$, respectively.

If the bounded operator L in Theorem I is self-adjoint and such that

$$L^2=I \tag{2.4}$$

$(I=I_{\mathscr{K}})$ then L is a continuous isomorphism of the Hilbert space $\mathscr{K}$. Conditions (2.2) and (2.3) amount to saying that the closed operator iLX is symmetric.

Real constants $M\geqslant 1$ and a exist such that: $\|T(t)\|\leqslant Me^{at}$ for $t\geqslant 0$; the half-plane $\{\zeta\in\mathbb{C}: \operatorname{Re}\zeta>a\}$ is contained in $r(X)$, and moreover

$$\|(\xi I-X)^{-m}\|\leqslant M(\xi-a)^{-m}$$

for all real $\xi>0$ and $m=1, 2, \ldots$.

Assume now that $L\mathscr{D}(X)=\mathscr{D}(X^*)$ and that (2.3) and (2.4) hold. By (2.3) $(LX)^*=-LX$, i.e., iLX is self-adjoint. $\sigma(X^*)$ is the image of $\sigma(X)$ by the map $\zeta\mapsto-\zeta$. On the other hand $\zeta\in r(X)$ if, and only if, $\bar\zeta\in r(X^*)$, and moreover

$$(\zeta I-X^*)^{-1}=(\bar\zeta I-X)^{-1*}.$$

Hence

$$(\bar\zeta I-X)^{-1*}=(\zeta I+LXL)^{-1}=L(\zeta I+X)^{-1}L,$$

and therefore

$$\begin{aligned}\|(\zeta I+X)^{-m}\|=\|L(\zeta I-X)^{-m*}L\|&=\|(\zeta I-X)^{-m*}\|\\&=\|(\zeta I-X)^{-m}\|\leqslant M(\zeta-a)^{-m},\end{aligned}$$

for all real $\zeta>a$ and $m=1, 2, \ldots$.

Thus, if $L\mathscr{D}(X)=\mathscr{D}(X^*)$, $-X$ generates a strongly continuous semigroup $S: t\mapsto S(t)$ $(t\geqslant 0)$ on $\mathscr{K}$, such that

$$S(t)^*LS(t)=L \qquad \text{for all } t\geqslant 0.$$

In conclusion X generates the strongly continuous group $R: \mathbb{R}\to\mathscr{L}(\mathscr{K})$ defined by $R(t)=T(t)$ for $t\geqslant 0$, $R(t)=S(-t)$ for $t\leqslant 0$. Hence

$$R(t)^*LR(t)=L \qquad \text{for all } t\in\mathbb{R}. \tag{2.5}$$

Vice versa, if X is the generator of a strongly continuous group, then there is $a>0$ such that $\{\zeta\in\mathbb{C}: |\operatorname{Re}\zeta|>a\}\subset r(X)$.

If $L\mathscr{D}(X)\subsetneqq\mathscr{D}(X^*)$ and if (2.4) and (2.3) hold, then X^* is a proper extension of the closed operator $Y=-LXL$, with domain $\mathscr{D}(Y)=J\mathscr{D}(X)$. Therefore [4, p. 56]

$$\begin{aligned}\{\zeta\in\mathbb{C}: \operatorname{Re}\zeta<-a\}\subset r(Y)\subset p\sigma(X^*)&\subset\{\zeta\in\mathbb{C}: \bar\zeta\in\sigma(X)\}\\&\subset\{\zeta\in\mathbb{C}: |\operatorname{Re}\zeta|\leqslant a\},\end{aligned}$$

and this is absurd. This proves

THEOREM II. *If L satisfies* (2.4) *and if the generator X of a strongly continuous semigroup satisfies* (2.2) *and* (2.3), *then X generates a strongly continuous one-parameter group R if, and only if, $L\mathscr{D}(X)=\mathscr{D}(X^*)$. For the group R* (2.5) *holds.*

3

Throughout the following $T: \mathbb{R}_+ \to \mathscr{L}(\mathscr{H}\oplus\mathbb{C})$ will be a strongly continuous (i.e., C_0) semigroup of bounded linear operators in the complex Hilbert space $\mathscr{K}=\mathscr{H}\oplus\mathbb{C}$ considered in Section 1, and henceforth the operator L in Theorem I will be the bounded self-adjoint operator J expressed by the matrix (1.2). Note that $J^2=I_{\mathscr{H}\oplus\mathbb{C}}$.

The semigroup T leaves the hermitian sesquilinear form $\mathfrak{a}$ invariant, i.e., it satisfies

$$T(t)^*\,JT(t)=J \qquad \text{for all } t\geqslant 0 \tag{3.1}$$

if, and only if, $T(t)$ is represented by a matrix

$$T(t)=\begin{pmatrix} A(t) & \xi(t) \\ \left(\cdot \,\middle|\, \dfrac{1}{a(t)}A(t)^*\,\xi(t)\right) & a(t)\end{pmatrix} \tag{3.2}$$

whose elements $a(t)\in\mathbb{C}$, $\xi(t)\in\mathscr{H}$, $A(t)\in\mathscr{L}(\mathscr{H})$ satisfy the conditions (similar to (1.5) and (1.6))

$$|a(t)|^2-\|\xi(t)\|^2=1 \tag{3.3}$$

$$A(t)^*\,A(t)=I+\frac{1}{|a(t)|^2}(\cdot \mid A(t)^*\,\xi(t))\,A(t)^*\,\xi(t) \qquad \text{for all} \quad t\geqslant 0. \tag{3.4}$$

If these conditions are fulfilled then the $\mathscr{H}$-valued function $\tilde{T}(t)$ defined on the open unit ball $B\subset\mathscr{H}$ by

$$\tilde{T}(t)(x)=\frac{1}{(x\mid(1/a(t))\,A(t)^*\,\xi(t))+a(t)}(A(t)\,x+\xi(t)) \tag{3.5}$$

is holomorphic on B, and in fact $T(t)\in\operatorname{Iso} B$.

The function $\tilde{T}: t\mapsto \tilde{T}(t)$ is thus a one-parameter semigroup of holomorphic isometries of B, which is continuous in the sense that, for every $x\in B$, $t\mapsto \tilde{T}(t)(x)$ is a continuous map of $\mathbb{R}_+$ into B, or equivalently $a(t)$ and $\xi(t)$ depend continuously on $t\geqslant 0$, and $t\mapsto A(t)$ is continuous for the strong operator topology. Actually $\tilde{T}(t)$ has a (unique) continuous extension $\hat{T}(t)$: $\bar{B}\to\bar{B}$ and $t\mapsto \hat{T}(t)(x)$ is continuous on $\mathbb{R}_+$ for every $x\in\bar{B}$.

By a theorem of Vigué [10, 2] Hol(B, B) and Aut B are a topological semigroup and a topological group for the topology of local uniform convergence on B.

Let the semigroup T satisfy condition (3.1). Setting

$$\eta(t)=\frac{1}{a(t)}\,\xi(t), \qquad C(t)=\frac{1}{a(t)}\,A(t),$$

the differential $d\tilde{T}(t)(0)$ of $T(t)$ at 0 is given by

$$d\tilde{T}(t)(0)\,v=C(t)\,v-(C(t)\,v\mid\eta(t))\,\eta(t) \qquad (v\in\mathscr{H}), \tag{3.6}$$

and the power series expansion of $\tilde{T}(t)$ in B is expressed by

$$\tilde{T}(t)(x)=\eta(t)+\sum_{n=0}^{+\infty}(-(C(t)\,x\mid\eta(t)))^n\,d\tilde{T}(t)(0)\,x \qquad (x\in B). \tag{3.7}$$

By (1.7)

$$\|C(t)\|\leqslant 1. \tag{3.8}$$

THEOREM III. *The C_0 semigroup T satisfying* (3.1) *is uniformly continuous if, and only if, $\tilde{T}: \mathbb{R}_+\to\operatorname{Iso} B$ is continuous for the topology of local uniform convergence on B.*

The basic ingredient in the proof of the theorem is the following

LEMMA 3.1. *If $\tilde{T}$ is continuous for the topology of local uniform convergence then*

$$\lim_{t\downarrow 0}\|A(t)-I\|=0.$$

Proof. First, it will be shown that

$$\lim_{t\downarrow 0}\|C(t)-I\|=0. \tag{3.9}$$

If this is not the case, there exist some $\varepsilon>0$ and two sequences $\{t_\nu\}$ and $\{x_\nu\}$ of positive numbers t_ν converging to 0 and of points $x_\nu\in B$, such that

$$\|C(t_\nu)\,x_\nu-x_\nu\|\geqslant\varepsilon \qquad \text{for} \quad \nu=1, 2,\ldots. \tag{3.10}$$

On the other hand, since $\tilde{T}(t_\nu)$ converges to the identity map for the topology of local uniform convergence as $\nu\to+\infty$, then [5, 1.5 Theorem]

$$\lim_{\nu\to+\infty}\|d\tilde{T}(t_\nu)(0)-I\|=0,$$

whence, by (3.6),

$$\lim_{\nu\to+\infty} \|C(t_\nu)x_\nu - (C(t_\nu)x_\nu \mid \eta(t_\nu))\,\eta(t_\nu) - x_\nu\|$$
$$\leqslant \lim_{\nu\to+\infty} \|C(t_\nu) - (C(t_\nu)\cdot \mid \eta(t_\nu))\,\eta(t_\nu) - I\| = 0.$$

Because the continuous linear form $(C(t)\cdot \mid \eta(t))$ has norm

$$\|(C(t)\cdot \mid \eta(t))\| = \|C(t)^*\,\eta(t)\| \leqslant \|C(t)^*\|\;\|\eta(t)\| = \|C(t)\|\;\|\eta(t)\| \leqslant \|\eta(t)\| \tag{3.11}$$

by (3.8), and because

$$\lim_{t\downarrow 0}\eta(t) = \lim_{t\downarrow 0}\tilde{T}(t)\,0 = 0,$$

then

$$\lim_{\nu\to+\infty} \|(C(t_\nu)x_\nu \mid \eta(t_\nu))\,\eta(t_\nu)\| \leqslant \lim_{\nu\to+\infty}\|\eta(t_\nu)\|^2 = 0.$$

Hence

$$\lim_{\nu\to+\infty}\|C(t_\nu)x_\nu - x_\nu\|$$
$$\leqslant \lim_{\nu\to+\infty} \|C(t_\nu)x_\nu - (C(t_\nu)x_\nu \mid \eta(t_\nu))\,\eta(t_\nu) - x_\nu\| + \lim_{\nu\to+\infty}\|(C(t_\nu)x_\nu \mid \eta(t_\nu))\,\eta(t_\nu)\| = 0,$$

contradicting (3.10) and thereby proving (3.9).

Because $\lim_{t\downarrow 0} a(t) = 1$, and

$$\|A(t) - I\| \leqslant |a(t)|\;\|C(t) - I\| + |a(t) - 1|,$$

(3.9) yields the conclusion. Q.E.D.

Proof of Theorem III. (a) Let $\tilde{T}$ be continuous for the topology of local uniform convergence on B.

For $p = (x, \tau)$ $(x \in \mathscr{H}, \tau \in \mathbb{C})$,

$$T(t)\,p - p = \left((A(t) - I)\,x + \tau\xi(t), \left(A(t)\,x \,\middle|\, \frac{1}{a(t)}\,\xi(t)\right) + (a(t) - 1)\,\tau\right), \tag{3.12}$$

whence

$$\begin{aligned}\|(T(t) - I_{\mathscr{H}\oplus\mathbb{C}})\,p\|^2 &\leqslant 2\left\{\|(A(t) - I)\,x\|^2 + |\tau|^2\,\|\xi(t)\|^2 + \left|\left(A(t)\,x \,\middle|\, \frac{1}{a(t)}\,\xi(t)\right)\right|^2 + |a(t) - 1|^2\,|\tau|^2\right\}\\ &\leqslant 2\left\{\left(\|A(t) - I\|^2 + \left(\frac{\|A(t)\|}{|a(t)|}\,\|\xi(t)\|\right)^2\right)\|x\|^2 + (\|\xi(t)\|^2 + |a(t) - 1|^2)\,|\tau|^2\right\}\\ &\leqslant 2\max\left\{\|A(t) - I\|^2 + \left(\frac{\|A(t)\|}{|a(t)|}\,\|\xi(t)\|\right)^2, \|\xi(t)\|^2 + |a(t) - 1|^2\right\}\|p\|^2.\end{aligned}$$

Since

$$\lim_{t\downarrow 0}\xi(t) = 0, \tag{3.13}$$

$$\lim_{t\downarrow 0} a(t) = 1 \tag{3.14}$$

and $\|A(t)\| \leqslant |a(t)|$, Lemma 3.1 implies then

$$\lim_{t\downarrow 0}\|T(t) - I_{\mathscr{H}\oplus\mathbb{C}}\| = 0, \tag{3.15}$$

showing that the semigroup T is uniformly continuous.

(b) Now let the semigroup T be uniformly continuous, i.e., let (3.15) hold. To prove that the homomorphism $\tilde{T}$: $\mathbb{R}_+ \to \operatorname{Iso} B$ is continuous for the topology of local uniform convergence on B, it suffices to show that $\tilde{T}(t)$ tends to the identity map for the latter topology as $t\downarrow 0$.

By (3.11), (3.13) and (3.14),

$$\lim_{t\downarrow 0}\|(C(t)\cdot \mid \eta(t))\| = 0. \tag{3.16}$$

Choosing $p = (x, 0)$ $(x \in \bar{B})$, (3.12) yields

$$\begin{aligned}\|A(t) - I\|^2 &= \sup\{\|(A(t) - I)\,x\|^2 \colon \|x\| \leqslant 1\}\\ &\leqslant \sup\{\|(A(t) - I)\,x\|^2 + |(A(t)\,x \mid \eta(t))|^2 \colon \|x\| \leqslant 1\}\\ &= \sup\{\|(T(t) - I_{\mathscr{H}\oplus\mathbb{C}})(x, 0)\|^2 \colon \|x\| \leqslant 1\}\\ &\leqslant \sup\{\|(T(t) - I_{\mathscr{H}\oplus\mathbb{C}})\,q\|^2 \colon \|q\| \leqslant 1\}\\ &= \|T(t) - I_{\mathscr{H}\oplus\mathbb{C}}\|^2.\end{aligned}$$

Hence, because

$$\|C(t)-I\| \leqslant \frac{1}{|a(t)|}\,\|A(t)-I\| + \left|\frac{1}{a(t)}-1\right|,$$

(3.14) and (3.15) imply

$$\lim_{t\downarrow 0} \|C(t)-I\| = 0$$

i.e., by (3.6) and (3.16),

$$\lim_{t\downarrow 0} \|d\tilde{T}(t)(0)-I\| = 0. \tag{3.17}$$

Since, by (3.6), (3.8) and (3.11),

$$\|d\tilde{T}(t)(0)\| \leqslant \|C(t)\| + \|(C(t)\cdot \mid \eta(t))\,\eta(t)\| \leqslant 1 + \|\eta(t)\|^2,$$

then by (3.16) every summand of degree $n \geqslant 1$ on the right-hand side of the power series expansion (3.7) tends to zero for the norm topology as $t\downarrow 0$. Thus (3.12) and (3.17) imply that, given any r with $0<r<1$,

$$\lim_{t\downarrow 0} \tilde{T}(t)\,x = \lim_{t\downarrow 0} \eta(t) + \lim_{t\downarrow 0} d\tilde{T}(t)(0)\,x = x$$

uniformly for $\|x\| \leqslant r$. Q.E.D.

In view of Theorem III, if condition (3.1) is satisfied, $\tilde{T}$ is continuous for the topology of local uniform convergence on B if, and only if, the infinitesimal generator X of the semigroup T is a bounded linear operator on $\mathscr{H}\oplus\mathbb{C}$. If that is the case, then $T(t)=\exp tX$. By consequence $T(t)$ is invertible; therefore $\tilde{T}(t)\in \operatorname{Aut} B$ and $\tilde{T}$ is the restriction to $\mathbb{R}_+$ of a continuous homomorphism $\mathbb{R}\to\operatorname{Aut} B$.

4

There are no non-constant holomorphic families of holomorphic isometries for the hyperbolic metric of $B\subset\mathscr{H}$. This fact—which was established in [9]—implies that there are no non-trivial holomorphic semigroups of holomorphic isometries of B.

This section is devoted to showing— independently of [9]—that there are no non-constant holomorphic families of linear operators in $\mathscr{H}\oplus\mathbb{C}$ leaving the form $\mathfrak{a}$ invariant.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two complex Hilbert spaces and let F_1 and F_2 be

two holomorphic maps of a domain $D\subset\mathbb{C}$, into $\mathscr{L}(\mathscr{H}_1)$ and $\mathscr{L}(\mathscr{H}_2)$, respectively. Let $F\in\operatorname{Hol}(D, \mathscr{L}(\mathscr{H}_1\oplus\mathscr{H}_2))$ be defined by the matrices

$$F(z)=\begin{pmatrix} F_1(z) & 0 \\ 0 & F_2(z)\end{pmatrix} \qquad (z\in D).$$

Denoting by I_1 and I_2 the identities in $\mathscr{L}(\mathscr{H}_1)$ and $\mathscr{L}(\mathscr{H}_2)$, let J_{12} be the matrix

$$J_{12}=\begin{pmatrix} I_1 & 0 \\ 0 & -I_2\end{pmatrix}.$$

Proposition 4.1. *If*

$$F(z)^*\,J_{12}F(z)=J_{12} \qquad \textit{for all } z\in D, \tag{4.1}$$

then the functions F_1 and F_2 are constant.

Proof. If $F_j(z)=\Sigma_{n=0}^{+\infty}(z-z_0)^n F_{j,n}$ ($F_{j,n}\in\mathscr{L}(\mathscr{H}_j)$, $j=1,2$) is the power series expansion of F_j in a neighborhood V of z_0 in D, $F(z)^*\,J_{12}F(z)$ is represented in V by the expansion

$$F(z)^*\,J_{12}F(z)=\sum_{m,n\geqslant 0}(z-z_0)^m\,(\overline{z-z_0})^n\,F_{m\bar n}$$

($F_{m\bar n}\in\mathscr{L}(\mathscr{H}_1\oplus\mathscr{H}_2)$), whose coefficient $F_{1\bar 1}$ is

$$F_{1\bar 1}=\begin{pmatrix} F^*_{1,1}F_{1,1} & 0 \\ 0 & F^*_{2,1}F_{2,1}\end{pmatrix}.$$

Condition (4.1) yields $F_{1\bar 1}=0$, i.e., $F_{1,1}=0$, $F_{2,1}=0$, showing that the differentials of F_1 and F_2 at any $z_0\in D$ vanish. Q.E.D.

Choosing $\mathscr{H}_1=\mathscr{H}$, $\mathscr{H}_2=\mathbb{C}$, the above proposition yields

Corollary 4.2. *Let T be a C_0 semigroup leaving the sesquilinear form $\mathfrak{a}$ invariant. The function $t\mapsto T(t)$ cannot be extended to a non-constant holomorphic map T of a neighborhood U of the positive real axis into $\mathscr{L}(\mathscr{H}\oplus\mathbb{C})$ such that $T(z)^*\,JT(z)=J$ for all $z\in U$.*

This statement is also a consequence of the following maximum principle, which may be independently interesting.

Proposition 4.3. *Let $f\in\operatorname{Hol}(D,\mathscr{H}\oplus\mathbb{C})$ and let $z_0\in D$ be a relative maximum point of the function $z\mapsto\mathfrak{a}(f(z),f(z))$ $(z\in D)$. If $\mathfrak{a}(f(z_0),f(z_0))<0$, f is constant.*

Proof. Let $g(z)$ and $h(z)$ be the components of $f(z)$ in $\mathscr{H}$ and $\mathbb{C}$, and let $g(z)=\sum_{n=0}^{+\infty}(z-z_0)^n g_n$ ($g_n\in\mathscr{H}$) and $h(z)=\sum_{n=0}^{+\infty}h_n(z-z_0)^n$ ($h_n\in\mathbb{C}$) be the power series expansions of the functions $g\in\mathrm{Hol}(D,\mathscr{H})$ and $h\in\mathrm{Hol}(D,\mathbb{C})$ in a circular neighborhood V of z_0 in D.

For any $z\in V$,

$$\begin{aligned}\mathfrak{a}(f(z),f(z))\\ &=\|g_0\|^2-|h_0|^2+2\,\mathrm{Re}[((g_1\mid g_0)-h_1\bar{h}_0)(z-z_0)]\\ &\quad+(\|g_1\|^2-|h_1|^2)\,|z-z_0|^2+2\,\mathrm{Re}[((g_2,g_0)-h_2\bar{h}_0)(z-z_0)^2]\\ &\quad+O(|z-z_0|^3).\end{aligned}\tag{4.2}$$

The fact that z_0 is a relative maximum for the function $z\mapsto\mathfrak{a}(f(z),f(z))$ implies that

$$(g_1\mid g_0)-h_1\bar{h}_0=0,\tag{4.3}$$

whence, by the Schwarz inequality,

$$|h_0h_1|\leqslant\|g_0\|\ \|g_1\|.$$

But then, because $0>\mathfrak{a}(f(z_0),f(z_0))=\|g_0\|^2-|h_0|^2$, either $g_1=0$, $h_1=0$ or

$$|h_1|<\|g_1\|.\tag{4.4}$$

Setting $z-z_0=\rho e^{i\theta}$ ($\rho\geqslant 0$) for $z\in V$, (4.2) becomes, in view of (4.3),

$$\begin{aligned}\mathfrak{a}(f(z),f(z))=\|g_0\|^2-|h_0|^2+\{\|g_1\|^2-|h_1|^2\\ +2\,\mathrm{Re}[e^{2i\theta}((g_2\mid g_0)-h_2\bar{h}_0)]\}\,\rho^2+O(\rho^3).\end{aligned}\tag{4.5}$$

Choosing θ in such a way that

$$\mathrm{Re}[e^{2i\theta}((g_2\mid g_0)-h_2\bar{h}_0)]\geqslant 0,$$

(4.4) and (4.5) imply that there is $\rho'>0$ so that whenever $0<\rho<\rho'$

$$\mathfrak{a}(f(z),f(z))>\|g_0\|^2-|h_0|^2=\mathfrak{a}(f(z_0),f(z_0)),\tag{4.6}$$

contradicting the fact that z_0 is a maximum point.

Assume inductively that

$$g_1=g_2=\cdots=g_m=0,\qquad h_1=h_2=\cdots=h_m=0$$

for some $m\geqslant 1$. Then

$$\begin{aligned}\mathfrak{a}(f(z),f(z))&=\|g_0\|^2-|h_0|^2\\ &\quad+2\sum_{n=1}^{m+1}\mathrm{Re}[e^{i(n+m)\theta}((g_{n+m}\mid g_0)-h_{n+m}\bar{h}_0)]\,\rho^{n+m}\\ &\quad+\{\|g_{m+1}\|^2-|h_{m+1}|^2+2\,\mathrm{Re}[e^{2i(m+1)\theta}\\ &\quad\times((g_{2m+2}\mid g_0)-h_{2m+2}\bar{h}_0)]\}\,\rho^{2m+2}+O(\rho^{2m+3}).\end{aligned}$$

The fact that z_0 is a relative maximum point yields

$$(g_{n+m}\mid g_0)-h_{n+m}\bar{h}_0=0\qquad\text{for}\quad n=1,\dots,m+1,\tag{4.7}$$

so that

$$\begin{aligned}\mathfrak{a}(f(z),f(z))=\|g_0\|^2-|h_0|^2+\{\|g_{m+1}\|^2-|h_{m+1}|^2\\ +2\,\mathrm{Re}[e^{2i(m+1)\theta}((g_{2m+2}\mid g_0)-h_{2m+2}\bar{h}_0)]\}\\ \times\rho^{2m+2}+O(\rho^{2m+3}).\end{aligned}\tag{4.8}$$

For $n=1$, condition (4.7) and the Schwarz inequality imply

$$|h_{m+1}h_0|\leqslant\|g_{m+1}\|\ \|g_0\|,$$

whence—because $\mathfrak{a}(f(z_0),f(z_0))<0$—either $g_{m+1}=0$, $h_{m+1}=0$ or

$$|h_{m+1}|<\|g_{m+1}\|.$$

In the latter case, choosing θ such that

$$\mathrm{Re}[e^{2i(m+1)\theta}((g_{2m+2}\mid g_0)-h_{2m+2}\bar{h}_0)]\geqslant 0,$$

(4.8) shows that there is $\rho''>0$ so that, whenever $0<\rho<\rho''$, (4.6) holds. That contradicts the fact that the function $z\mapsto\mathfrak{a}(f(z),f(z))$ has a relative maximum for $z=z_0$. Q.E.D.

COROLLARY 4.3. *Let $f\in\mathrm{Hol}(D,\mathscr{H}\oplus\mathbb{C})$ be such that $\mathfrak{a}(f(z),f(z))=k$ for some real constant k and all $z\in D$. If $k<0$, f is constant.*

THEOREM IV. *Let T be a map of D into the set of all linear maps of $\mathscr{H}\oplus\mathbb{C}$ into $\mathscr{H}\oplus\mathbb{C}$ such that for every $p\in\mathscr{H}\oplus\mathbb{C}$ with $\mathfrak{a}(p,p)<0$ the function $z\mapsto T(z)p$ is holomorphic on D, and*

$$\mathfrak{a}(T(z)p,T(z)p)\leqslant\mathfrak{a}(p,p)\qquad\textit{for all } z\in D.$$

If for every $p \in \mathscr{H} \oplus \mathbb{C}$ with $\mathfrak{a}(p, p) < 0$ there exists some $z(p) \in D$ such that

$$\mathfrak{a}(T(z(p))\, p, T(z(p))\, p) = \mathfrak{a}(p, p), \tag{4.9}$$

then $T(z)$ is independent of z and is continuous on $\mathscr{H} \oplus \mathbb{C}$ leaving the form $\mathfrak{a}$ invariant.

Proof. For $p \in \mathscr{H} \oplus \mathbb{C}$ with $\mathfrak{a}(p, p) < 0$ let $f_p \in \mathrm{Hol}(D, \mathscr{H} \oplus \mathbb{C})$ be defined by $f_p(z) = T(z)\, p$ $(z \in D)$. By Proposition 4.3, f_p is independent of z for all p such that $\mathfrak{a}(p, p) < 0$. Since for any $p = (x, \tau) \in \mathscr{H} \oplus \mathbb{C}$ $(x \in \mathscr{H}, \tau \in \mathbb{C})$ there is $\sigma \in \mathbb{C}$ such that for $q = (x, \sigma)$ one has $\mathfrak{a}(q, q) < 0$, it is readily seen that $T(z)\, p$ is independent of z for all $p \in \mathscr{H} \oplus \mathbb{C}$. Setting $T(z) = T^0$, to prove that the operator T^0 is continuous on $\mathscr{H} \oplus \mathbb{C}$, let

$$T^0 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

be the matrix representation of T^0, where T_{11} maps linearly $\mathscr{H}$ into $\mathscr{H}$, $T_{12} \in \mathscr{H}$, $T_{22} \in \mathbb{C}$, and T_{21} is a linear form on $\mathscr{H}$.

Let $p = (x, \tau) \in \mathscr{H} \oplus \mathbb{C}$ $(x \in \mathscr{H}, \tau \in \mathbb{C})$ with $\mathfrak{a}(p, p) = \|x\|^2 - |\tau|^2 < 0$. Condition (4.9) reads

$$\begin{aligned} &\|T_{11}x\|^2 - |T_{21}x|^2 - \|x\|^2 + 2\,\mathrm{Re}[\bar{\tau}((T_{11}x \mid T_{12}) - \bar{T}_{22} T_{21} x)] \\ &\quad + |\tau|^2(\|T_{12}\|^2 - |T_{22}|^2 + 1) = 0, \end{aligned} \tag{4.10}$$

and holds for all $x \in \mathscr{H}$, $\tau \in \mathbb{C}$ such that $\|x\| < |\tau|$. Thus

$$\|T_{11}x\|^2 = |T_{21}x|^2 + \|x\|^2,$$

$$(T_{11}x \mid T_{12}) = \bar{T}_{22} T_{21} x,$$

for all $x \in \mathscr{H}$, and

$$|T_{22}|^2 = \|T_{12}\|^2 + 1.$$

By the Schwarz inequality, these identities yield

$$|T_{21}x| \leqslant \|T_{12}\|\ \|x\|$$

for all $x \in \mathscr{H}$, implying that the linear form T_{21} is continuous. Thus the first identity shows that $T_{11} \in \mathscr{L}(\mathscr{H})$.

Because condition (4.10) is identically satisfied, (4.9) holds for all $p \in \mathscr{H} \oplus \mathbb{C}$. Q.E.D.

5

This section is devoted to characterizing the infinitesimal generator X of a strongly continuous linear semigroup T on the Hilbert space $\mathscr{K} = \mathscr{H} \oplus \mathbb{C}$ satisfying (3.1), where J is expressed by (1.2).

By Theorem I, $J\mathscr{D}(X) \subset \mathscr{D}(X^*)$, and condition (2.3) holds with $L = J$, i.e., iJX is a symmetric operator.

Let $P_1 : \mathscr{K} \to \mathscr{H}$, $P_2 = \mathscr{K} \to \mathbb{C}$ be the linear maps defined by the orthogonal projectors $\mathscr{H} \oplus \mathbb{C} \to \mathscr{H} \oplus \{0\} \cong \mathscr{H}$, $\mathscr{H} \oplus \mathbb{C} \to \{0\} \oplus \mathbb{C} \cong \mathbb{C}$.

Then $\mathscr{D} = P_1(\mathscr{D}(X))$ is a dense linear manifold in $\mathscr{H}$. It will be shown that

$$\mathscr{D}(X) = \mathscr{D} \oplus \mathbb{C}, \tag{5.1}$$

i.e., $\{0\} \oplus \mathbb{C} \subset \mathscr{D}(X)$.

Assume that that is not the case, i.e., that $\{0\} \oplus \mathbb{C} \not\subset \mathscr{D}(X)$. Then, for every $x \in \mathscr{D}$, there is a unique $\zeta \in \mathbb{C}$ such that $(x, \zeta) \in \mathscr{D}(X)$. The map $\lambda : x \mapsto \zeta$ is a linear form on $\mathscr{D}$.

Lemma 5.1. *Let $\mathscr{D}$ be a dense linear manifold in $\mathscr{H}$, and let λ be a linear form on $\mathscr{D}$. The set $\Lambda = \{(x, \lambda(x)) : x \in \mathscr{D}\}$ is dense in $\mathscr{H} \oplus \mathbb{C}$ if, and only if, λ is not continuous.*

Proof. Let λ be discontiuous on $\mathscr{D}$ and let $(y, \tau) \perp \Lambda$ $(y \in \mathscr{H}, \tau \in \mathbb{C})$, i.e.,

$$(x \mid y) + \lambda(x)\, \bar{\tau} = 0 \qquad \text{for all } x \in \mathscr{D}.$$

Then, because λ is discontinuous, $\tau = 0$ and therefore $y \perp \mathscr{D}$, whence $y = 0$.

If λ is continuous on $\mathscr{D}$, then λ is the restriction to $\mathscr{D}$ of a continuous linear form $\tilde{\lambda}$ on $\mathscr{H}$. The set $\{(x, \zeta) \in \mathscr{H} \oplus \mathbb{C} : \zeta = \tilde{\lambda}(x)\}$ is a closed proper linear subspace of $\mathscr{H} \oplus \mathbb{C}$. Its complement is open and non-empty. A fortiori the complement of Λ in $\mathscr{H} \oplus \mathbb{C}$ contains a non-empty open set. Q.E.D.

Because the domain $\mathscr{D}(X)$ is dense, Lemma 5.1 implies that the linear form $\lambda : x \mapsto \zeta$ is discontinuous.

Let $X_1 : \mathscr{D} \to \mathscr{H}$, $X_2 : \mathscr{D} \to \mathbb{C}$ be the linear operators defined by $X_1 x = P_1 \circ X(x, \lambda(x))$, $X_2 x = P_2 \circ X(x, \lambda(x))$ for $x \in \mathscr{D}$ Let $Y_1 : \mathscr{D}(X^*) \to \mathscr{H}$, $Y_2 : \mathscr{D}(X^*) \to \mathbb{C}$ be the linear maps defined by $Y_1 = P_1 \circ X^*$, $Y_2 = P_2 \circ X^*$. For $p \in \mathscr{D}(X)$, $q \in \mathscr{D}(X^*)$, let $x = P_1 p$, $y = P_1 q$, $\tau = P_2 q$. Equality

$$(Xp, q) = (p, X^* q) \tag{5.2}$$

can be written

$$(X_1 x \mid y) + \bar{\tau} X_2 x = (x \mid Y_1 q) + \lambda(x) \overline{Y_2 q}. \tag{5.3}$$

Since $J\mathscr{D}(X) \subset \mathscr{D}(X^*)$, then $(x, -\lambda(x)) \in \mathscr{D}(X^*)$ for all $x \in \mathscr{D}$. Choosing $q \in \mathscr{D}(X)$, then $y \in \mathscr{D}$, $Jq \in \mathscr{D}(X^*)$ and (2.3) with $L = J$ yields

$$Y_1 \circ Jq + X_1 y = 0, \qquad Y_2 \circ Jq - X_2 y = 0.$$

Replacing q by Jq in (5.3) one has

$$(X_1 x \mid y) - \overline{\lambda(y)}\, X_2 x = -(x \mid X_1 y) + \lambda(x) \overline{X_2 y} \qquad \text{for all } x, y \text{ in } \mathscr{D}. \tag{5.4}$$

If $\lambda(x) = 0$, then $(x, 0)$ is invariant by J, and therefore $(x, 0) \in \mathscr{D}(X^*)$, i.e., the linear form

$$y \mapsto (X(y, \lambda(y)), (x, 0)) = (X_1 y \mid x)$$

is continuous on $\mathscr{D}$. Hence, by (5.4), if $\lambda(x) = 0$ and $X_2 x \neq 0$, $y \mapsto \lambda(y)$ is continuous on $\mathscr{D}$. This contradiction proves that $\lambda(x) = 0$ implies $X_2 x = 0$, i.e., $X_2 = c\lambda$ for some $c \in \mathbb{C}$. Thus X is represented by the matrix

$$\begin{pmatrix} X_1 & 0 \\ 0 & c \end{pmatrix} \tag{5.5}$$

on the domain $\mathscr{D}(X) = \{(x, \lambda(x)) \colon x \in \mathscr{D}\}$.

Since X is the infinitesimal generator of a strongly continuous semigroup, the operator on $\mathscr{D}(X)$ represented by the matrix

$$\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix},$$

which is obtained from X by the bounded perturbation

$$\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix},$$

also generates a strongly continuous semigroup. Hence there exists $b \in \mathbb{R}$ such that

$$\operatorname{Re} \zeta > b \Rightarrow \zeta \in r(X_1). \tag{5.6}$$

Let X' be the operator represented by the matrix (5.5) on the domain $\mathscr{D} \oplus \mathbb{C}$. The point spectrum $p\sigma(X')$ is expressed by

$$p\sigma(X') = \{c\} \cup p\sigma(X_1). \tag{5.7}$$

On the other hand, X' is a proper extension of the closed operator X, and therefore $r(X) \subset p\sigma(X')$. Thus, by (5.7) and in view of the fact that X is the infinitesimal generator of a strongly ontinuous semigroup, there is $a \in R$ such that, if $\operatorname{Re} \zeta > a$, $\zeta \in p\sigma(X_1)$. This contradicts (5.6) and thereby proves (5.1).

Hence X is represented by a matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \tag{5.8}$$

where X_{11} is a linear operator on $\mathscr{H}$ with domain $\mathscr{D}(X_{11}) = \mathscr{D}$; $X_{12} \in \mathscr{H}$; $X_{22} \in \mathbb{C}$ and X_{21} is a linear form on $\mathscr{D}$.

LEMMA 5.2. *The operator X_{11} is closed.*

Proof. Let $\{x_\nu\}$ be a sequence in $\mathscr{D}$ converging to $x \in \mathscr{H}$ and such that $\{X_{11} x_\nu\}$ converges to $y \in \mathscr{H}$. Setting $p_\nu = (x_\nu, 0)$ then $X p_\nu = (X_{11} x_\nu, X_{21} x_\nu)$.

If $\lim_{\nu \to +\infty} |X_{21} x_\nu| = +\infty$, let $z_\nu = (1/X_{21} x_\nu)\, x_\nu$ for $\nu \gg 0$. Then $\lim_{\nu \to +\infty} z_\nu = 0$ and $\lim_{\nu \to +\infty} X_{11} z_\nu = 0$. Therefore, setting $q_\nu = (z_\nu, 0)$, $\lim_{\nu \to +\infty} X q_\nu = \lim_{\nu \to +\infty} (X_{11} z_\nu, 1) = (0, 1)$. Because X is closed, then $X0 = (0, 1)$. This contradiction shows that there is a sequence of indices $0 < \nu_1 < \nu_2 < \cdots$ such that $\{X_{21} x_\nu\}$ converges to some $\mu \in \mathbb{C}$. Hence $\{X p_{\nu_j}\}$ converges to (y, μ). Because the operator X is closed, then $(x, 0) \in \mathscr{D}(X)$, i.e., $x \in \mathscr{D}$, and $X(x, 0) = (y, \mu)$, whence $y = X_{11} x$. Q.E.D.

Condition $J(\mathscr{D} \oplus \mathbb{C}) \subset \mathscr{D}(X^*)$ implies that $\mathscr{D} \oplus \mathbb{C} \subset \mathscr{D}(X^*)$, and therefore X^* is represented by a matrix

$$X^* = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \tag{5.9}$$

where Y_{11} is a linear operator on $\mathscr{H}$ with dense domain $\mathscr{D}(Y_{11}) = P_1 \mathscr{D}(X^*) \supset \mathscr{D}$, $Y_{12} \in \mathscr{H}$, $Y_{22} \in \mathbb{C}$, and Y_{21} is a linear form on $\mathscr{D}(Y_{11})$. The same argument as that in Lemma 5.2 shows that Y_{11} is closed.

For $x \in \mathscr{D}(X_{11})$, $y \in \mathscr{D}(Y_{11})$, ζ and τ in $\mathbb{C}$, setting $p = (x, \zeta)$, $q = (y, \tau)$, condition (5.2) now reads, in view of (5.8) and (5.9),

$$\begin{aligned} &(X_{11} x \mid y) + \zeta (X_{12} \mid y) + X_{21}(x)\, \bar{\tau} + \zeta \bar{\tau} X_{22} \\ &\quad = (x \mid Y_{11} y) + \bar{\tau} (x \mid Y_{12}) + \zeta \overline{Y_{21}(y)} + \zeta \overline{\tau Y_{22}} \end{aligned}$$

for all ζ, τ in $\mathbb{C}$. This condition is equivalent to

$$(X_{11} x \mid y) = (x \mid Y_{11} y) \qquad \text{for all } x \in \mathscr{D}(X_{11}),\ y \in \mathscr{D}(Y_{11}), \tag{5.10}$$

$$(y \mid X_{12}) = Y_{21}(y) \qquad \text{for all } y \in \mathscr{D}(Y_{11}), \tag{5.11}$$

$$X_{21}(x)=(x \mid Y_{12}) \qquad \text{for all } x\in\mathscr{D}(X_{11}), \tag{5.12}$$

$$X_{22}=\overline{Y_{22}}. \tag{5.13}$$

By (5.12) and (5.11), X_{21} and Y_{21} are restrictions to $\mathscr{D}$ of the continuous linear forms $(\cdot \mid Y_{12})$ and $(\cdot \mid X_{12})$. Furthermore $y\in\mathscr{D}(Y_{11})$ if, and only if, $(y, 0)\in\mathscr{D}(X^*)$, i.e., if, and only if, the linear form

$$(x,\zeta)\mapsto (X(x,\zeta),(y,0))=(X_{11}x \mid y)+\zeta(X_{12}\mid y)$$

is continuous on $\mathscr{D}(X)$. Thus $y\in\mathscr{D}(Y_{11})$ if, and only if, the linear form $x\mapsto(X_{11}x \mid y)$ is continuous on $\mathscr{D}=\mathscr{D}(X_{11})$, i.e., if, and only if, $y\in\mathscr{D}(X_{11}^*)$. This shows that

$$Y_{11}=X_{11}^*. \tag{5.14}$$

Conditions (5.1) and $J\mathscr{D}(X)\subset\mathscr{D}(X^*)$ imply that $\mathscr{D}(X_{11})\subset\mathscr{D}(X_{11}^*)$, while (2.3) (with $L=J$), (5.14), and (5.13) yield

$$X_{11}+X_{11}^*=0 \qquad \text{on } \mathscr{D}=\mathscr{D}(X_{11}),$$

$$X_{12}-Y_{12}=0, \qquad \operatorname{Re} X_{22}=0.$$

Summing up, the following proposition holds.

PROPOSITION 5.3. *Let X be the infinitesimal generator of a strongly continuous semigroup T. Then T leaves the sesquilinear form $\mathfrak{a}$ invariant if, and only if, the following two conditions are fulfilled:*

(1) *there exists a dense linear manifold $\mathscr{D}$ in $\mathscr{H}$ for which* (5.1) *holds;*

(2) *the operator X is represented by the matrix*

$$X=\begin{pmatrix} X_{11} & X_{12}\\ (\cdot\mid X_{12}) & iX_{22}\end{pmatrix}, \tag{5.15}$$

where $X_{22}\in\mathbb{R}$, $X_{12}\in\mathscr{H}$ and iX_{11} is a closed symmetric operator with domain $\mathscr{D}(X_{11})=\mathscr{D}$.

If conditions (1) and (2) hold, the operator X^* is represented by the matrix

$$X^*=\begin{pmatrix} X_{11}^* & X_{12}\\ (\cdot\mid X_{12}) & -iX_{22}\end{pmatrix}.$$

The operator X expressed by (5.15) and the operator

$$X'=\begin{pmatrix} X_{11} & 0\\ 0 & iX_{22}\end{pmatrix} \tag{5.16}$$

with domain $\mathscr{D}(X)$, differ by the bounded perturbation

$$K=\begin{pmatrix} 0 & X_{12}\\ (\cdot\mid X_{12}) & 0\end{pmatrix}. \tag{5.17}$$

Thus X is the infinitesimal generator of a strongly continuous semigroup if, and only if, X' is the infinitesimal generator of a strongly continuous semigroup.

Let $\Pi_l=\{\zeta\in\mathbb{C}: \operatorname{Re}\zeta<0\}$, $\Pi_r=\{\zeta\in\mathbb{C}: \operatorname{Re}\zeta>0\}$.

If iX_{11} is closed and symmetric, X' generates a strongly continuous semigroup (which turns out to be a semigroup of contractions of $\mathscr{H}\oplus\mathbb{C}$ provided that $X_{22}\in\mathbb{R}$) if, and only if $\Pi_r\subset r(X_{11})$.

Since the norm of the operator (5.17) is equal to $\|X_{12}\|$, the following theorem summarizes the results obtained so far.

THEOREM V. *Let X be a linear operator on $\mathscr{H}\oplus\mathbb{C}$. Then X is the infinitesimal generator of a strongly continuous linear semigroup T on $\mathscr{H}\oplus\mathbb{C}$, leaving the sesquilinear form $\mathfrak{a}$ invariant if, and only if, there is a dense linear manifold $\mathscr{D}$ in $\mathscr{H}$ such that* (5.1) *holds and X is represented by* (5.15), *where X_{12} and X_{22} are arbitrarily chosen in $\mathscr{H}$ and $\mathbb{R}$, and where iX_{11} is any closed symmetric operator on $\mathscr{H}$ with domain $\mathscr{D}$, such that $r(X_{11})\supset\{\zeta\in\mathbb{C}: \operatorname{Re}\zeta>0\}$.*

The semigroup T satisfies the estimate

$$\|T(t)\|\leqslant e^{\|X_{12}\|t} \qquad \textit{for all } t\geqslant 0. \tag{5.18}$$

The following theorem is a consequence of Theorems II and V.

THEOREM VI. *The linear operator X expressed by* (5.15) *is the infinitesimal generator of a strongly continuous group leaving the sesquilinear form $\mathfrak{a}$ invariant if, and only if, iX_{11} is self-adjoint and $X_{22}\in\mathbb{R}$.*

6

In Sections 6–8 the spectral structure of the infinitesimal generator X of a strongly continuous semigroup T leaving the form $\mathfrak{a}$ invariant will be investigated.

Following the notations of Section 5, $\mathscr{D}(X)$ and X will be represented by (5.1) and by the matrix (5.15), where iX_{11} is a closed symmetry operator with domain $\mathscr{D}(X_{11})=\mathscr{D}$ and resolvent set $r(X_{11})\supset\Pi_r$, $X_{12}\in\mathscr{H}$, $X_{22}\in\mathbb{R}$.

For $\zeta \in r(X)$ the linear continuous operator $(\zeta I - X)^{-1}$ is represented on $\mathscr{H} \oplus \mathbb{C}$ by a matrix

$$Z = Z(\zeta) = (\zeta I - X)^{-1} = \begin{pmatrix} Z_{11} & Z_{12} \\ (\cdot \mid Z_{21}) & Z_{22} \end{pmatrix},$$

where $Z_{22} = Z_{22}(\zeta) \in \mathbb{C}$, $Z_{12} = Z_{12}(\zeta)$, $Z_{21} = Z_{21}(\zeta)$ are contained in $\mathscr{H}$ and $Z_{11} = Z_{11}(\zeta) \in \mathscr{L}(\mathscr{H})$.

Denoting by $\mathrm{Ran}(X)$ the range of an operator X, condition $\mathrm{Ran}\,(\zeta I - X)^{-1} \subset \mathscr{D}(X)$ is equivalent to $\mathrm{Ran}\, Z_{11} \subset \mathscr{D}$, and $Z_{12} \in \mathscr{D}$. More specifically

$$(\zeta I - X) \circ Z = I_{\mathscr{H} \oplus \mathbb{C}} \qquad \text{on } \mathscr{H} \oplus \mathbb{C}$$

if, and only if, $\mathrm{Ran}(Z_{11}) \subset \mathscr{D}$, $Z_{12} \in \mathscr{D}$, and

$$(\zeta I - X_{11}) \circ Z_{11} - (\cdot \mid Z_{21})\, X_{12} = I \qquad \text{on } \mathscr{H}, \tag{6.1}$$

$$(\zeta I - X_{11})(Z_{12}) - Z_{22} X_{12} = 0, \tag{6.2}$$

$$-(Z_{11} \cdot \mid X_{12}) + (\zeta - iX_{22})(\cdot \mid Z_{21}) = 0 \qquad \text{on } \mathscr{H}, \tag{6.3}$$

$$-(Z_{12} \mid X_{12}) + (\zeta - iX_{22})\, Z_{22} = 1. \tag{6.4}$$

Similarly

$$Z \circ (\zeta I - X_{11}) = I_{\mathscr{H} \oplus \mathbb{C}} \qquad \text{on } \mathscr{D}(X)$$

if, and only if,

$$Z_{11} \circ (\zeta I - X_{11}) - (\cdot \mid X_{12})\, Z_{12} = I \qquad \text{on } \mathscr{D}, \tag{6.5}$$

$$-Z_{11}(X_{12}) + (\zeta - iX_{22})\, Z_{12} = 0, \tag{6.6}$$

$$((\zeta I - X_{11}) \cdot \mid Z_{21}) - Z_{22}(\cdot \mid X_{12}) = 0 \qquad \text{on } \mathscr{D}, \tag{6.7}$$

$$-(X_{12} \mid Z_{21}) + (\zeta - iX_{22})\, Z_{22} = 1. \tag{6.8}$$

If $\zeta \in r(X) \cap r(X_{11})$, (6.2) yields

$$Z_{12} = Z_{22}(\zeta I - X_{11})^{-1} X_{12}, \tag{6.9}$$

and therefore (6.4) becomes

$$Z_{22}[\zeta - iX_{22} - ((\zeta I - X_{11})^{-1} X_{12} \mid X_{12})] = 1. \tag{6.10}$$

Let ϕ be the holomorphic function on $r(X_{11}) \supset \Pi_r$ defined by

$$\phi(\zeta) = \zeta - iX_{22} - ((\zeta I - X_{11})^{-1} X_{12} \mid X_{12}).$$

For $\zeta \in r(X_{11})$ let $Y_{12} = Y_{12}(\zeta) \in \mathscr{D}$ be the vector defined by $Y_{12} = (\zeta I - X_{11})^{-1} X_{12}$. Since for $\zeta \in r(X_{11})$ and $\mathrm{Re}\, \zeta \neq 0$

$$\|(\zeta I - X_{11})^{-1}\| \leqslant |\mathrm{Re}\, \zeta|^{-1}, \tag{6.11}$$

then

$$\|(Y_{12}(\zeta)\| \leqslant |\mathrm{Re}\, \zeta|^{-1}\, \|X_{12}\| \qquad (\zeta \in r(X_{11}),\ \mathrm{Re}\, \zeta \neq 0). \tag{6.12}$$

Let C be the zero set of ϕ,

$$C = \{\zeta \in r(X_{11}) \colon \phi(\zeta) = 0\}. \tag{6.13}$$

Since for $\zeta \in r(X_{11})$

$$\begin{aligned} \phi(\zeta) &= \zeta - \bar{\zeta}\, \|Y_{12}(\zeta)\|^2 - iX_{22} + (Y_{12}(\zeta) \mid X_{11} Y_{12}(\zeta)) \\ &= (1 - \|Y_{12}(\zeta)\|^2)\, \mathrm{Re}\, \zeta \\ &\quad + i[(1 + \|Y_{12}(\zeta)\|^2)\, \mathrm{Im}\, \zeta - X_{22} - i(Y_{12}(\zeta) \mid (X_{11} Y_{12}(\zeta))] \end{aligned}$$

(where, because $(Y_{12}(\zeta) \mid X_{11} Y_{12}(\zeta)) \in i\mathbb{R}$, the summand in square brackets is real), then $\zeta \in C$ is purely imaginary unless $\|Y_{12}(\zeta)\| = 1$, in which case, in view of (6.12),

$$|\mathrm{Re}\, \zeta| \leqslant \|X_{12}\|.$$

Hence

$$C \subset \{\zeta \in \mathbb{C} \colon |\mathrm{Re}\, \zeta| \leqslant \|X_{12}\|\}. \tag{6.14}$$

Since the closed symmetric operator iX_{11} is the generator of a C_0 semigroup, either $r(X_{11}) = \Pi_r$ or $r(X_{11}) \supset \{\zeta \in \mathbb{C} \colon \mathrm{Re}\, \zeta \neq 0\}$. Hence, (6.14) implies that ϕ is not constant on any connected component of $r(X_{11})$, and therefore C is a discrete set in $r(X_{11})$.

It will be shown later that the part of C not contained in $i\mathbb{R}$ contains two points at most.

For $\zeta \notin \sigma(X_{11}) \cup C$ define Z_{22} by (6.10) and then Z_{12} by (6.9). Define then Z_{11} and $(\cdot \mid Z_{21})$ by (6.5) and by (6.7), obtaining, for all $\zeta \notin \sigma(X_{11}) \cup C$,

$$Z_{22} = \phi(\zeta)^{-1}, \tag{6.15}$$

$$Z_{12} = (\phi(\zeta)(\zeta I - X_{11}))^{-1} X_{12}, \tag{6.16}$$

$$Z_{11} = (\zeta I - X_{11})^{-1} + ((\zeta I - X_{11})^{-1} \cdot \mid X_{12})\, Z_{12}, \tag{6.17}$$

and $(\cdot \mid Z_{21}) = Z_{22}((\zeta I - X_{11})^{-1} \cdot \mid X_{12})$, which is equivalent to

$$Z_{21} = \overline{Z_{22}}(\bar{\zeta} I - X_{11}^*)^{-1} X_{12}. \tag{6.18}$$

A direct computation shows that Z_{22}, Z_{12}, Z_{11}, Z_{21} as defined by (6.15), (6.16), (6.17), and (6.18) satisfy (6.1), (6.2), (6.3), (6.4), (6.6), and (6.8) for all $\zeta \notin \sigma(X_{11}) \cup C$. Thus the latter condition implies that $\zeta \in r(X)$, and consequently $\sigma(X) \setminus \sigma(X_{11}) \subset C$. On the other hand, by (6.10), if $\zeta \in r(X) \cap r(X_{11})$ then $\zeta \notin C$. Thus, by (6.10), if $\zeta \in C$ then $\zeta \in \sigma(X)$, and in conclusion

$$\sigma(X) \setminus \sigma(X_{11}) = C. \tag{6.19}$$

Since C is discrete in $r(X_{11})$, every $\zeta_0 \in C$ has a neighborhood U in $r(X_{11})$ such that $U \cap C = \{\zeta_0\}$ and such that, if U is sufficiently small, for every $\zeta \in U$,

$$\begin{aligned}\phi(\zeta) &= \zeta - iX_{22} - ((((\zeta - \zeta_0) I + \zeta_0 I - X_{11})^{-1} X_{12} \mid X_{12}) \\ &= \zeta - iX_{22} - ((((\zeta - \zeta_0)(\zeta_0 I - X_{11})^{-1} + I)^{-1} (\zeta_0 I - X_{11})^{-1} X_{12} \mid X_{12}) \\ &= (\zeta - \zeta_0) \left\{ 1 + \sum_{n=0}^{+\infty} (\zeta_0 - \zeta)^n ((\zeta_0 I - X_{11})^{-(n+2)} X_{12} \mid X_{12}) \right\}.\end{aligned}$$

By (6.15)–(6.18), this proves that ζ_0 is an isolated pole of $(\zeta I - X)^{-1}$, i.e.,

LEMMA 6.1. *Every point of C is an isolated point of $\sigma(X)$, at which $(\zeta I - X)^{-1}$ has a pole.*

Let $\zeta \in \mathbb{C}$ and let $p = (x, \tau) \in \mathscr{D}(X) \setminus \{0\}$ $(x \in \mathscr{H},\ \tau \in \mathbb{C})$ be such that $Xp = \zeta p$. Since $\mathscr{D}(X) = \mathscr{D}(X_{11}) \oplus \mathbb{C} = J\mathscr{D}(X) \subset \mathscr{D}(X^*)$, then $Jp \in \mathscr{D}(X^*)$ and, by (2.3) (with $L = J$),

$$\begin{aligned}2 \operatorname{Re} \zeta (Jp, p) &= (\zeta + \bar{\zeta})(Jp, p) = (J\zeta p, p) + (Jp, \zeta p) \\ &= (JXp, p) + (Jp, Xp) = ((JX + X^*J)\, p, p) = 0.\end{aligned}$$

Thus, if $\operatorname{Re} \zeta \neq 0$, then $(Jp, p) = \|x\|^2 - |\tau|^2 = 0$. Because $T(t)\, p = e^{\zeta t} p$ for all $t \geq 0$ [4, Theorem 16.7.2, pp. 467–469], then for any $t > 0$ the point $z = (1/\tau)\, x \in \partial B$ is an isolated fixed pont of the continuous extension $\hat{T}(t)$ of $\tilde{T}(t) \in \operatorname{Iso} B$.

Proposition 1.3, Lemma 1.4, and (1.9) yield

PROPOSITION 6.2. *The set $C' = C \setminus i\mathbb{R}$ of points of C outside the imaginary axis is either empty or consists of two points at most. In the latter case the two points are symmetric with respect to the imaginary axis.*

The second part of the above proposition can be established also by noting that, if iX_{11} is self-adjoint and if $\zeta \in r(X_{11})$, then

$$\phi(-\bar{\zeta}) = -\bar{\zeta} - iX_{22} - ((-\bar{\zeta} I - X_{11})^{-1} X_{12} \mid X_{12}) = -\overline{\phi(\zeta)}.$$

7

Further information on the spectrum $\sigma(X)$ of X depends on the structure of the set of fixed points of the semigroup $\tilde{T}$.

Let μ' and μ'' be two eigenvalues of X and let $\mathscr{E}(\mu') \subset \mathscr{D}(X)$, $\mathscr{E}(\mu'') \subset \mathscr{D}(X)$ be the corresponding eigenspaces. If $\mu' + \overline{\mu''} \neq 0$, then for $p' \in \mathscr{E}(\mu')$, $p'' \in \mathscr{E}(\mu'')$,

$$\begin{aligned}(Jp', p'') &= \frac{1}{\mu' + \overline{\mu''}} (\mu' + \overline{\mu''})(Jp', p'') \\ &= \frac{1}{\mu' + \overline{\mu''}} ((JXp', p'') + (Jp', Xp'')) \\ &= \frac{1}{\mu' + \overline{\mu''}} ((JX + X^*J)\, p', p'') = 0,\end{aligned}$$

proving thereby

LEMMA 7.1. *If $\mu' + \overline{\mu''} \neq 0$ then* $\mathfrak{a}(\mathscr{E}(\mu'), \mathscr{E}(\mu'')) = (J\mathscr{E}(\mu'), \mathscr{E}(\mu'')) = \{0\}$.

The map $\hat{T}: t \mapsto \hat{T}(t)$ $(t \geq 0)$ is a semigroup of continuous mappings $\hat{T}(t): \bar{B} \to \bar{B}$.

If $\operatorname{Fix} \hat{T}(t)$ indicates the set of fixed points of $\hat{T}(t)$, $\operatorname{Fix} \hat{T}(t) = \{z \in \bar{B}: \hat{T}(t)\, z = z\}$, $\operatorname{Fix} \hat{T}(t) \cap B$ is the set $\operatorname{Fix} \tilde{T}(t)$.

By Corollary 1.2, $\operatorname{Fix} \tilde{T}(t)$, if not empty, is the intersection of B with a closed affine subspace of $\mathscr{H}$.

For every $t \geq 0$, $\operatorname{Fix} \tilde{T}(t)$ and $\operatorname{Fix} \hat{T}(t)$ are invariant subsets of $\tilde{T}(s)$ and $\hat{T}(s)$ for all $s > 0$. The restrictions of the semigroups $\tilde{T}$ and $\hat{T}$ to $\operatorname{Fix} \tilde{T}(t)$ and to $\operatorname{Fix} \hat{T}(t)$ are periodic with period t.

Let $\operatorname{Fix} \tilde{T} = \{z \in B: \tilde{T}(t)\, z = z \text{ for all } t \geq 0\}$, $\operatorname{Fix} \hat{T} = \{z \in B: \hat{T}(t)\, z = z \text{ for all } t \geq 0\}$, be the set of fixed points of the semigroups $\tilde{T}$ and $\hat{T}$, respectively.

If $z \in B$ $(z \in \bar{B})$ then the point $p = (z, 1) \in \mathscr{H} \oplus \mathbb{C}$ is such that $(Jp, p) < 0$ $((Jp, p) \leq 0$, respectively). Vice versa, if $p = (x, \tau) \in \mathscr{H} \oplus \mathbb{C}$, $p \neq 0$, is such that $(Jp, p) \leq 0$, $((Jp, p) < 0)$, then the point $z = (1/\tau)\, x$ is contained in $\bar{B}$ (in B, respectively).

Thus looking for fixed points of $\tilde{T}(t)$ or of $\hat{T}(t)$ is the same as looking for eigenvectors p of $T(t)$ corresponding to non-vanishing eigenvalues and such that $(Jp, p) < 0$ or $(Jp, p) \leq 0$, respectively.

Since by (1.8), Fix $\hat{T}(t) \neq \emptyset$ for all $t \geq 0$, then, given $t > 0$ there exist $\mu \in \mathbb{C}$, $p \in \mathscr{H} \oplus \mathbb{C}$ such that $p \neq 0$, $(Jp, p) \leq 0$, and

$$T(t)\, p = e^{\mu t} p. \tag{7.1}$$

There is some $k \in \mathbb{Z}$ such that $\mu_k = \mu + 2k\pi i/t \in p\sigma(X)$, and p is contained in the closed linear extension of the linearly independent closed subspaces $\mathrm{Ker}(\mu_k I - X)$ for all $\mu_k \in p\sigma(X)$

$$p \in V\{\mathrm{Ker}(\mu_k I - X) \colon k \in \mathbb{Z},\ \mu_k \in p\sigma(X)\} \tag{7.2}$$

[4, Theorem 16.7.2, pp. 467–469]. This proves

LEMMA 7.2. *The point spectrum of X is non-empty.*

In view of (7.2) let $p_k \in \mathrm{Ker}(\mu_k I - X)$ be such that

$$p = \sum p_k. \tag{7.3}$$

By Lemma 7.1

$$(Jp, p) = \sum (Jp_k, p_k). \tag{7.4}$$

Since $(Jp, p) \leq 0$, there is some $k \in \mathbb{Z}$, such that $\mu_k \in p\sigma(X)$, $p_k \neq 0$, and $(Jp_k, p_k) \leq 0$. If moreover $(Jp, p) < 0$, then $(Jp_k, p_k) < 0$. Since

$$T(s)\, p_k = e^{\mu_k s} p_k \qquad \text{for all } s \geq 0, \tag{7.5}$$

then

$$\mathrm{Fix}\, \hat{T} \neq \emptyset, \tag{7.6}$$

and furthermore the following proposition holds.

PROPOSITION 7.3. *If* Fix $\tilde{T}(t) \neq \emptyset$ *for some* $t > 0$, *then* Fix $\tilde{T} \neq \emptyset$.

If Fix $\tilde{T}(t) = \emptyset$ for some $t > 0$, then Fix $\tilde{T}(s) = \emptyset$ for all $s > 0$. By Proposition 1.3, Fix $\hat{T}(t)$ consists of one or two points contained in ∂B. If z is one of them, setting $p = (z, 1)$, there is $\mu \in \mathbb{C}$ satisfying (7.1). Since Fix $\tilde{T}(s) = \emptyset$ for all $s > 0$, all eigenvectors p_k appearing in (7.4) are such that $(Jp_k, p_k) \geq 0$. Because, on the other hand, $(Jp, p) = 0$, then $(Jp_k, p_k) = 0$ for all p_k appearing in (7.4). Since for all $p_k = (x_k, \tau_k) \neq 0$ in (7.4), $z_k = (1/\tau_k)\, x_k \in \partial B$, and eigenvectors $p_k \neq 0$ corresponding to different k are linearly independent, then the fact that the cardinality of Fix $\hat{T}(t)$ is at most two implies that there are at most two integral values of k corresponding to $z_k \in$ Fix $\hat{T}(t)$. If there are two such integers, k_1, k_2, $k_1 \neq k_2$, the affine complex line joining the two distinct points z_{k_1} and z_{k_2} has a non-empty intersection with B. Since by (7.5) this intersection consists of fixed points of $\tilde{T}(t)$, that contradicts the hypothesis Fix $\tilde{T}(t) = \emptyset$. Hence the right-hand side of (7.3) reduces to one summand $p_k = p$, and (7.5) becomes

$$T(s)\, p = e^{\mu s} p \qquad \text{for all } s \geq 0,$$

This proves

PROPOSITION 7.4. *If* Fix $\tilde{T}(t) = \emptyset$ *for some* $t > 0$, *then, for every* $s > 0$, Fix $\hat{T}(s) =$ Fix $\hat{T}(t)$ *and the latter set consists of one or two points contained in* ∂B.

8

To give a more detailed description of $\sigma(X)$ the two cases in which iX_{11} is self-adjoint or is symmetric but not self-adjoint will now be considered separately.

Assume first that iX_{11} is symmetric but not self-adjoint. Then $\Pi_r = r(X_{11})$. If $\mathrm{Re}\, \zeta < 0$, ζ and $\bar{\zeta}$ are contained in $r(-X_{11})$. Because $X_{11}^* = -X_{11}$ on $\mathscr{D}(X_{11}) = \mathscr{D}$ and X_{11} is closed, then $\bar{\zeta} \in p\sigma(X_{11}^*)$, i.e., there is some $x \in \mathscr{D}(X_{11}^*) \setminus \{0\}$ for which

$$(\bar{\zeta} I - X_{11}^*)\, x = 0. \tag{8.1}$$

Suppose now that $\mathrm{Re}\, \zeta < 0$ and that $\zeta \in r(X)$. Then (6.1) yields

$$(Z_{11}\, y \mid (\bar{\zeta} I - X_{11}^*)\, x) - (y \mid Z_{21})(X_{12} \mid x) = (y, x)$$

for all $y \in \mathscr{H}$, i.e., by (8.1),

$$(x \mid X_{12})(Z_{21} \mid y) + (x \mid y) = 0 \qquad \text{for all } y \in \mathscr{H},$$

whence

$$(x \mid X_{12})\, Z_{21} + x = 0, \tag{8.2}$$

implying

$$(x \mid X_{12}) \neq 0 \tag{8.3}$$

and

$$0 \neq Z_{21} \in \mathscr{D}(X_{11}^*),$$

so that, by (8.2) and (8.3), Z_{21} is an eigenvector of X_{11}^* with eigenvalue $\bar{\zeta}$. Thus, by (6.7)

$$Z_{22}(\cdot \mid X_{12})=0 \qquad \text{on } \mathscr{D}(X_{11}),$$

and therefore either $Z_{22}=0$ or (because $\mathscr{D}(X_{11})$ is dense in $\mathscr{H}$) $X_{12}=0$. But this would contradict (8.3). Hence

$$Z_{22}=0 \tag{8.4}$$

and therefore, by (6.2),

$$(\zeta I - X_{11})(Z_{12})=0. \tag{8.5}$$

Because $\bar{\zeta} \in p\sigma(X_{11}^*)$, either $\zeta \in p\sigma(X_{11})$ or ζ is contained in the residual spectrum $r\sigma(X_{11})$ of X_{11} [8, Theorem 4.15, p. 143]. But the first possibility cannot occur because $p\sigma(X_{11}) \subset i\mathbb{R}$ [8, Theorem 4.13, p. 143]. Hence $\zeta \in r\sigma(X_{11})$, and therefore (8.5) yields $Z_{12}=0$, which, together with (8.4), contradicts (6.4). In view of (6.19) and of Proposition 6.2 the following proposition has been proved.

PROPOSITION 8.1. *If iX_{11} is symmetric but not self-adjoint, one of the following two cases necessarily occurs*:

(1) *$C' = \varnothing$ and $\sigma(X) = \overline{\Pi_l}$*;

(2) *C' consists of one point, c, which is an eigenvalue of X, and $\sigma(X) = \overline{\Pi_l} \cup \{c\}$.*

Let E be either Π_l or $\Pi_l \setminus \{-\bar{c}\}$ according to whether case (1) or case (2) occurs.

LEMMA 8.2. *If iX_{11} is symetric but not self-adjoint E is contained in the residual spectrum of X.*

Proof. By Proposition 8.1, $E \subset \sigma(X)$. If $\zeta \in \Pi_l$ is contained in $p\sigma(X)$, there is a some $p \in \mathscr{D}(X) \setminus \{0\}$ satisfying the equation $Xp = \zeta p$, which is equivalent to $JXJJp = \zeta Jp$, i.e.,

$$X^* Jp = -\zeta Jp.$$

Because the operator X is closed, $r(X^*)$ is the image of $r(X)$ by the reflection on the real axis. Hence, because $-\zeta$ is an eigenvalue of X^*, then $-\bar{\zeta} \in \sigma(X) \cap \Pi_r = C \cap \Pi_r = C'$. Thus

$$E \cap p\sigma(X) = \varnothing. \tag{8.6}$$

Now, if $\zeta \in E$, then $-\bar{\zeta} \in r(X)$, that is, $\bar{\zeta} \in r(-JXJ)$.

Since X^* is a proper extension of the closed operator $-JXJ$, then $\bar{\zeta} \in p\sigma(X^*)$, and consequently [8, Theorem 4.15, p. 143] either $\zeta \in p\sigma(X)$ or $\zeta \in r\sigma(X)$, and (8.6) yields the conclusion. Q.E.D.

Proposition 8.1 gives a characterization of the case in which $T(t_0)$ is a compact operator for some $t_0 > 0$. Indeed, as a consequence of Theorem 2.20 in [1, p. 47], (i) $\sigma(X)$ consists of a countable discrete set of eigenvalues each of finite multiplicity, and (ii) if the space $\mathscr{H}$ is infinite-dimensional, then

$$\sigma(T(t)) = \{0\} \cup e^{t\sigma(X)}.$$

Let $\mathscr{H}$ be infinite-dimensional. In view of (i), Proposition 8.1 implies that iJX is self-adjoint. But then, by Theorem II, $T(t)$ is invertible in $\mathscr{L}(\mathscr{H} \oplus \mathbb{C})$, contradicting (ii). This proves

PROPOSITION 8.3. *If there exists $t_0 > 0$ such that $T(t_0)$ is a compact operator, then $n = \dim_{\mathbb{C}} \mathscr{H} < \infty$, and T is (the restriction to $\mathbb{R}_+$ of) a continuous one-parameter subgroup of the classical group $U(n, 1)$.*

Before considering the case in which iX_{11} is self-adjoint, it will be useful to note that the function ϕ is closely related to a Weinstein–Aronszajn determinant.

The operator X is a perturbation of the closed operator X', with domain $\mathscr{D}(X') = \mathscr{D}(X)$, given by (5.16), by the degenerate operator K, expressed by (5.17), whose range (if $X_{12} \neq 0$) is the two-dimensional complex subspace $\mathbb{C}X_{12} \oplus \mathbb{C}$ of $\mathscr{H} \oplus \mathbb{C}$. For any $\zeta \in r(X') = r(X_{11}) \setminus \{iX_{22}\}$ the operator

$$K(\zeta I - X)^{-1} = \begin{pmatrix} 0 & \dfrac{1}{\zeta - iX_{22}} X_{12} \\ ((\zeta I - X_{11})^{-1} \cdot \mid X_{12}) & 0 \end{pmatrix}$$

also has range $\mathbb{C}X_{12} \oplus \mathbb{C}$. Hence the Weinstein–Aronszajn determinant $\omega(\zeta, X', K)$ associated to X' and K (i.e., the determinant of the restriction of $I_{\mathscr{H} \oplus \mathbb{C}} - K(\zeta I - X')^{-1}$ to $\mathbb{C}X_{12} \oplus \mathbb{C}$), is given, for $\zeta \in r(X_{11})$, $\zeta \neq iX_{22}$, by

$$\omega(\zeta, X', K) = 1 - \frac{((\zeta I - X_{11})^{-1} X_{12} \mid X_{12})}{\zeta - iX_{22}} = \frac{\phi(\zeta)}{\zeta - iX_{22}}.$$

Now let iX_{11} be self-adjoint, or equivalently, let iJX be self-adjoint. First, Proposition 6.2 implies that C is symmetric with respect to the imaginary axis. Therefore $\sigma(X)$ is also symmetric with respect to the imaginary axis. Proposition 6.2 then implies

PROPOSITION 8.4. *If iX_{11} is self-adjoint, one of the following two cases necessarily occurs*:

(1) $C' = \varnothing$ *and* $\sigma(X) \subset i\mathbb{R}$;

(2) *there exists one point* $c \in \Pi_r$ *such that* $\sigma(X) \setminus i\mathbb{R} = \{c, -\bar{c}\}$.

The fact that $X = X' + K$, where K is compact, implies that X and X' have the same essential spectrum. Since X' is self-adjoint, any $\zeta \in \sigma(X')$ belongs to the essential spectrum of X' unless ζ is an isolated eigenvalue of $\sigma(X')$ of finite multiplicity.

By Proposition 8.3, further investigation of the structure of $\sigma(X)$ can be restricted to the imaginary axis and carried out by direct inspection of the Weinstein–Aronszajn formula [6, Theorem 6.2, p. 247].

9

Let X be the infinitesimal generator of a strongly continuous linear semigroup T leaving the hermitian sesquilinear form $\mathfrak{a}$ invariant, i.e., satisfying condition (3.1).

For $p_0 \in \mathscr{D}(X)$ the initial value problem

$$\frac{dp(t)}{dt} = Xp(t) \qquad (t > 0) \tag{9.1}$$

$$p(0) = p_0$$

for a continuously differentiable function $p: \mathbb{R}_+ \to \mathscr{D}(X)$ has a unique solution, expressed by

$$p(t) = T(t)\, p_0 \tag{9.2}$$

for $t \geqslant 0$. If iJX is self-adjoint, the strongly continuous group T generated by X defines, by means of (9.2), the unique solution of the initial value problem (9.1) for all $t \in \mathbb{R}$.

Setting $p(t) = (x(t), \tau(t))$, $p_0 = (x_0, \tau_0)$ $(x(t), x_0 \in \mathscr{H}$ $\tau(t), \tau_0 \in \mathbb{C})$ then $t \mapsto x(t)$ and $t \mapsto \tau(t)$ are continuously differentiable maps $\mathbb{R}_+ \to \mathscr{H}$, $\mathbb{R}_+ \to \mathbb{C}$. By Theorem V, X is expressed by (5.15), where $X_{22} \in \mathbb{R}$, $X_{12} \in \mathscr{H}$, and iX_{11} is a closed symmetric operator on $\mathscr{H}$ whose resolvent set $r(X_{11}) \supset \Pi_r$. The initial value problem (9.1) is expressed by

$$\frac{dx(t)}{dt} = X_{11}x(t) + \tau(t)\, X_{12}, \tag{9.3}$$

$$\frac{d\tau(t)}{dt} = (x(t) \mid X_{12}) + i\tau(t)\, X_{22}, \qquad x(0) = x_0, \quad \tau(0) = \tau_0. \tag{9.4}$$

If $T(t)$ is represented by (3.2) then (9.2) becomes for $t \geqslant 0$

$$\begin{aligned} x(t) &= A(t)\, x_0 + \tau_0 \xi(t), \\ \tau(t) &= \left(x_0 \,\middle|\, \frac{1}{\overline{a(t)}} A(t)^* \, \xi(t) \right) + \tau_0 a(t). \end{aligned} \tag{9.5}$$

The dense space $\mathscr{D}(X) = \mathscr{D}(X_{11}) \oplus \mathbb{C}$ is complete for the norm

$$p \mapsto \|p\| + \|Xp\|. \tag{9.6}$$

Moreover $T(t)\, \mathscr{D}(X) \subset \mathscr{D}(X)$ for all $t \geqslant 0$, and the restriction of T to $\mathscr{D}(X)$ is a C_0 semigroup for the norm (9.6).

Similarly the dense space $\mathscr{D}(X_{11}) \subset \mathscr{H}$ is complete for the norm

$$|\,\|x\|\,| = \|x\| + \|X_{11}x\| \qquad (x \in \mathscr{D}(X_{11})).$$

Since $t \mapsto x(t)$ is continuously differentiable on $\mathbb{R}_+$ and $t \mapsto \tau(t)$ is continuous, given $t_0 \geqslant 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $t \geqslant 0$ and $|t - t_0| < \delta$, then by (9.3)

$$\|X_{11}(x(t) - x(t_0)) + (\tau(t) - \tau(t_0))\, X_{12}\| < \varepsilon$$

and

$$\|(\tau(t) - \tau(t_0))\, X_{12}\| = |\tau(t) - \tau(t_0)|\ \|X_{12}\| < \varepsilon.$$

Hence for $t \geqslant 0$ and $|t - t_0| < \delta$

$$\begin{aligned} \|X_{11}(x(t) - x(t_0))\| &\leqslant \|X_{11}(x(t) - x(t_0)) + (\tau(t) - \tau(t_0))\, X_{12}\| \\ &\quad + \|(\tau(t) - \tau(t_0))\, X_{12}\| < 2\varepsilon, \end{aligned}$$

and this proves

LEMMA 9.1. *For all* $x_0 \in \mathscr{D}(X_{11})$, $\tau_0 \in \mathbb{C}$, *the function* x: $t \mapsto x(t)$ *defined by* (9.5) *for* $t \geqslant 0$ *maps* $\mathbb{R}_+$ *into* $\mathscr{D}(X_{11})$ *and is continuous for the norm* $|\,\|\ \|\,|$.

The $\mathfrak{a}$-invariance of T yields

$$\|x(t)\|^2 - |\tau(t)|^2 = \|x_0\|^2 - |\tau_0|^2 \qquad \text{for} \quad t \geqslant 0.$$

Hence, if $\|x_0\| < |\tau_0|$, then $\|x(t)\| < |\tau(t)|$ for all $t \geqslant 0$. Setting $z_0 = (1/\tau_0)\, x_0$, $z(t) = (1/\tau(t))\, x(t)$, then $z_0 \in B \cap \mathscr{D}(X_{11})$ and z: $t \mapsto z(t)$ is a continuous map $\mathbb{R}_+ \to B \cap \mathscr{D}(X_{11})$ such that

$$z(0) = z_0. \tag{9.7}$$

Equations (9.3) and (9.4) imply that z is a continuously differentiable map of $\mathbb{R}_+$ into $\mathscr{H}$ and satisfies the Riccati equation

$$\frac{dz(t)}{dt} = X_{11} z(t) - ((z(t) \mid X_{12}) + iX_{22})\, z(t) + X_{12}. \tag{9.8}$$

The function z is expressed by

$$z(t) = \tilde{T}(t)\, z_0, \tag{9.9}$$

for $t \geqslant 0$, where $\tilde{T}(t)$ is given by (3.5).

Because for $t_0 \geqslant 0$ and $t \geqslant 0$

$$\begin{aligned}\|X_{11}(z(t)-z(t_0))\| &= \frac{1}{|\tau(t)\,\tau(t_0)|} \|\tau(t_0)\, X_{11}(x(t)-x(t_0)) \\ &\quad + (\tau(t_0)-\tau(t))\, X_{11} x(t_0)\| \\ &\leqslant \frac{1}{|\tau(t)|} \|X_{11}(x(t)-x(t_0))\| \\ &\quad + \left|\frac{1}{\tau(t)} - \frac{1}{\tau(t_0)}\right| \|X_{11} x(t_0)\|,\end{aligned}$$

the vact that $t \mapsto 1/\tau(t)$ is a continuous map of $\mathbb{R}_+$ into $\mathbb{C}\setminus\{0\}$ and Lemma 9.1 imply

LEMMA 9.2. *For $z_0 \in B$ the function $z: \mathbb{R}_+ \to B \cap \mathscr{D}(X_{11})$ expressed by (9.9) for $t \geqslant 0$ is a solution of the Riccati equation (9.8) with initial condition (9.7), which is continuous for the norm $| \;\|\;\;\|\; |$.*

It will be shown now that z is the unique solution of the initial value problem (9.8), (9.9) which is continuous for the norm $| \;\|\;\;\|\; |$.

More exactly the following theorem holds.

THEOREM VII. *For any $\gamma > 0$ and any choice of $z_0 \in B \cap \mathscr{D}(X_{11})$ the function $z: [0, \gamma] \to \mathscr{D}(X_{11})$ defined by (9.9) for $0 \leqslant t \leqslant \gamma$ is the unique continuously differentiable map of $[0, \gamma]$ into $\mathscr{H}$, with $z([0, \gamma]) \subset \mathscr{D}(X_{11})$ which is continuous for the norm $| \;\|\;\;\|\; |$ and satisfies the Riccati equation (9.8) with initial condition (9.7).*

Proof. Let $u: [0, \gamma] \to \mathscr{D}(X_{11})$ be a solution of (9.8) satisfying all the requirements stated in the theorem. The function $y: [0, \gamma] \to \mathscr{D}(X_{11})$ defined by $y(t) = \tau(t)\, u(t)$ is continuous for the norm $| \;\|\;\;\|\; |$. Moreover, y is a map of class C^1 of $[0, \gamma]$ into $\mathscr{H}$, and satisfies the equation

$$\frac{dy(t)}{dt} = X_{11}\, y(t) + ((x(t)-y(t)) \mid X_{12})\, u(t) + \tau(t)\, X_{12}, \tag{9.10}$$

with initial condition

$$y(0) = x(0) = x_0.$$

The function $w: t \mapsto w(t) = y(t) - x(t)$ is a map of $[0, \gamma]$ into $\mathscr{D}(X_{11})$ which is continuous for the norm $| \;\|\;\;\|\; |$. Furthermore w is a map of class C^1 of $[0, \gamma]$ into $\mathscr{H}$, and satisfies the evolution equation

$$\frac{dw(t)}{dt} = Z(t)\, w(t) \tag{9.11}$$

with initial condition

$$w(0) = 0, \tag{9.12}$$

where the linear operator $Z(t) = X_{11} + (\cdot \mid X_{12})\, u(t)$, with domain $\mathscr{D}(Z(t)) = \mathscr{D}(X_{11})$ is a perturbation of X_{11} by the bounded operator $(\cdot \mid X_{12})\, u(t)$, whose norm is

$$\|(\cdot \mid X_{12})\, u(t)\| = \|X_{12}\|\; \|u(t)\| \leqslant \|X_{12}\| \max\{\|u(t)\| : 0 \leqslant t \leqslant \gamma\}.$$

Since X_{11} generates a C_0 semigroup of contractions, and therefore defines a stable family of generators, then [7, Theorem 2.3, p. 132] $\{Z(t): 0 \leqslant t \leqslant \gamma\}$ is a stable family of generators of C_0 semigroups, with stability constants 1 and $\kappa = \|X_{12}\| \max\{\|u(t)\| : 0 \leqslant t \leqslant \gamma\}$. Because $u: [0, \gamma] \to \mathscr{H}$ is continuously differentiable, for any $x \in \mathscr{D}(X_{11})$ $t \mapsto Z(t)\, x$ is a continuously differentiable map of $[0, \gamma]$ into $\mathscr{H}$. Hence [7, Theorems 4.8, 4.3, pp. 145, 141] there exists a unique evolution system $\{U(t, s): 0 \leqslant s \leqslant t \leqslant \gamma\}$ such that

$$\|U(t, s)\| \leqslant e^{\kappa(t-s)} \qquad \text{for} \quad 0 \leqslant s \leqslant t \leqslant \gamma;$$

$$\frac{\partial^+}{\partial t} U(t, s)\, v\,|_{t=s} = Z(s)\, v \qquad \text{for} \quad v \in \mathscr{D}(X_{11}), \quad 0 \leqslant s \leqslant \gamma;$$

$$\frac{\partial}{\partial s} U(t, s)\, v = -U(t, s)\, Z(s)\, v \qquad \text{for} \quad v \in \mathscr{D}(X_{11}), \quad 0 \leqslant s \leqslant t \leqslant \gamma;$$[1]

$$U(t, s)\, \mathscr{D}(X_{11}) \subset \mathscr{D}(X_{11}) \qquad \text{for} \quad 0 \leqslant s \leqslant t \leqslant \gamma;$$

for $v \in \mathscr{D}(X_{11})$, $t \mapsto V(t, s)\, v$ is continuous in $\mathscr{D}(X_{11})$ for $0 \leqslant s \leqslant t \leqslant \gamma$ with respect to the norm $| \;\|\;\;\|\; |$;[2]

for every $v \in \mathscr{D}(X_{11})$, $w(t) = U(t, s)\, v$ is the unique solution of (9.11) on $[s, \gamma]$ with initial condition $w(s) = v$, which is continuous for the norm $| \;\|\;\;\|\; |$ on $\mathscr{D}(X_{11})$.

[1] The right derivative $\partial^+/\partial t$ and the derivative $\partial/\partial s$ are in the strong sense in $\mathscr{H}$.

[2] The norm which appears in Theorems 4.8 and 4.3 of [7] is not $| \;\|\;\;\|\; |$ but $\|\circ\| + \|Z(0)\circ\|$. However, these two norms are equivalent.

Hence $w = 0$ is the unique solution of (9.11) with initial condition (9.12) which is a continuously differentiable map of $[0, \gamma]$ into $\mathscr{H}$, whose values belong to $\mathscr{D}(X_{11})$ and which is continuous for the norm $|\,\|\;\|\,|$. Hence $y(t) = x(t)$ for all $t \in [0, \gamma]$. Q.E.D.

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