Strongly continuous one-parameter groups of holomorphic unit ball automorphisms

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24/09/2015, Ploiești

BASIC NOTATIONS

$\mathbf{H}, \langle\cdot|\cdot\rangle$ Hilbert space, $\mathbf{B} := \{x : \|x\| < 1\}$, $a^* := [x \mapsto \langle x|a\rangle]$

$x \oplus \xi \equiv \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$, $\quad \mathcal{L}(\mathbf{H} \oplus \mathbb{C}) \equiv \left\{ \left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] : \ A \in \mathcal{L}(\mathbf{H}),\ b, c \in \mathbf{H},\ d \in \mathbb{C} \right\}$

Unbounded linear operators

$A : \mathbf{S}_1 \to \mathbf{H}$, $c : \mathbf{S}_2 \to \mathbb{C}$ linear maps, $b \in \mathbf{H}, d \in \mathbb{C}$
$\mathcal{S}$ is a linear submanifold in $\mathbf{H} \oplus \mathbb{C}$,

$\mathcal{A} := \left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] \Big| \mathcal{S} : x \oplus \xi \mapsto (Ax + b) \oplus (c^* x + d)$

$\mathrm{dom}(\mathcal{A}) := \{x \oplus \xi \in \mathcal{S} : x \in \mathbf{S}_1 \cap \mathbf{S}_2\}$.

FRACTIONAL LINEAR TRANSFORMATIONS

$\mathcal{A} = \left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right]$

$F(\mathcal{A}) := \left[x \mapsto (c^*x + d)^{-1}(Ax + b)\right]$

$\mathrm{dom}\big(F(\mathcal{A})\big) = \{x : c^*x + d \neq 0\}$

Projective shifts:
$\mathcal{T}_x := \left[\begin{smallmatrix} I & x \\ 0 & 1 \end{smallmatrix}\right]$ $(x \in \mathbf{H})$.

Notice: $\mathcal{T}_x^{-1} = \mathcal{T}_{-x}$.

MÖBIUS TRANSFORMATIONS

For $c \in \mathbf{B}$,

$$P_c := \big[\text{orthogonal projection } \mathbf{H} \to \mathbb{C}c\big] = \|c\|^{-2}cc^* : x \mapsto \langle x|c\rangle c$$
$$\beta(c) := \sqrt{1 - \|x\|^2}$$
$$Q_c := P_c + \beta(c)\overline{P}_c \text{ with } \overline{P}_c := I - P_c$$

Möbius transformation
$$\Theta_c := \big[x \mapsto (1 + \langle x|c\rangle)^{-1}(Q_c x + c)\big] = F\big(\mathcal{M}_c\big)$$

lifted Möbius transformation
$$\mathcal{M}_c := \beta(c)^{-1}\left[\begin{smallmatrix} Q_c & c \\ c^* & 1 \end{smallmatrix}\right]$$

Notice: $\mathcal{M}_{-c} = \mathcal{M}_c^{-1}, \quad \Theta_{-c} = \Theta_c^{-1}.$

VESENTINI 1987-89

$\ell(x \oplus \xi) := \|x\|^2 - |\xi|^2$ skew metrics on $H \oplus \mathbb{C}$
$\mathfrak{U} := \Big[$str.cont. $\mathbb{C}$-lin. 1-pr semigroups of ℓ-isometries$\Big]$

$d_{\mathbf{B}} := \big[$Caratódory distance on $\mathbf{B}\big]$
$\mathfrak{U} := \Big[$str.cont. 1-pr semigroups of holomorphic δ-isometries$\Big]$
Theorem 1. $F^{\#} : \big[\mathcal{U}^t : t \in \mathbb{R}_+\big] \mapsto \big[F(\mathcal{U}^t)|\mathbf{B} : t \in \mathbb{R}_+\big]$ is $\mathfrak{U} \leftrightarrow \mathfrak{F}$

Theorem 2. Given $[\mathcal{U}^t : t \in \mathbb{R}_+] \in \mathfrak{U}$ with inf. generator $\mathcal{A}$,
for the corresponding non-linear objects $\Psi^t := F(\mathcal{U}^t)$ we have

$$\big\{p \in \mathbf{B} : t \mapsto \Psi^t(p) \text{ is differentiable}\big\} = \big\{x \in \mathbf{B} : \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathrm{dom}(\mathcal{A})\big\}$$

which is dense in the ball **B**.

NON-LINEAR GENERATORS

Theorem 2 $\longrightarrow$

infinitesimal generator for a non-linear semigroup

$[\Psi^t : t \in \mathbb{R}_+] = [F^{\#}\mathcal{U}^t : t \in \mathbb{R}_+]$ as the vector field

$$\Gamma(x) := \frac{d}{dt}\Big|_{t=0+} F\mathcal{U}^t \begin{bmatrix} x \\ 1 \end{bmatrix} = -\Big[\mathcal{A}\begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbb{C}} x + \Big[\mathcal{A}\begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbf{H}}$$

with $\quad x \in \mathrm{dom}(\Gamma) := \big\{x \in \mathbf{B} : \ \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathrm{dom}(\mathcal{A})\big\}.$

MÖBIUS-EQUIVALENCE

$\Psi \sim \Phi$ in $\mathrm{Aut}(\mathbf{B})$: $\quad \exists\, \Theta_{c_1}, \ldots, \Theta_{c_n}$ Möbius trf. with

$$\Psi = [\Theta_{c_1} \circ \cdots \circ \Theta_{c_n}] \circ \Phi[\Theta_{-c_n} \circ \cdots \circ \Theta_{-c_1}]$$

$[\Psi^t : t \in \mathbb{R}_+] \approx [\Phi^t : t \in \mathbb{R}_+]$: $\quad \exists\, \Theta_{c_1}, \ldots, \Theta_{c_n}$ Möbius trf. with

$$\Psi = [\Theta_{c_1} \circ \cdots \circ \Theta_{c_n}] \circ \Phi[\Theta_{-c_n} \circ \cdots \circ \Theta_{-c_1}]$$

Example. $\mathbf{H} = \mathbb{C}$, $[\Psi^t : t \in \mathbb{R}_+]$ (str.)cont 1-prsg in $\mathrm{Aut}(\mathbf{B}) \Longrightarrow \exists\, \lambda \in \mathbb{R}$
$[\Psi^t : t \in \mathbb{R}_+] \approx \Big\{$ either $\big[e^{i\lambda t} z : t \in \mathbb{R}_+\big]$ or $\big[\Theta_{\tanh(\lambda t)} : t \in \mathbb{R}_+\big]$
or $\big[\frac{1-i\lambda t}{1+i\lambda t} \Theta_{i\lambda t/(1-i\lambda t)} : t \in \mathbb{R}_+\big]$ $\Big\}$

DIFFERENTIABLE 0-ORBIT

Theorem 3 (Vesentini).
The orbit $t \mapsto \Psi^t(0)$ is differentiable $\iff$ the lin. gen. has the form

$$\mathcal{A} = \begin{bmatrix} iR + \nu I & b \\ b^* & \nu \end{bmatrix} \qquad \begin{array}{c} \nu \in \mathbb{C}, \quad r \in \mathbf{H} \\ iR = \big[\text{gen. of a str-cont. 1-prsg of } \mathbf{H}\text{-isometries}\ \big] \end{array}$$

Corollary. Taking $c \in \mathrm{dom}(\Gamma)$, $\Phi^t := \Theta_{-c} \circ \Psi^t \circ \Theta_c$, the orbit $t \mapsto \Phi^t(0)$

is differentiable Thus any str-cont. 1-prsg $[\Psi^t : t \in \mathbb{R}_+] \in \mathfrak{F}$ **Möbius equivalent** to a semigroup $[\Phi^t : t \in \mathbb{R}_+] \in \mathfrak{F}$ with inf. gen.

$$\Gamma(x) = b - \langle x | b \rangle x + iRx, \quad x \in \mathrm{dom}(R) \cap \mathbf{B}$$

Conversely: if $iR = \big[\text{gen. of a str-cont. 1-prsg of } \mathbf{H}\text{-isometries}\ \big]$, $b \in \mathbf{H}$, then, $\Gamma = \big[\text{gen. of a str-cont. 1-prsg of hol. Carathéodory } \mathbf{B}\text{-isometries}\ \big]$

JORDAN SETTING

KAUP 1978-83

Inf. gen. of **uniformly continuous one-parameter groups** of hol. Carathéodory isometries on **bounded symmetric domains**:

Up to a biholomorphic transformation, they are equivalent to transformation groups obtained by integration of vector fields of the form

$$x \mapsto b - \{xb^*x\} + iRx$$

with some JB*-triple product $\{xy^*z\}$ and JB*-hermitian op. R

Case of the unit ball of a Hilbert space: $\{xy^*z\} = \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x$

Kaup's formula for the integration of the vector fields $x \mapsto b - \{xb * x\}$:

$$\Theta_c(x) = c + B(c)^{1/2}\big[1 + L(x,c)\big]^{-1} x \qquad \begin{array}{c} B(c) := [\text{Bergman op.}], \\ L(x,c)z := \{xc^*z\} \end{array}$$

Remark: Kaup's method seems unsuitable in treating the cases $R \neq 0$.

BOUNDARY EXTENSIONS

Brau-Kaup-Upmeier, Vigué 1978
E Banach space **D** bounded circular domain
$\mathbf{D}_0 := \big\{\Psi(0) : \Psi \in \mathrm{Aut}(\mathbf{D})\big\}$ unit ball of a JB*-triple in $\mathbf{E}_0 := \mathbb{C}\mathbf{D}_0$

Partial JB*-triple $\big(\mathbf{E}, \mathbf{E}_0, \{\ldots\}_{\mathbf{D}}\big)$ (Complete axioms **Stachó 1991**)
$\mathrm{Aut}(\mathbf{D}) = \big\{\exp(b - \{zb^*z\}_{\mathbf{D}}) \circ \Lambda : b \in \mathbf{D},\ \Lambda \text{ lin. } \mathbf{D}\text{-unitary op.}\big\}$

Kaup's Möbius formula $\Longrightarrow$
If $\mathbf{E} = \mathbf{E}_0$ is a JB*-triple, $\mathbf{D} = \big[\text{unit ball}\big]$ then
$\Psi \in \mathrm{Aut}(\mathbf{D})$ extends holomorphically to $\rho_\Psi \mathbf{D}$, $\rho_\Psi := \tanh^{-1} \|\Psi(0)\|$

In particular, for $\Psi \in \mathrm{Aut}(\mathbf{B})$,

$$\overline{\Psi} := \big[\text{cont. extension of } \Psi \text{ to } \overline{\mathbf{B}}\big] \text{ well-def.}$$

STRONG CONTINUITY OF BOUNDARY EXTENSI0NS

Proposition. Let $\mathbf{K}$ be a domain in a Banach space $\mathbf{E}$ and let $f_t : \mathbf{D}_t \to \mathbf{E}$ $(t \in \mathbb{R}_+)$ be a family of holomorphic maps defined on open neighborhoods of $\overline{\mathbf{K}}$ such that the restrictions $[f_t|\mathbf{K} : t \in \mathbb{R}_+]$ form a strongly continuous one-parameter semigroup. Assume that for every boundary point $x \in \partial\mathbf{K}$ there exists a 1-dimensional complex disc Δ_x centered in x and intersecting $\mathbf{K}$ such that $\Delta_x \subset \bigcap_{t\in[0,\delta_x]} \mathbf{D}_t$ and $\bigcup_{t\in[0,\delta_x]} f_t(\Delta_x)$ is a bounded set for some $\delta_x > 0$. Then $[f_t|\overline{\mathbf{K}} : t \in \mathbb{R}]$ is also a strongly continuous one-parameter semigroup.

Corollary. If $\mathbf{E}$ is a JB*-triple, $\mathbf{K}$ is its open unit ball and $[f_t : t \in \mathbb{R}]$ is a strongly continuous one-parameter subgroup of $\mathrm{Aut}(\mathbf{K})$ then the maps $\overline{f}_t$ obtained with graph closure from the respective f_t, form a strongly continuous one-parameter group of maps $\overline{\mathbf{K}} \to \overline{\mathbf{K}}$.

FIXED POINTS

Lemma. $\mathcal{K}$ compact top., $[f_t: \ t \in \mathbb{R}_+]$ 1-prsg of cont. maps $\mathcal{K} \to \mathcal{K}$ admitting fixed points, all the functions $t \mapsto f_t(x)$ are continuous.
$\Longrightarrow \bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(f_t) \neq \emptyset$.

Lemma. **E** Banach space with predual, **K** bounded open convex subset in **E**, $[f_q: \ t \in \mathbb{Q}_+]$ 1-prsg of w*-continuous maps $f_t : \overline{\mathbf{K}} \to \overline{\mathbf{K}}$.
$\Longrightarrow \ \bigcap_{q\in\mathbb{Q}_+} \mathrm{Fix}(f_q) \neq \emptyset$.

Hayden-Suffridge-Vesentini. Hilbert case: For $\Psi \in \mathrm{Aut}(\mathbf{B})$, $\mathrm{Fix}(\overline{\Psi}) =$
$= \Big\{ \big[1 \text{ point in } \overline{\mathbf{B}}\big] \text{ or } \big[2 \text{ points in } \partial\mathbf{B}\big] \text{ or } \big[\dim{>}0 \text{ closed aff. subsp.}\big] \cap \overline{\mathbf{B}} \Big\}$

Corollary. Similar alternatives for $\bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\overline{\Psi^t})$ if $[\Psi^t : t \in \mathbb{R}_+] \in \mathfrak{F}$

LEMMA ON TRIANGULAR GENERATORS

Lemma. Let $\mathbf{E}_1, \mathbf{E}_2$ be Banach spaces,
$[W^t : t \in \mathbb{R}_+]$ str.cont. 1-prsg in $\mathcal{L}(\mathbf{E}_1)$ with infinitesimal generator G,
$\mathcal{G}_1 := \begin{bmatrix} G & 0 \\ G_1 & 0 \end{bmatrix}$, $\mathcal{G}_2 := \begin{bmatrix} G & G_2 \\ 0 & 0 \end{bmatrix}$ where $G_1 \in \mathcal{L}(\mathbf{E}_2, \mathbf{E}_1)$, $G_2 \in \mathcal{L}(\mathbf{E}_1, \mathbf{E}_2)$.
Then

$$\left[\begin{bmatrix} W^t & 0 \\ G_1 \int_0^t W^\tau d\tau & I \end{bmatrix} : t \in \mathbb{R}_+\right], \quad \left[\begin{bmatrix} W^t & \int_0^t W^\tau d\tau G_2 \\ 0 & I \end{bmatrix} : t \in \mathbb{R}_+\right]$$

are strongly continuous one-parameter subsemigroups of $\mathcal{L}(\mathbf{E}_1 \oplus \mathbf{E}_2)$ with the infinitesimal generators $\mathcal{G}_1, \mathcal{G}_2$, respectively.

FIXED POINTS AND TRIANGULARIZATION

Assume:
$[\Phi^t : t \in \mathbb{R}_+]$ str.cont. 1-prsg in $\mathrm{Iso}_{Carath}(\mathbf{B})$, $\quad \overline{x} \in \partial\mathbf{B} \cap \bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\overline{\Phi^t})$
$\mathrm{gen}[\Phi^t : t \in \mathbb{R}_+] = [b - \langle x|b\rangle + iRx]\partial/\partial x$.

That is
$\Phi^t = F(\mathcal{U}^t)$, $[\mathcal{U}^t : t \in \mathbb{R}_+]$ str.cont. 1-prsg in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$,
$\mathcal{R} := \mathrm{gen}[\mathcal{U}^t : t \in \mathbb{R}_+] = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix}, \quad \mathrm{dom}(\mathcal{R}) = \mathrm{dom}(R) \oplus \mathbb{C}$
$iR = \mathrm{gen}[W^t : t \in \mathbb{R}_+]$ str.cont. 1-prsg of **isometries** in $\mathcal{L}(\mathbf{H})$,
$\overline{x} \oplus 1 = \left[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\right]$ joint eigenvector $[\mathcal{U}^t : t \in \mathbb{R}_+]$, $\quad \exists\, \nu \in \mathbb{C}$,

$$\overline{x} \in \mathrm{dom}(R), \quad \mathcal{R}[\overline{x} \oplus 1] = \nu[\overline{x} \oplus 1], \quad b = \nu\overline{x} - iR\overline{x}, \quad \nu = \langle \overline{x}|b\rangle$$

FIXED POINTS AND TRIANGULARIZATION (cont.)

Triangularization with the projective shift $\mathcal{T} : x \oplus \xi \mapsto (x + \xi\overline{x}) \oplus \xi$

$$\mathcal{V}^t := e^{-\nu t}\mathcal{T}^{-1}\mathcal{U}^t\mathcal{T}, \quad \mathcal{B} = \mathcal{T}^{-1}\mathcal{A}\mathcal{T}$$

$\mathcal{B} = \mathrm{gen}[\mathcal{V}^t : t \in \mathbb{R}_+]$ bded perturb of $(iR) \oplus 1$, $\mathrm{dom}(\mathcal{B}) = \mathrm{dom}(R) \oplus \mathbb{C}$.

$$\mathcal{B} = \begin{bmatrix} I & -\overline{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} iR - \nu I & b \\ b^* & -\nu \end{bmatrix} \begin{bmatrix} I & \overline{x} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B & 0 \\ b^* & 0 \end{bmatrix}, \qquad \begin{aligned} B &:= iR - \nu I - \overline{x}b^* \\ &= \mathrm{gen}[V^t : t \in \mathbb{R}_+] \end{aligned}$$

Lemma of Triangular Generators $\Longrightarrow$

$$\mathcal{V}^t = \begin{bmatrix} V^t & 0 \\ b^* \int_0^\tau V^\tau d\tau & 1 \end{bmatrix}$$

SECOND TRIANGULARIZATION

Quadrature formula for V^t with space decomposition

$$\mathbf{H}_0 := \mathbf{H} \ominus (\mathbb{C}\overline{x}), \quad P := \mathrm{P}_{\mathbb{C}\overline{x}} = \overline{x}\,\overline{x}^*, \quad P_0 := \mathrm{P}_{\mathbf{H}_0} = I - P,$$

By setting $\lambda := \mathrm{Re}(\nu)$, $\mu := \mathrm{Im}(\nu) = \langle \overline{x} \big| R\overline{x} \rangle / 2$, $S := R - \mu I$,
$b_0 := P_0 b = -iP_0 R\overline{x} = b - \nu\overline{x}$, $S_0 := P_0 S P_0 \big| \mathbf{H}_0$, $I_0 := \mathrm{Id}_{\mathbf{H}_0}$

in terms of $\mathbf{H}_0 \oplus (\mathbb{C}\overline{x})\big]$-matrices,

$$B = \begin{bmatrix} P_0[iS - \lambda I] \big| \mathbf{H}_0 & iP_0 S\overline{x} \\ 0 & -2\lambda \end{bmatrix} = \begin{bmatrix} iS_0 + \lambda I_0 & -b_0 \\ 0 & 0 \end{bmatrix} - 2\lambda \begin{bmatrix} I_0 & 0 \\ 0 & 1 \end{bmatrix}$$

$P_0[iS]P_0 = \big[\text{bded } iR\text{-pert}\big]$, $\Rightarrow iS_0 \big| \mathbf{H}_0 = \mathrm{gen}[V_0^t : t \in \mathbb{R}_+]$ $\mathbf{H}_0$-isometries

Lemma of Tri. Gen. $\Longrightarrow$

$$V^t = e^{-2\lambda t} \begin{bmatrix} e^{\lambda t} V_0^t & \int_0^t e^{\lambda\tau} V_0^\tau b_0 \, d\tau \\ 0 & 1 \end{bmatrix}$$

MAIN RESULT

Theorem. Assume $[\Phi^t : t \in \mathbb{R}_+]$ str.cont. 1-prsg in $\mathrm{Iso}_{Carath}(\mathbf{B})$,

$$\mathrm{gen}[\Phi^t : t \in \mathbb{R}_+] = [b - \langle x|b\rangle x + iR]\partial/\partial x, \quad \overline{x} \in \partial\mathbf{B} \cap \bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\overline{\Phi^t}).$$

Then, with $P := \mathrm{P}_{\mathbb{C}\overline{x}}$, $P_0 := I - P$, $\mathbf{H}_0 := P_0\mathbf{H}$, we have

$$\overline{x} \in \mathrm{dom}(R), \quad iP_0R\Big|\mathbf{H}_0 = \mathrm{gen}[U_0^t : \ t \in \mathbb{R}_+] \text{ of } \mathbf{H}_0\text{-isometries.}$$

There exist constants $\lambda, \mu \in \mathbb{R}$ such that, by setting

$$S := R - i\mu I, \quad S_0 := P_0S\Big|\mathbf{H}_0, \quad V_0^t := e^{i\mu t}U_0^t, \quad b_0 := P_0(iSb),$$

for every point $x \in \mathbf{B}$ with $x_0 := P_0x$ and $Px = \xi\overline{x}$ we can write

$$P\Phi^t(x) = \Big[\varphi_{\lambda,\mu}(t, x_0, \xi)^{-1}(\xi - 1)e^{-2\lambda t} + 1\Big]\overline{x},$$

$$P_0\Phi^t(x) = \varphi_{\lambda,\mu}(t, x_0, \xi)^{-1}\Big[e^{-\lambda t}V_0^t x_0 - (\xi - 1)e^{-2\lambda t}\big(\textstyle\int_0^t e^{\lambda\tau}V_0^\tau \, d\tau\big)\, b_0\Big],$$

$$\varphi_{\lambda,\mu}(t, x_0, \xi) := \Big\langle\big(\textstyle\int_0^t e^{-\lambda\tau}V_0^\tau d\tau\big)x_0\Big|b_0\Big\rangle + (\xi - 1)\big(\lambda + i\mu\big)e^{-2\lambda t} + 1 -$$

$$-(\xi - 1)\Big\langle\big(\textstyle\int_0^t e^{-2\lambda\tau}\int_0^\tau e^{\lambda\sigma}V_0^\sigma d\sigma d\tau\big)b_0\Big|b_0\Big\rangle \ .$$

CASE OF TWO BOUNDARY FIXED POINTS

Corollary. Each str.cont. 1-prsg $[\Psi^t : t \in \mathbb{R}_+]$ of hol. Carathéodory **B**-isometries with exactly two joint boundary fixed points is Möbius equivalent to some semigroup $[\Phi^t : t \in \mathbb{R}_+]$ of the form

$$P\Phi^t(x) = \frac{(1-\lambda)(1-\xi)e^{-2\lambda t} + 1}{1-(1-\xi)\lambda e^{-2\lambda t}}\overline{x},$$

$$P_0\Phi^t(x) = \frac{e^{-\lambda t}}{1-(1-\xi)\lambda e^{-2\lambda t}} V_0^t x_0 \ .$$

Remark. It seems that no analogous Möbius equivalent simplification is available in the case of a unique common fixed point lying in the boundary of the unit ball.

EXAMPLES

1-DIM ($\sim$folklore) $\quad$ $\mathbf{B} = \{\eta \in \mathbb{C} : |\zeta| < 1\}$,

$L_u : \zeta \mapsto u\zeta, \quad M_a : \zeta \mapsto (\zeta + a)/(1 + \overline{a}\zeta)$

Alternatives: For some $\lambda \in \mathbb{R}$,
$[\Psi^t : t \in \mathbb{R}]$ str. cont. 1-**prg** *Möbius equivalent* to

a) $\left[L_{e^{i\lambda t}} : \ t \in \mathbb{R}\right]$,

b) $\left[M_{\tanh(\lambda t)} : t \in \mathbb{R}\right]$,

c) $\left[\exp\left(t\lambda\left(i(1-z)^2\right)\frac{\partial}{\partial z}\right) : \ t \in \mathbb{R}\right] = \left[L_{\frac{1-i\lambda t}{1+i\lambda t}} \circ M_{\frac{i\lambda t}{1-i\lambda t}} : \ t \in \mathbb{R}\right]$.

EXAMPLES

$2 \le N < \infty$ DIM. ($\sim$NEW)
$\mathbf{B} = \left\{ (\zeta_1, \dots, \zeta_N)^{\mathrm{T}} \in \mathbb{C}^N : \sum_k |\zeta_k|^2 < 1 \right\}$

For some $\lambda, \mu \in \mathbb{R}$,

$$S_0 = \mathrm{diag}(\sigma_1, \dots, \sigma_{N-1}) \text{ real},$$
$$b_0 = (\beta_1, \dots, \beta_N)^{\mathrm{T}} \in \mathbb{C}^{N-1},$$

$[\Psi^t : t \in \mathbb{R}]$ str. cont. 1-**prg** *Möbius equivalent* to

$$\Phi^t = \mathrm{Fractional}\,\mathsf{exp} \left(t \begin{bmatrix} i(S_0 + \mu) & -b_0 & b_0 \\ b_0^* & 2i\mu & \lambda - i\mu \\ b_0^* & \lambda + i\mu & 0 \end{bmatrix} \right) \qquad (t \in \mathbb{R}),$$

Φ^t can be written in a $\mathbb{C} \times \mathbb{C}^{N-1}$ split form
with the functions $f_1, f_2, \phi_{\lambda,\mu}$ of the Dilation Theorem next.

DILATION

$\exists$ unitary dilation $[\widehat{V}_0^t : r \in \mathbb{R}]$ of the isometry semigroup $[V_0^t : t \in \mathbb{R}_+]$:

Thm. $\exists$ a str. cont. 1-**prg** $[\widehat{\Psi}^t : t \in \mathbb{R}_+]$ of **surjective** hol. Carathéodory isom. of the unit ball $\widehat{\mathbf{B}}$ of some Hilbert space $\widehat{\mathbf{H}} \supset \mathbf{H}$ (as subspace) such that $\Psi^t = \widehat{\Psi}^t\big|\mathbf{B}$ $(t \in \mathbb{R}_+)$.

Functional calculus of the skew self-adj. gen. $i\widehat{S}_0$ of $[\widehat{V}_0^t : t \in \mathbb{R}]$, $\Longrightarrow$

Cor. In the main theorem

$$\varphi_{\lambda,\mu}(t, x_0, \xi) = \Big\langle x_0 \Big| f_1(t, -\lambda, -\widehat{S}_0) b_0 \Big\rangle - (\xi - 1)\Big\langle \big(f_2(t, \lambda, \widehat{S}_0) b_0 \Big| b_0 \Big\rangle +$$

$$+(\xi - 1)\big(\lambda + i\mu\big) e^{-2\lambda t} + 1,$$

$$P_0\Phi^t(x) = \varphi_{\lambda,\mu}(t, x_0, \xi)^{-1}\Big[e^{-\lambda t}\exp(it\widehat{S}_0)x_0 - (\xi - 1)e^{-2\lambda t} f_1(t, \lambda, \widehat{S}_0) b_0\Big].$$

with the bded anal. functions $f_j(t, \lambda, \cdot) : \mathbb{R} \to \mathbb{C}$ $(j = 1, 2;\ \lambda, t \in \mathbb{R})$

$$f_1(t,\lambda,\mathbf{s}) := \frac{e^{-t(\lambda+i\mathbf{s})}-1}{\lambda+i\mathbf{s}} = t\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}\big[t(\lambda+i\mathbf{s})\big]^k,$$

$$f_2(t,\lambda,\mathbf{s}) := \frac{e^{-2\lambda t}(\lambda-i\mathbf{s})-2\lambda e^{-t(\lambda-i\mathbf{s})}+i\mathbf{s}}{2\lambda(\lambda^2+\mathbf{s}^2)(\lambda-i\mathbf{s})} =$$

$$= \sum_{n=0}^{\infty}\Big[\sum_{k=0}^{n}\frac{(-2\lambda)^{n-k}}{(n-k)!}\,\frac{(\lambda+i\mathbf{s})^k}{(k+1)!}\Big]\frac{t^{n+2}}{n+2}.$$

OPEN PROBLEMS

- No Hille-Yosida type theory as far for the generators of strongly continuous **non-linear** transformation groups

- *Given a str.cont 1-prsg* $[\Psi^t : t \in \mathbb{R}_+]$ *of Carathéodory isom. of the unit ball* $\mathbf{B}$ *of JB*-triple* $(\mathbf{E}, \{\ldots\})$*, are the orbits* $t \mapsto \Psi^t(x)$ *differentiable for a dense subset in* $\mathbf{B}$*?*

- In particular, if the answer is affirmative and the orbit $t \mapsto \Psi^t(0)$ is differentiable, can we write

$$\left.\frac{d\Psi^t(x)}{dt}\right|_{t=0+} = b - \{xb^x\} + iRx$$

with some generator iR of a str.cont isom. 1-prsg of $\mathbf{E}$?

- Analogous questions for the partial JB*-triples of bounded circular domains.

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- *Given a str.cont 1-prsg* $[\Psi^t : t \in \mathbb{R}_+]$ *of Carathéodory isom. of the unit ball* $\mathbf{B}$ *of JB*-triple* $(\mathbf{E}, \{\ldots\})$*, are the orbits* $t \mapsto \Psi^t(x)$ *differentiable for a dense subset in* $\mathbf{B}$*?*

- In particular, if the answer is affirmative and the orbit $t \mapsto \Psi^t(0)$ is differentiable, can we write

$$\frac{d\Psi^t(x)}{dt}\Big|_{t=0+} = b - \{xb^x\} + iRx$$

with some generator iR of a str.cont isom. 1-prsg of $\mathbf{E}$?

- Analogous questions for the partial JB*-triples of bounded circular domains.

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ACKNOWLEDGEMENTS

Our related research was supported by the projects

TAMOP-4.2.2.A-11/1/KONV-2012-0060
Impulse Lasers for Use in Materials Science and Biophotonics
of the European Union and co-financed by the European Social Fund.

and

OTKA K 109782 — Global Dynamics of Differential Equations
of the Hungarian Academy of Sciences