Throughout this work $\mathbf{H}$ denotes an arbitrarily fixed complex Hilbert space with the scalar product $\langle x|y\rangle$ which is linear in x and conjugate linear in y, giving rise to the norm $\|x\| = \langle x|x\|^{1/2}$. For any vector $c \in \mathbf{H}$, we shall write $c^* := [x \mapsto \langle x|a\rangle]$ for its associated functional. The open ball of radius $\rho > 0$ and centered in 0 in $\mathbf{H}$ will be denoted with $\rho\mathrm{Ball}(\mathbf{H})$, and we write $\rho\overline{\mathrm{Ball}}(\mathbf{H})(= \{x : \ \|x\| \le \rho\})$ for its closure.

Recall [3, Ch. VI] that the group of $\mathrm{Aut}\,\mathrm{Ball}(\mathbf{H})$ all holomorphic automorphisms of $\mathrm{Ball}(\mathbf{H})$ admits a matrix representation in the following sense: Each element Ψ of $\mathrm{Aut}\,\mathrm{Ball}(\mathbf{H})$ has the fractional linear form

$$(R) \qquad \Psi(x) = \frac{Ax+b}{\langle x|c\rangle + d}\ , \quad A \in \mathcal{L}(\mathbf{H}),\ b,c \in \mathbf{H},\ d \in \mathbb{C}$$

and we have

$$\Psi_1 \circ \Psi_2(x) = \Psi_1\big(\Psi_2(x)\big) = \frac{Ax+b}{\langle x|c\rangle + d} \quad \text{whenever} \quad \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} = \begin{bmatrix} A_1 & b_1 \\ c_1^* & d_1 \end{bmatrix}\begin{bmatrix} A_2 & b_2 \\ c_2^* & d_2 \end{bmatrix}.$$

This representation is unique up to a constant, since in (R) we necessarily have

$$\begin{bmatrix} A & b \\ c^* & d \end{bmatrix} = d\begin{bmatrix} (\mathrm{P}_a + \beta\overline{\mathrm{P}}_a) & a \\ a^* & 1 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{where} \qquad \begin{array}{l} a := \Psi(0),\ \beta = \sqrt{1 - \|a\|^2}, \\ U = (\beta^2\mathrm{P}_a + \beta\overline{\mathrm{P}}_a)^{-1}\Psi'(0) \end{array}$$

in terms of the projections P_a onto $\mathbb{C}a$ resp. $\overline{\mathrm{P}}_a := 1 - \mathrm{P}_a$ and the Fréchet derivative Ψ'. Any automorphism $\Psi \in \mathrm{Aut}\,\mathrm{Ball}(\mathbf{H})$ extends holomorphically to $(1 - \|\Psi(0)\|)^{-1}\mathrm{Ball}(\mathbf{H})$ and hence admits a continuous extension $\overline{\Psi}$ to the closed unit ball $\overline{\mathrm{Ball}}(\mathbf{H})$. We write $\mathrm{Aut}\,\overline{\mathrm{Ball}}(\mathbf{H}) := \{\overline{\Psi} : \ \psi \in \mathrm{Aut}\,\mathrm{Ball}(\mathbf{H})\}$ for their collection.

Recall [3, Ch.VI] that any automorphism $\overline{\Psi} \in \mathrm{Aut}\,\overline{\mathrm{Ball}}(\mathbf{H})$ is weakly continuous and preserves the family $\mathrm{Aff}(\overline{\mathrm{Ball}}(\mathbf{H}))$ of all complex affine* closed subspaces intersected with $\overline{\mathrm{Ball}}(\mathbf{H})$. By Schauder's fixed point theorem, $\mathrm{Fix}(\overline{\Psi}) \neq \emptyset$, since $\overline{\mathrm{Ball}}(\mathbf{H})$ is weakly compact. Moreover we have the following alternatives:

(1) $\mathrm{Fix}(\overline{\Psi}) \in \mathrm{Aff}(\overline{\mathrm{Ball}}(\mathbf{H}))$, (2) $\mathrm{Fix}(\overline{\Psi})$ consists of two boundary points.

In case (2) from the proof of [3, Thm.VI.4.8] we see even that $\overline{\Psi} = \overline{\Phi} \circ \overline{\Theta}_a \circ \overline{\Phi}^{-1}$ with a suitable automorphism $\Phi \in \mathrm{Aut}\,\overline{\mathrm{Ball}}(\mathbf{H})$ and a Möbius shift

$$(M) \qquad \Theta_a : x \mapsto \frac{\mathrm{P}_a x + a + \beta_a \overline{\mathrm{P}}_a x}{1 + \langle x|a\rangle}, \qquad \beta_a = \sqrt{1 - \|a\|^2}$$

for some $0 \neq a \in \mathrm{Ball}(\mathbf{H})$ such that $\mathrm{Fix}(\overline{\Theta})_a = \{-e, e\}$ where $e := a/\|a\|$.

The next result is an infinite dimensional extension for a simple special case of a far reaching theorem of Abate [4] established for finite dimensional uniformly convex domains. It

* If $x = \sum_{k=1}^2 \lambda_k x_k$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\sum_{k=1}^2 \lambda_k = 1$ then $\overline{\Psi}(x) = \sum_{k=1}^2 \alpha_k \overline{\Psi}(x_k)$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$ with $\sum_{k=1}^2 \alpha_k = 1$ $\big($namely with $\alpha_k = \lambda_k[1 + \langle x_k|U^*a\rangle]/[1 + \langle \lambda_1 x_1 + \lambda_2 x_2|U^*a\rangle]\big)$.

seems that Dineen's bidual embedding method [5] is suitable in proving a complete infinite dimensional analogy with uniformly convex domains in Banach spaces. Instead, below we give a short direct proof on the basis of the special algebraic form (R).

Lemma. *Abelian subsets of* $\mathrm{Aut}\,\overline{\mathrm{Ball}}(\mathbf{H})$ *admit common fixed points.*

Proof. Assume $\{\overline{\Psi}_j : j \in \mathcal{J}\} \subset \mathrm{Aut}\,\overline{\mathrm{Ball}}(\mathbf{H})$ with $\overline{\Psi}_j \circ \overline{\Psi}_k = \overline{\Psi}_k \circ \overline{\Psi}_j$ $(j,k \in \mathcal{J})$. By weak continuity, for any index family $\mathcal{K} \subset \mathcal{J}$, the set $\bigcap_{k\in\mathcal{K}} \mathrm{Fix}(\overline{\Psi}_k)$ of common fixed points is weakly compact. Thus, according to Riesz' intersection theorem, it suffices to see that $\bigcap_{k\in\mathcal{K}} \mathrm{Fix}(\overline{\Psi}_k) \neq \emptyset$ for finite index families $\mathcal{K}$. By proceeding to contradiction, let $\mathcal{K} = \{k_1, \dots, k_N\}$ be a subset of $\mathcal{J}$ with minimal cardinality such that $\bigcap_{k\in\mathcal{K}} \mathrm{Fix}(\overline{\Psi}_k) = \emptyset$. Necessarily $N > 1$ and $\mathbf{S} := \bigcap_{n=1}^{N-1} \mathrm{Fix}(\overline{\Psi}_{k_n}) \neq \emptyset$ is either a weakly compact convex subset of $\overline{\mathrm{Ball}}(\mathbf{H})$ or it consists of two boundary points. Since $\overline{\Psi}_{k_N}$ commutes with all the maps $\overline{\Psi}_{k_n}$ $(n < N)$, we have $\overline{\Psi}_{k_N}(\mathbf{S}) \subset \mathbf{S}$. Indeed, if $x \in \mathbf{S}$ then $\overline{\Psi}_{k_n}\big(\overline{\Psi}_{k_N}(x)\big) = \overline{\Psi}_{k_N}\big(\overline{\Psi}_{k_n}(x)\big) = \overline{\Psi}_{k_N}(x) \Rightarrow \overline{\Psi}_{k_N}(x) \in \mathrm{Fix}(\overline{\Psi}_{k_n})$ $(n < N)$. Hence Schauder's fixed point theorem excludes the case of $\mathbf{S}$ being convex. Suppose $\mathbf{S} = \{p,q\} \subset \partial\mathrm{Ball}(\mathbf{H})$. Then necessarily $\mathrm{Fix}(\overline{\Psi}_{k_M}) = \{p,q\}$ for some index $M < N$ and $\overline{\Psi}_{k_N} : p \leftrightarrow q$. However, in this case we can write $\overline{\Psi}_{k_N} = \overline{\Phi}\circ\overline{\Theta}_a\circ\overline{\Phi}^{-1}$ with a suitable automorphism $\overline{\Phi} \in \mathrm{Aut}\,\overline{\mathrm{Ball}}(\mathbf{H})$ and a Möbius shift $\overline{\Theta}_a$ of the form (M) where $0 \neq a \in \mathrm{Ball}(\mathbf{H})$. Then, by setting $\overline{\Omega} := \overline{\Phi}^{-1}\circ\overline{\Psi}_{k_M}\circ\overline{\Phi}$ and $e := a/|a\|$, we have $\{\pm e\} = \mathrm{Fix}(\overline{\Theta}_a)$ and $\overline{\Omega} : e \leftrightarrow -e$. On the other hand, it is immediate that $e = \lim_{n\to\infty} \overline{\Theta}_a^n(x)$ for every point $x \in \overline{\mathrm{Ball}}(\mathbf{H}) \setminus \{e\}$. Taking any point $f \in \mathrm{Fix}(\overline{\Omega})$ we get the contradiction $e = \lim_{n\to\infty} \overline{\Theta}_a^n(f) = \lim_{n\to\infty} \overline{\Theta}_a^n \circ \overline{\Omega}(f) = \lim_{n\to\infty} \overline{\Omega} \circ \overline{\Theta}_a^n(f) = \overline{\Omega}(e) = -e$. Q.e.d.

Henceforth we keep fixed the notation $\big[\Psi^t : t \in \mathbb{R}\big]$ for an arbitrarily given strongly continuous one-parameter group in $\mathrm{Aut}\,\mathrm{Ball}(\mathbf{H})$. That is

$$(P) \qquad \Psi^t(x) = \frac{(P_t + \beta_t \overline{P}_t)U^t x + a_t}{1 + \langle U_t x | a_t \rangle} = \big[\Theta_{a_t} \circ U_t\big](x)$$

where $a_t = \Psi_t(0) \in \mathrm{Ball}(\mathbf{H})$, $U_t \in \mathcal{L}(\mathbf{H})$ is a unitary operator and, with the notations in (M), $\beta_t := \beta_{a_t}, P_t := \mathrm{P}_{a_t}, \overline{P}_t := \overline{\mathrm{P}}_{a_t}$ for short. In particular $P_t x = [\langle x|a_t\rangle / \langle a_t|a_t\rangle] a_t$, $\overline{P}_t = I - P_t$. According to the previous lemma, the continuous extensions $\overline{\Psi}^t$ $(t \in \mathbb{R})$ to $\overline{\mathrm{Ball}}(\mathbf{H})$ have a common fixed point which we shall denote with $\overline{x}$:

$$\overline{x} \in \overline{\mathrm{Ball}}(\mathbf{H}), \qquad \overline{\Psi}^t(\overline{x}) = \overline{x} \quad (t \in \mathbb{R}).$$

Remark. In finite dimensions, it is customary to normalize (R) by requiring $\det \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} = 1$. Thus, in case of $\dim(\mathbf{H}) = N$, in this manner one can establish a canonical identification of $\mathrm{Aut}\,\mathrm{Ball}(\mathbf{H})$ with a subgroup of the classical matrix group $\mathrm{SL}(N+1)$. Though in infinite dimensions such a normalization is not available, for one-parameter groups with common fixed point there an alternative way as follows.

Proposition. *With the notations (P), the (R)-type representation matrices*

$$\widehat{\Psi}^t := \frac{1}{1+\langle U_t\overline{x}|a_t\rangle}\begin{bmatrix}(P_t+\beta_t\overline{P}_t)U_t & a_t\\ [U_t^*a_t]^* & 1\end{bmatrix} = \frac{1}{1+\langle U_t\overline{x}|a_t\rangle}\begin{bmatrix}P_t+\beta_t\overline{P}_t & a_t\\ a_t^* & 1\end{bmatrix}\begin{bmatrix}U_t & 0\\ 0 & 1\end{bmatrix}$$

of $\big[\Psi_t:\ t\in\mathbb{R}\big]$ *form a strongly continuous one-parameter group of operators in* $\mathbf{H}\oplus\mathbb{C}$.

Proof. Since $\Psi^t\circ\Psi^s=\Psi^{t+s}$ $(t,s\in\mathbb{R})$, for the representation matrices we have $\widehat{\Psi}^t\widehat{\Psi}^s = d_{t,s}\widehat{\Psi}^{t+s}$ with suitable constants $d_{t,s}\in\mathbb{C}$. The fixed point property $\overline{\Psi}^t(\overline{x})=\overline{x}$ implies

$$\widehat{\Psi}^t\begin{bmatrix}\overline{x}\\ 1\end{bmatrix}=\begin{bmatrix}\overline{x}\\ 1\end{bmatrix}\qquad (t\in\mathbb{R}).$$

Hence necessarily $d_{t,s}=1$ $(t,s\in\mathbb{R})$, thus $\big[\widehat{\Psi}^t : t\in\mathbb{R}\big]$ is a one-parameter matrix group.
By assumption, the function $t\mapsto a_t=\Psi^t(0)$ is norm-continuous $\mathbb{R}\to\mathrm{Ball}(\mathbf{H})$. Hence we can deduce the strong continuity of the $\mathbf{H}$-unitary operator valued function $t\mapsto U_t$. Namely consider any vector $x\in\mathbf{H}$. To establish the norm-continuity of the function $t\mapsto U_t$, we may assume without loss of generality that $x\in\mathrm{Ball}(\mathbf{H})$. Then, by the aid of the Möbius shifts (M) we can write

$$U_tx=\big[\Theta_{a_t}^{-1}\circ\Psi_t\big](x)=\Theta_{-a_t}\big(\Psi(x)\big)\qquad (t\in\mathbb{R}).$$

Observe that the norm continuity of $t\mapsto a_t$ implies the continuity of $t\mapsto\langle x|a_t\rangle$ and $t\mapsto\beta_t\in[0,1)$ entailing the norm-continuity of $t\mapsto P_t+\beta_t\overline{P}_t\in\overline{\mathrm{Ball}}\big(\mathcal{L}(\mathbf{H})\big)$. Hence the required norm-continuity of $t\mapsto U_tx=(1-\langle x|a_t\rangle)^{-1}[(P_tx-a_t+\beta_t\overline{P}_tx]$ is immediate. In general, the product of two bounded strongly continuous linear operator valued functions $\mathbb{R}\to\mathcal{L}(\mathbf{X})$ over a normed space $\mathbf{X}$ is strongly continuous. Hence we conclude that the entries $(1,1),(1,2),(2,1)$ resp. $(2,2)$ of the matrices $\widehat{\Psi}^t$ are strongly continuous functions $\mathbb{R}\to\mathcal{L}(\mathbf{H})$, $\mathbb{R}\to\mathbf{H}$, $\mathbb{R}\to\mathbf{H}^*\simeq\mathbf{H}$ resp. $\mathbb{R}\to\mathbb{R}$ which completes the proof. Q.e.d.

Remark. It is worth to notice that the term $\langle U_t\overline{x}|a_t\rangle$ is actually independent of u_t as

$$(Ux)\qquad \langle U_t\overline{x}|a_t\rangle=\frac{\langle\overline{x}-a_t|a_t\rangle}{1-\langle\overline{x}|a_t\rangle},\qquad \widehat{\Psi}^t=\frac{1-\langle\overline{x}|a_t\rangle}{1-\langle a_t|a_t\rangle}\begin{bmatrix}P_t+\beta_t\overline{P}_t & a_t\\ a_t^* & 1\end{bmatrix}\begin{bmatrix}U_t & 0\\ 0 & 1\end{bmatrix}.$$

Proof: In general we have $P_ty=\langle y|a_t\rangle\langle a_t|a_t\rangle^{-2}a_t$ $(0\neq a_t,y\in\mathbf{H})$. It follows $\langle P_tU_t\overline{x}|a_t\rangle = \langle U_t\overline{x}|a_t\rangle$ with $\langle\overline{P}_tU_t\overline{x}|a_t\rangle=0$ for any $t\in\mathbb{R}$. Thus multiplying the fixed point equation $\overline{x}=\overline{\Psi}^t(\overline{x})=(1+\langle U_t\overline{x}|a_t\rangle)^{-1}(P_t+\beta_t\overline{P}_t)U_t\overline{x}$ with $|a_t\rangle$, we get $(1+\langle U_t\overline{x}|a_t\rangle)^{-1}\langle U_t\overline{x}+a_t|a_t\rangle = \langle\overline{x}|a_t\rangle$ whence the relations (Ux) are immediate.

Recalling the Hille–Yosida theorem [6, Kap.10] on strongly continuous one-parameter groups of bounded linear operators in Banach spaces, the above proposition ensures the existence of a not necessarily bounded linear map $\mathcal{A} : \mathcal{D} \to \mathbf{H} \oplus \mathbb{C}$ (the infinitesimal generator of $\big[\widehat{\Psi}^t : t \in \mathbb{R}\big]$) such that

$$(Aa)\qquad \begin{aligned} &\mathcal{D} = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbf{H} \oplus \mathbb{C} : \frac{d}{dt}\Big|_{t=0} \widehat{\Psi}^t \begin{bmatrix} x \\ \xi \end{bmatrix} \text{ exists} \right\}, \\ &\frac{d}{dt} \widehat{\Psi}^t = \mathcal{A} \widehat{\Psi}^t = \widehat{\Psi}^t \mathcal{A} \qquad \text{on } \mathcal{D}, \\ &\mathcal{A} \begin{bmatrix} x \\ \xi \end{bmatrix} = \frac{d}{dt}\Big|_{t=0} \widehat{\Psi}^t \begin{bmatrix} x \\ \xi \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathcal{D}. \end{aligned}$$

Henceforth we fix the notations $\mathcal{A}, \mathcal{D}$ for the infinitesimal generator of $\big[\widehat{\Psi}^t : t \in \mathbb{R}\big]$ and its domain, respectively. It is well-known that $\mathcal{A}$ has closed graph and $\mathcal{D}$ is a dense linear submanifold of $\mathbf{H} \oplus \mathbb{C}$ being invariant under the maps $\widehat{\Psi}^t$. We shall write $\big\{ \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] \big\}_{\mathbf{H}} := x$ resp. $\big\{ \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] \big\}_{\mathbb{C}} := \xi$ for the canonical projections in $\mathbf{H} \oplus \mathbb{C}$. We also introduce the notations

$$(N)\qquad \begin{aligned} &\mathcal{D}_0 := \left\{ x \in \mathbf{H} : \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{D} \right\}, \quad \mathcal{B} := \big(\overline{x} + \mathcal{D}_0\big) \cap \overline{\mathrm{Ball}}(\mathbf{H}) \ ; \\ &Bz := \left\{ \mathcal{A} \begin{bmatrix} z \\ 0 \end{bmatrix} \right\}_{\mathbf{H}}, \quad \Lambda z := \left\{ \mathcal{A} \begin{bmatrix} z \\ 0 \end{bmatrix} \right\}_{\mathbb{C}} \quad \text{for} \ \ z \in \mathcal{D}_0 \ . \end{aligned}$$

Proposition. *$\mathcal{D}_0$ is a dense linear submanifold in $\mathbf{H}$ with $\mathcal{D} = \big[\begin{smallmatrix} \mathcal{D}_0 \\ 0 \end{smallmatrix}\big] + \mathbb{C}\big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big]$. We have $\mathcal{B} = \big\{ x \in \mathrm{Ball}(\mathbf{H}) : \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D} \big\}$. The set $\mathcal{B}$ is $\big[\Psi^t : t \in \mathbb{R}\big]$-invariant and dense in $\overline{\mathrm{Ball}}(\mathbf{H})$ such that*

$$\frac{d}{dt}\Big|_{t=0} \overline{\Psi}^t(x) = [\Lambda(\overline{x} - x)]x + B(x - \overline{x}) \qquad (x \in \mathcal{B}).$$

Proof. The first statement in (Aa) implies $\big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$ with $\mathcal{A}\big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big] = 0$ since $\widehat{\Psi}^t \big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big] \equiv \big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big]$. Since $\mathcal{D}$ is closed for linear combinations, it follows $\big[\begin{smallmatrix} \mathcal{D}_0 \\ 0 \end{smallmatrix}\big] + \mathbb{C}\big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big] \subset \mathcal{D}$. Given any vector $\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] \in \mathcal{D}$, we have $x - \xi \overline{x} \in \mathcal{D}_0$ because $\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] - \xi \big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$. Hence also $\mathcal{D} \subset \big[\begin{smallmatrix} \mathcal{D}_0 \\ 0 \end{smallmatrix}\big] + \mathbb{C}\big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big]$.
As for the density of $\mathcal{D}_0$ in $\mathbf{H}$ and hence also the density of $\mathcal{B}$ in $\overline{\mathrm{Ball}}(\mathbf{H})$, consider an arbitrary vector $x \in \mathbf{H}$. Since $\mathcal{D}$ is dense in $\mathbf{H} \oplus \mathbb{C}$, there is a sequence $\big[\begin{smallmatrix} x_1 \\ \xi_1 \end{smallmatrix}\big], \big[\begin{smallmatrix} x_2 \\ \xi_2 \end{smallmatrix}\big], \ldots \in \mathcal{D}$ with $\big[\begin{smallmatrix} x_n \\ \xi_n \end{smallmatrix}\big] \to \big[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\big]$ that is $\xi_n \to 0$, $x_n \to 0$ (in norm) whence $\mathcal{D}_0 \ni x_n - \xi_n \overline{x} \to x$.
To see $\mathcal{B} = \big\{ x \in \overline{\mathrm{Ball}}(\mathbf{H}) : \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D} \big\}$ it suffices to notice that in general

$$(*)\qquad x \in \mathcal{D}_0 + \overline{x} \iff \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathcal{D} \qquad (x \in \mathbf{H}).$$

Proof of $(*)$: $x \in \mathcal{D}_0 \iff \big[\begin{smallmatrix} x - \overline{x} \\ 0 \end{smallmatrix}\big] \in \mathcal{D} \iff \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] - \big[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\big] \in \mathcal{D} \iff \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$.
Using $(*)$, we show the $\big[\overline{\Psi}^t : t \in \mathbb{R}\big]$-invariance of the set $\mathcal{B}$ as follows. Since $\widehat{\Psi}^t : \mathcal{D} \leftrightarrow \mathcal{D}$ and $\overline{\Psi}^t : \overline{\mathrm{Ball}}(\mathbf{H}) \leftrightarrow \overline{\mathrm{Ball}}(\mathbf{H})$, we can deduce consecutively the equivalence of the statements (i) $\overline{\Psi^t(x} \in \mathcal{B}$, (ii) $x \in \mathrm{Ball}(\mathbf{H})$ and $\big[\begin{smallmatrix} \Psi^t(x) \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$, (iii) $x \in \overline{\mathrm{Ball}}(\mathbf{H})$ and $\big\{ \widehat{\Psi}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \big\}_{\mathbb{C}}^{-1} \widehat{\Psi}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$, (iv) $x \in \overline{\mathrm{Ball}}(\mathbf{H})$ and $\widehat{\Psi}^t \big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$, (v) $x \in \overline{\mathrm{Ball}}(\mathbf{H})$ and $\big[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\big] \in \mathcal{D}$, (vi) $x \in \mathcal{B}$.

Finally, given any vector $x \in \mathcal{B}$, by $(*)$ we have $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathcal{D}$ whence $\mathcal{A}\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] = \frac{d}{dt}\big|_{t=0} \widehat{\Psi}^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$. Since $\overline{\Psi}^t(x) = \big\{\widehat{\Psi}^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbb{C}}^{-1} \big\{\widehat{\Psi}^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbf{H}}$ and since $\widehat{\Psi}^0 = \mathrm{Id}$, $\mathcal{A}\left[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\right] = \frac{d}{dt}\big|_{t=0} \widehat{\Psi}^t \left[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\right] = \frac{d}{dt}\big|_{t=0} \left[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\right] = 0$, hence we get

$$\frac{d}{dt}\Big|_{t=0} \overline{\Psi}^t(x) = -\Big\{\widehat{\Psi}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbb{C}}^{-2} \Big[\frac{d}{dt}\Big|_{t=0} \Big\{\widehat{\Psi}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbb{C}}\Big] \Big\{\widehat{\Psi}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbf{H}} + \Big\{\widehat{\Psi}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbb{C}}^{-1} \frac{d}{dt}\Big|_{t=0} \Big\{\widehat{\Psi}^0 \begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbf{H}} =$$

$$= -\Big\{\mathcal{A}\begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbb{C}} x + \Big\{\mathcal{A}\begin{bmatrix} x \\ 1 \end{bmatrix}\Big\}_{\mathbf{H}} = -\Big\{\mathcal{A}\begin{bmatrix} x - \overline{x} \\ 0 \end{bmatrix}\Big\}_{\mathbb{C}} x + \Big\{\mathcal{A}\begin{bmatrix} x - \overline{x} \\ 0 \end{bmatrix}\Big\}_{\mathbf{H}}. \quad \text{Q.e.d.}$$

Corollary. *Given any vector $v \in \mathcal{D}_0$ we have*

$$\mathrm{Re}\Big(-\Lambda v + \langle Bv | \overline{x} \rangle + \langle Bv | v \rangle \Big) = 0 \quad \text{whenever} \quad \|\overline{x} + v\| = 1\,.$$

Proof. Let $v \in \mathcal{D}_0$, $u := \overline{x} + v$ and assume $\|u\| = 1$. Since, as it is well-known [3], $\overline{\Psi}\big(\partial\mathrm{Ball}(\mathbf{H})\big) = \partial\mathrm{Ball}(\mathbf{H})$ for any holomorphic automorphism Ψ of the unit ball, we have $\|\overline{\Psi}^t(u)\| = 1$ $(t \in \mathbb{R})$. It follows

$$0 \;=\; \frac{d}{dt}\Big|_{t=0} \Big\langle \overline{\Psi}^t(u) \,\Big|\, \overline{\Psi}^t(u) \Big\rangle \;=\; 2\,\mathrm{Re} \Big\langle \frac{d}{dt}\Big|_{t=0} \overline{\Psi}^t(u) \;\Big|\; u \Big\rangle.$$

Applying the proposition with $x := u$ and $v := x - \overline{x}$, this means that $\mathrm{Re}\,\big\langle [-\Lambda v]u + Bv \,\big|\, u \big\rangle = 0$. Taking into account that $\langle u|u \rangle = 1$, the statement is immediate. Q.e.d.

Proposition. *For some constant $\lambda \in \mathbb{R}$ we have*

$$\Lambda z = \big\langle (B - 2\lambda I) z \,\big|\, \overline{x} \big\rangle, \quad \mathrm{Re}\big\langle \big(B - \lambda I\big) z \,\big|\, z \big\rangle = 0 \qquad (z \in \mathcal{D}_0).$$

In particular, if $\|\overline{x}\| < 1$ then necessarily $\lambda = 0$ above.

Proof. Consider any vector $z \neq 0$. Let $\zeta \in \mathbb{C}$ be the (unique) constant such that $\overline{x} + \zeta z \perp z$ and define $\varrho := \sqrt{1 - \|\overline{x} + \zeta z\|^2}$. Actually we have $\zeta = -\langle \overline{x}|z \rangle / \langle z|z \rangle$ and $1 \geq \|\overline{x}\|^2 = \|\overline{x} + \zeta z\|^2 + \|-\zeta z\|^2$ showing that both ζ and ϱ are well-defined. Consider the unit vectors

$$v_\varphi := \overline{x} + \zeta z + e^{i\varphi} \varrho z \qquad (\varphi \in \mathbb{R}).$$

According to the last corollary,

$$\mathrm{Re}\Big((\zeta + e^{i\varphi}\varrho)\big[-\Lambda z + \langle Bz \,|\, \overline{x} \rangle \big] + |\zeta + e^{i\varphi}\varrho|^2 \langle Bz \,|\, z \rangle \Big) = 0 \qquad (\varphi \in \mathbb{R}).$$

That is we have

$$\begin{aligned} &\mathrm{Re}\big(\alpha + \beta e^{i\varphi} + \gamma e^{-i\varphi}\big) = 0 \quad (\varphi \in \mathbb{R}) \\ &\text{where } \ \alpha := \zeta\big[-\Lambda z + \langle Bz \,|\, \overline{x} \rangle \big] + (|\zeta|^2 + \varrho^2)\langle Bz \,|\, z \rangle, \\ &\qquad \beta := \varrho\,\big[-\Lambda z + \langle Bz \,|\, \overline{x} \rangle + \overline{\zeta}\langle Bz \,|\, z \rangle \big], \quad \gamma := \varrho\zeta\langle Bz \,|\, z \rangle. \end{aligned}$$

Since $2\,\mathrm{Re}\big(\alpha + \beta e^{i\varphi} + \gamma e^{-i\varphi}\big) = 2\,\mathrm{Re}(\alpha) + (\beta + \overline{\gamma})e^{i\varphi} + (\overline{\beta} + \gamma)e^{-i\varphi}$, we have necessarily

$$\mathrm{Re}(\alpha) = \beta + \overline{\gamma} = 0 \ .$$

As $\beta + \overline{\gamma} = 0$, it follows

$$(B) \qquad \Lambda z - \langle Bz \,|\, \overline{x} \rangle = 2\,\overline{\zeta}\,\mathrm{Re}\,\langle Bz \,|\, z \rangle = -2\,\langle z \,|\, \overline{x} \rangle\, \frac{\mathrm{Re}\,\langle Bz \,|\, z \rangle}{\langle z \,|\, z \rangle} \ ,$$

and substituting this into the relation $0 = \mathrm{Re}(\alpha)$, we get

$$(B') \qquad 0 = (\varrho^2 - |\zeta|^2)\,\mathrm{Re}\,\big\langle Bz \,\big|\, z \big\rangle = \big(1 - \|\overline{x}\|^2\big)\,\mathrm{Re}\,\big\langle Bz \,\big|\, z \big\rangle \ .$$

From (B) we see that

$$z \mapsto \frac{\mathrm{Re}\big\langle Bz \,\big|\, z \big\rangle}{\big\langle z \,\big|\, z \big\rangle} = -\frac{1}{2}\, \frac{\Lambda z - \big\langle Bz \,\big|\, \overline{x} \big\rangle}{\big\langle z \,\big|\, \overline{x} \big\rangle}$$

is a real valued Gâteaux holomorphic function on the algebraically open and in $\mathcal{D}_0$ algebraically dense domain $\{z \in \mathcal{D}_0 : \ z \not\perp \overline{x}\}$ which is possible only if being constant on $\mathcal{D}_0$. By writing λ for this constant value, the statement is immediate from (B) and (B'). Q.e.d.

Taking int account the obvious reversibility properties of the steps of the proof, we conclude immediately the following technical statement.*

Corollary. *Given a possibly unbounded linear functional* $\Lambda : \mathcal{D}_0 \to \mathbb{C}$ *along with a possibly unbounded linear operator* $B : \mathcal{D}_0 \to \mathbf{H}$, *we have* $\big[\Lambda(\overline{x} - x)\big]x + B(x - \overline{x}) \perp x$ *for any* $x \in [\overline{x} + \mathcal{D}_0] \cap \partial\mathrm{Ball}(\mathbf{H})$ *if and only if* $\Lambda z = \big\langle (B - 2\lambda I)z \,\big|\, \overline{x} \big\rangle$, $\mathrm{Re}\big\langle \big(B - \lambda I\big)z \,\big|\, z \big\rangle = 0$ $(z \in \mathcal{D}_0)$ *with some real constant* λ *if* $\|\overline{x}\| = 1$ *and with* $\lambda = 0$ *in case of* $\|\overline{x}\| < 1$.

Lemma. *The operator* $B : \mathcal{D}_0 \to \mathcal{D}_0$ *is closed.*

Proof. Consider a convergent sequence $\mathcal{D}_0 \ni z_n \to z$ with limit point in $\mathbf{H}$ such that $Bz_n \to w$ for some $w \in \mathbf{H}$. We have to show that $(*)$ $z \in \mathcal{D}_0$ with $Bz = w$, For $(*)$, it suffices to observe that

$$\Lambda z_n = \big\langle (B - 2\lambda I)z_n \big| \overline{x} \big\rangle \to \big\langle w - 2\lambda z \big| \overline{x} \big\rangle,$$

* Direct proof. Given any $\lambda \in \mathbb{R}$ and a vector $z \in \mathbf{H}$ with $\|\overline{x} + z\| = 1$, we have the following chain of equivalences: $0 = \mathrm{Re}\big\langle -\langle (iA - \lambda)z|\overline{x}\rangle(\overline{x} + z) + (iA + \lambda I)z\big|\overline{x} + z\big\rangle$ iff $0 = \mathrm{Re}\big[-\langle (iA - \lambda)z|\overline{x}\rangle + \langle (iA + \lambda I)z|\overline{x} + z\rangle\big]$ iff $0 = \mathrm{Re}\big[2\lambda\langle z|\overline{x}\rangle + \langle iAz|z\rangle + \lambda\langle z|z\rangle\big]$ iff $0 = \mathrm{Re}\big[\lambda\big(\langle \overline{x} + z|\overline{x} + z\rangle - \langle \overline{x}|\overline{x}\rangle - \langle \overline{x}|z\rangle + \langle z|\overline{x}\rangle\big) + \langle iAz|z\rangle\big]$ iff $0 = \mathrm{Re}\big[\lambda(1 - \|\overline{x}\|^2) + \langle iAz|z\rangle\big]$.

recalling that the not necessarily bounded linear operator $\mathcal{A}\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] = \left[\begin{smallmatrix} B(x-\xi\overline{x}) \\ \Lambda(x-\xi\overline{x}) \end{smallmatrix}\right]$ mapping the linear manifold $\mathcal{D} := \left[\begin{smallmatrix} \mathcal{D}_0 \\ 0 \end{smallmatrix}\right] + \mathbb{C}\left[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\right]$ into itself is closed. Indeed we have

$$\begin{bmatrix} \mathcal{D}_0 \\ 0 \end{bmatrix} \ni \begin{bmatrix} z_n \\ 0 \end{bmatrix} \to \begin{bmatrix} z \\ 0 \end{bmatrix}, \quad \mathcal{A}\begin{bmatrix} z_n \\ 0 \end{bmatrix} = \begin{bmatrix} Bz_n \\ \Lambda z_n \end{bmatrix} \to \begin{bmatrix} w \\ \langle w - 2\lambda z|\overline{x}\rangle \end{bmatrix}$$

implying $(*)$ along with the relation $\Lambda z = \langle w - 2\lambda z|\overline{x}\rangle = \langle (B - 2\lambda I)z|\overline{x}\rangle$. Q.e.d.

Definition. For convenience, we shall write $A := -i(B + \lambda I)$. Thus henceforth

$$(A) \qquad Bz = (iA + \lambda I)z\,, \quad \Lambda z = \big\langle (iA - \lambda I)z\big|\overline{x}\big\rangle \qquad (z \in \mathcal{D}_0)$$

where $A : \mathcal{D}_0 \to \mathcal{D}_0$ is a densely defined closed symmetric linear operator and λ is some real number. We know also that any vector field of the form $\overline{x}+\mathcal{D}_0 \ni \overline{x}+z \mapsto -[\Lambda z](\overline{x}+z)+Bz$ is tangent to the unit sphere if and only if B and Λ are of the form (A) with a symmetric operator A and any real number λ if $\|\overline{x}\| = 1$ and with $\lambda = 0$ in case of $\|\overline{x}\| = 1$.

The case $0 \in \mathcal{B}$ with $\|\overline{x}\| = 1$

Throughout this section we assume that a common fixed point $\overline{x}$ of $\left[\overline{\Psi}^t : t \in \mathbb{R}\right]$ lyes in the boundary of the unit ball and the origin belongs to the domain $\mathcal{B}$ of the non-linear generator that is we have

$$(Ab) \qquad 0 \in \mathcal{B}, \quad \|\overline{x}\| = 1.$$

Remark, Since $\mathcal{B} = \overline{x} + \mathcal{D}_0 \cap \overline{\mathrm{Ball}}(\mathbf{H})$ and since $\mathcal{D}_0$ is a complex linear submanifold of $\mathbf{H}$, we have $0 \in \mathcal{B}$ if and only if $\overline{x} \in \mathcal{D}_0$ which is equivalent to $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \overline{x} \\ 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \overline{x} \\ 0 \end{smallmatrix}\right] \in \mathcal{D}$. In this case the terms $B\overline{x}, \Lambda\overline{x}$ are also well-defined and in their terms we can write the infinitesimal generator $\mathcal{A}$ in a matrix form
as

$$\mathcal{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathcal{A}\left(\begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} - \begin{bmatrix} \overline{x} \\ 0 \end{bmatrix}\right) = -\begin{bmatrix} B\overline{x} \\ \Lambda\overline{x} \end{bmatrix},$$
$$\mathcal{A}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} B(x - \xi\overline{x}) \\ \Lambda(x - \xi\overline{x}) \end{bmatrix} = \begin{bmatrix} B & -B\overline{x} \\ \Lambda & -\Lambda\overline{x} \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix}.$$

Notice that in case of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \notin \mathcal{D}$ we have only $\mathcal{A}\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] = \left[\begin{smallmatrix} B \\ \Lambda \end{smallmatrix}\right]\Big([I \;\; -1]\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]\Big)$ while the formal product $\left[\begin{smallmatrix} B \\ \Lambda \end{smallmatrix}\right][I \;\; -1]$ cannot be defined.

Theorem. *Under the hypothesis (Ab), $\mathcal{B}$ is the intersection of the dense complex linear submanifold $\mathcal{D}_0$ of $\mathbf{H}$ with the closed unit ball and $\mathcal{V}$ admits the Jordan form*

$$(J) \qquad \mathcal{V}(x) = b - \{xb^*x\} + iAx \qquad (x \in \mathcal{B})$$

in terms of the Jordan triple product $\{xc^*y\} := \frac{1}{2}\langle x|c\rangle y + \frac{1}{2}\langle y|c\rangle x$ *where*

$$b := \frac{d}{dt}\Big|_{t=0}\Psi^t(0) = \lim_{t\to 0} a_t/t, \quad Ax := \lim_{t\to 0}\frac{1}{it}(U_t - I)x - \mathrm{Im}\langle\overline{x}|b\rangle x \quad (x \in \mathcal{D}_0).$$

Proof. By the definition of the domain $\mathcal{D}$,

$$\mathcal{A}\begin{bmatrix}0\\1\end{bmatrix} = \lim_{t\to 0}\frac{1}{t}\big(\widehat{\Psi}^t - \widehat{\Psi}^0\big)\begin{bmatrix}0\\1\end{bmatrix} = \lim_{t\to 0}\frac{1}{t}\Big\{\frac{1}{1+\langle U_t\overline{x}|a_t\rangle}\begin{bmatrix}a_t\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\Big\}.$$

Since $\lim_{t\to 0} a_t = 0$ and $\|U_t\overline{x}\| \le 1$ $(t \in \mathbb{R})$, taking (Ux) into account, we see that the limit

$$b := \lim_{t\to 0}\frac{1}{t}a_t = \frac{d}{dt}\Big|_{t=0}\Psi^t(0)$$

is well-defined and

$$B\overline{x} = -\Big\{\mathcal{A}\begin{bmatrix}0\\1\end{bmatrix}\Big\}_{\mathbf{H}} = -b, \quad \Lambda\overline{x} = -\Big\{\mathcal{A}\begin{bmatrix}0\\1\end{bmatrix}\Big\}_{\mathbb{C}} = \langle\overline{x}\,|\,b\rangle.$$

As a consequence we also have

$$\begin{aligned}\beta_t &= \sqrt{1-\langle a_t|a_t\rangle} = \sqrt{1-\|tb+o(t)\|^2} = \\ &= \sqrt{1-t^2\|b\|^2+o(t^2)} = 1 - \frac{1}{2}\|b\|^2t^2 + o(t^2).\end{aligned}$$

Hence we deduce that $1-\beta_t = o(t)$ and therefore, for any vector $z \in \mathcal{D}_0$.

$$\begin{aligned}\frac{1}{t}\big(\widehat{\Psi}^t - I\big)\begin{bmatrix}z\\0\end{bmatrix} &= \frac{1}{t}\begin{bmatrix}(P_t+\beta_t\overline{P}_t)U_tz - \langle U^t\overline{x}|a_t\rangle z\\ t\Lambda z + o(t)\end{bmatrix} = \\ &= \begin{bmatrix}t^{-1}(U_t-I)z + t^{-1}(1-\beta_t)\overline{P}_tU_tz - \langle\overline{x}|b\rangle z\\ \Lambda z\end{bmatrix} + o(1).\end{aligned}$$

Since, by definition, $\left[\begin{smallmatrix}Bz\\ \Lambda z\end{smallmatrix}\right] = \lim_{t\to 0}\frac{1}{t}\big(\widehat{\Psi}^t - I\big)\left[\begin{smallmatrix}z\\0\end{smallmatrix}\right]$, we get

$$Bz = \lim_{t\to 0}\frac{1}{t}(U_t - I)z - \big\langle\overline{x}\,\big|b\big\rangle z \qquad (z \in \mathcal{D}_0).$$

In particular, from the relations (Ab) we obtain that for any unit vector $z \in \mathcal{D}_0$ we have $i\langle Az|z\rangle + \lambda = \lim_{t\to 0} t^{-1}\langle(U_t - I)z|z\rangle - \langle\overline{x}|b\rangle$ whence the the symmetry of A along with the properties $0 = \frac{d}{dt}_{t=0}\|U_tz\|^2 = 2\,\mathrm{Re}\lim_{t\to 0} t^{-1}\langle(U_t - I)z|z\rangle$ and $\lambda \in \mathbb{R}$ implies

$$\lambda = -\mathrm{Re}\big\langle b\,\big|\,\overline{x}\big\rangle, \quad A = \lim_{t\to 0}^{\text{strong}}\frac{1}{it}\big(U_t - I\big) - \mathrm{Im}\big\langle\overline{x}\,\big|\,b\big\rangle I\,.$$

Finally we calculate $\mathcal{V} = \frac{d}{dt}\big|_{t=0}\Psi^t$ by substituting the previous results int the form $\mathcal{V}(x) = \big[\Lambda(\overline{x} - x)\big]x + B(x - \overline{x})$. Namely, given any vector $x \in \mathcal{B}$, taking into account the antisymmetry of the operator $B - \lambda I$, we can write

$$\begin{aligned}\mathcal{V}(x) &= \big[\Lambda\overline{x}\big]x - \big[\Lambda x\big]x + Bx - B\overline{x}) = \\ &= \langle\overline{x}|b\rangle - \langle(B-2\lambda I)x|\overline{x}\rangle x + Bx + b = \\ &= \langle\overline{x}|b\rangle - \langle(B-\lambda I)x|\overline{x}\rangle x + \langle(B-\lambda I)x|\overline{x}\rangle x + Bx + b = \\ &= b + \big[\langle\overline{x}|b\rangle I + B\big]x + \langle x|(B+\lambda I)\overline{x}\rangle x = \\ &= b + \big[\langle\overline{x}|b\rangle I + B\big]x + \langle x|(-b+2\lambda\overline{x})\rangle x = \\ &= b - \langle x|b\rangle x + \big[(\langle\overline{x}|b\rangle + \lambda + (B-\lambda I)\big]x = \\ &= b - \langle x|b\rangle x + \big[i\langle A\overline{x}|\overline{x}\rangle I + iA\big]x\end{aligned}$$

whence the Jordan form (J) is immediate. Q.e.d.

Lemma. *We have* $(1+\langle U_t\overline{x}|a_t\rangle)^{-1}\begin{bmatrix} P_t+\beta_t\overline{P}_t & a_t \\ a_t^* & 1\end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1\end{bmatrix} + t\begin{bmatrix} -\langle\overline{x}|b\rangle I & b \\ b^* & -\langle\overline{x}|b\rangle\end{bmatrix} + o^{\text{norm}}(t)$.

Proof. This is a direct consequence of the facts that $a_t = tb + o^{\text{norm}}(t)$, $\beta_t = 1 + o(t)$, $(1+\langle U_t\overline{x}|a_t\rangle)^{-1} = (1-\langle\overline{x}|a_t\rangle)(1-\langle a_t|a_t\rangle)^{-1} = (1-t\langle\overline{x}|b\rangle + o(t))\beta^{-2} = 1 - t\langle\overline{x}|b\rangle + o(t)$ and $P_t + \beta_t\overline{P}_t = (1-\beta_t)P_t + \beta_t(P_t + \overline{P}_t) = I + o^{\text{norm}}(t)$ as $t \to 0$. Q.e.d.

Corollary. *We have* $\mathcal{D}_0 = \operatorname{dom}\big(\lim_{t\to 0} t^{-1}(U_t - I)\big)$, *that is* $x \in \mathcal{D}_0$ *if and only if the limit* $\lim_{t\to 0} t^{-1}(U^t x - x)$ *exists.*

Proof. Recall that $\mathcal{D}_0 = \big\{x \in \mathbf{H} : \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t\big[\begin{smallmatrix} x \\ 0\end{smallmatrix}\big] \text{ is well-defined}\big\}$. From the theorem we know that $iAx = \lim_{t\to 0} t^{-1}(U_t - I)x$ is well-defined for every vector $x \in \mathcal{D}_0$. Conversely, suppose $u = \lim_{t\to 0} t^{-1}(U_t x - x)$ is well-defined. Then $U_t x = x + tu + o^{\text{norm}}(t)$ and

$$\begin{aligned}\widehat{\Psi}^t\begin{bmatrix} x \\ 0\end{bmatrix} &= (1+\langle U_t\overline{x}|a_t\rangle)\begin{bmatrix} P_t+\beta_t\overline{P}_t & a_t \\ a_t^* & 1\end{bmatrix}\begin{bmatrix} U_t & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} x \\ 0\end{bmatrix} = \\ &= \Big(\begin{bmatrix} I & 0 \\ 0 & 1\end{bmatrix} - t\begin{bmatrix} \langle\overline{x}|a_t\rangle & -b \\ -b^* & \langle\overline{x}|a_t\rangle\end{bmatrix} + o^{\text{norm}}(1)\Big)\begin{bmatrix} x + tu + o^{\text{norm}}(t) \\ 0\end{bmatrix} = \\ &= \begin{bmatrix} x \\ 0\end{bmatrix} + t\begin{bmatrix} u - \langle\overline{x}|a_t\rangle x \\ \langle x|b\rangle\end{bmatrix} + o^{\text{norm}}(t). \qquad \text{Q.e.d.}\end{aligned}$$

Lemma. *We have* $U_{-t} = U_t^{-1} = U_t^*, \;\; a_{-t} = -U_t^* a_t \quad (t \in \mathbb{R})$.

Proof. Given any parameter $t \in \mathbb{R}$, we have $\Psi^{-t} = \Psi_t^{-1}$ that is $\Theta_{a_{-t}}U_{-t} = \big[\Theta_{a_t}U_t\big]^{-1} = U_t^{-1}\Theta_{a_t}^{-1} = U_t^{-1}\Theta_{-a_t} = \big[U_t^{-1}\Theta_{-a_t}U_t\big]U_t^{-1} = \Theta_{U_t^{-1}(-a_t)}U_t^{-1}$. Using the uniqueness of the decomposability of holomorphic automorphisms of circular domains into Möbius and unitary parts [7], hence we deduce that $\Theta_{a_{-t}} = \Theta_{-U_t^{-1}a_t}$ and $U_{-t} = U_t^{-1}$. Q.e.d.

Proposition. *The operator A in (J) is self-adjoint.*

Proof. In view of the previous lemmas we can conclude that for any vector $\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] \in \mathcal{D}$,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \widehat{\Psi}^t &= \frac{d}{dt}\Big|_{t=0} (1 + \langle U_t \overline{x} | a_t \rangle)^{-1} \begin{bmatrix} P_t + \beta_t \overline{P}_t & a_t \\ a_t^* & 1 \end{bmatrix} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} = \\ &= \frac{d}{dt}\Big|_{t=0} (1 + \langle U_t \overline{x} | a_t \rangle)^{-1} \begin{bmatrix} P_t + \beta_t \overline{P}_t & a_t \\ a_t^* & 1 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \frac{d}{dt}\Big|_{t=0} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} = \\ &= \begin{bmatrix} iA - \langle \overline{x} | b \rangle I & b \\ b^* & -\langle \overline{x} | b \rangle \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \end{aligned}$$

The linear operator in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$ with matrix $\left[\begin{smallmatrix} -\langle \overline{x}|b\rangle I & b \\ b^* & -\langle \overline{x}|b\rangle \end{smallmatrix}\right]$ is bounded. Since $\mathcal{A} = \frac{d}{dt}\big|_{t=0} \widehat{\Psi}^t$ with domain $|\mathcal{D}$ is the generator of a strongly continuous semigroup in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$, by the theorem of bounded perturbations [8], also the operator with matrix $\left[\begin{smallmatrix} iA & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with domain $\mathcal{D}$ is the generator of a strongly continuous group in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$ strongly continuous one-parameter subgroup of $\mathcal{L}(\mathbf{H})$ entailing that iA in (J) is the generator of a strongly continuous group $\big[V_t : t \in \mathbb{R}\big]$ in $\mathcal{L}(\mathbf{H})$. Since $U_{-t} = U_t^{-1} = U_t^*$, the arguments on sun adjoint semigroups in [8, p. 69] show that $\lim_{t\to 0} t^{-1}(U_t^* - I) = -iA$ is the generator of the sun adjoint group $\big[V_t^* : t \in \mathbb{R}\big] = \big[V_{-t} : t \in \mathbb{R}\big]$ and we have $-iA = (iA)^*$. Q.e.d.

Theorem. *Any vector field of the form (J) where A is a not necessarily bounded self-adjoint operator with dense domain $\mathcal{D}_0 \subset \mathbf{H}$, is the infinitesimal generator defined on $\mathcal{B} := \mathcal{D}_0 \cap \overline{\mathrm{Ball}}(\mathbf{H})$ of a pointwise continuous one-parameter group $\Phi^t : \ t \in \mathbb{R}]$ of holomorphic automorphisms of $\overline{\mathrm{Ball}}(\mathbf{H})$.*[*]

Proof. It suffices to see that there is a strongly (i.e. pointwise) continuous one-parameter group $\big[V^t : \ t \in \mathbb{R}\big]$ of bounded linear operators of the space $\mathbf{H} \oplus \mathbb{C}$ such that

$$\frac{d}{dt}\Big|_{t=0} V^t \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \quad (x \in \mathcal{D}_0, \ \xi \in \mathbb{C}), \qquad V^t \mathbf{K} \subset \mathbf{K} := \Big\{ \begin{bmatrix} x \\ \xi \end{bmatrix} : \ \|x\|^2 \geq |\xi|^2 \Big\}.$$

Namely, in this case the maps

$$\Phi^t(x) := \Big\{ V^t \begin{bmatrix} x \\ 1 \end{bmatrix} \Big\}_{\mathbb{C}}^{-1} \Big\{ V^t \begin{bmatrix} x \\ 1 \end{bmatrix} \Big\}_{\mathbf{H}} \qquad (t \in \mathbb{R}, \ x \in \mathcal{B})$$

suit the requirements of the theorem since $x \in \mathcal{B} \Rightarrow \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathbf{K} \Rightarrow V^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \Rightarrow \Phi^t(x) \in \mathcal{B}$ and $x \in \mathcal{B} \Rightarrow \frac{d}{dt}\big|_{t=0} \Phi^t(x) = -\big\{V^0 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbb{C}}^{-2} \frac{d}{dt}\big|_{t=0} \big\{V^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbb{C}} \big\{V^0 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbf{H}} + \big\{V^0 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbb{C}}^{-1} \frac{d}{dt}\big|_{t=0} \big\{V^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbf{H}}$ $= -\big\{\left[\begin{smallmatrix} iA & b \\ b^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbb{C}} x + \big\{\left[\begin{smallmatrix} iA & b \\ b^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbf{H}} = -\langle x|b\rangle x + iAx + b = \mathcal{V}(x)$.

[*] That is, for all $x \in \mathcal{B} := \mathcal{D}_0 \cap \overline{\mathrm{Ball}}(\mathbf{H})$, the functions $t \mapsto \Phi^t(x)$ range in $\mathcal{D}_0$, they are differentiable and satisfy the identity $\frac{d}{dt}\Phi^t(x) = \mathcal{V}\big(\Phi^t(x)\big)$ $(t \in \mathbb{R})$.

Notice that a strongly continuous one parameter group of linear operator leaves the cone $\mathbf{K}$ invariant if all its members map the boundary $\partial\mathbf{K}=\left\{\left[\begin{smallmatrix}x\\ \xi\end{smallmatrix}\right]: \|x\|=|\xi|\right\}=\left\{\left[\begin{smallmatrix}x\\ e^{i\tau}\|x\|\end{smallmatrix}\right]: x\in\mathbf{H},\ \tau\in\mathbb{R}\right\}$ into itself.* Therefore it suffices to check that there is a (necessarily unique)strongly continuous one-parameter group in $\mathcal{L}(\mathbf{H}\oplus\mathbb{C})$ with domain $\mathcal{D}_0\oplus\mathbb{C}$ such that

$$\frac{d}{dt}V^t\begin{bmatrix}x\\ \xi\end{bmatrix}=\begin{bmatrix}iA & b\\ b^* & 0\end{bmatrix}V^t\begin{bmatrix}x\\ \xi\end{bmatrix},\quad \left\|\left\{V^t\begin{bmatrix}x\\ \|x\|\end{bmatrix}\right\}_{\mathbf{H}}\right\|=\left|\left\{V^t\begin{bmatrix}x\\ \|x\|\end{bmatrix}\right\}_{\mathbb{C}}\right| \qquad (x\in\mathcal{D}_0,\ t\in\mathbb{R}).$$

By Stone's theorem, the $\mathbf{H}\oplus\mathbb{C}$-unitary operators $W^t\left[\begin{smallmatrix}x\\ \xi\end{smallmatrix}\right]:=\left[\begin{smallmatrix}\exp(itA)x\\ \xi\end{smallmatrix}\right]$ form a strongly continuous one-parameter group whose infinitesimal generator is defined on $\mathrm{dom}(A)\oplus\mathbb{C}=\mathcal{D}_0\oplus\mathbb{C}$ with the diagonal matrix $\left[\begin{smallmatrix}iA & 0\\ 0 & 0\end{smallmatrix}\right]$. Since the matrix $\left[\begin{smallmatrix}0 & b\\ b^* & 0\end{smallmatrix}\right]$ represents a bounded linear operator in $\mathbf{H}\oplus\mathbb{C}$, by the theorem of bounded perturbations 8], there is a strongly continuous one-parameter group $\left[V^t : t\in\mathbb{R}\right]$ whose generator is defined on $\mathcal{D}_0\oplus\mathbb{C}$ with the matrix $\left[\begin{smallmatrix}iA & b\\ b^* & 0\end{smallmatrix}\right]$. In particular $\frac{d}{dt}V^t\left[\begin{smallmatrix}x\\ \xi\end{smallmatrix}\right]=\left[\begin{smallmatrix}iA & b\\ b^* & 0\end{smallmatrix}\right]V^t\left[\begin{smallmatrix}x\\ \xi\end{smallmatrix}\right]$ $(t\in\mathbb{R},\ x\in\mathcal{D}_0)$. To complete the proof, we show that necessarily

$$\frac{d}{dt}\left[\left\|\left\{V^t\begin{bmatrix}x\\ \|x\|\end{bmatrix}\right\}_{\mathbf{H}}\right\|^2-\left|\left\{V^t\begin{bmatrix}x\\ \|x\|\end{bmatrix}\right\}_{\mathbb{C}}\right|^2\right]=0 \qquad (t\in\mathbb{R},\ x\in\mathcal{D}_0).$$

Consider any vector $x\in\mathcal{D}_0$ and write $\left[\begin{smallmatrix}x_t\\ \xi_t\end{smallmatrix}\right]:=V^t\left[\begin{smallmatrix}x\\ \|x\|\end{smallmatrix}\right]$ for all parameters $t\in\mathbb{R}$. Then

$$\begin{aligned}\frac{d}{dt}\left[\|x_t\|^2-|\xi_t|^2\right]&=2\,\mathrm{Re}\Big[\langle dx_t/dt|x_t\rangle-\big(d\xi_t/dt\big)\overline{\xi_t}\Big]=\\ &=2\,\mathrm{Re}\left[\left\langle\left\{\begin{bmatrix}iA & b\\ b^* & 0\end{bmatrix}\begin{bmatrix}x_t\\ \xi_t\end{bmatrix}\right\}_{\mathbf{H}}\Big|x_t\right\rangle-\left\{\begin{bmatrix}iA & b\\ b^* & 0\end{bmatrix}\begin{bmatrix}x_t\\ \xi_t\end{bmatrix}\right\}_{\mathbb{C}}\big(d\xi_t/dt\big)\overline{\xi_t}\right]=\\ &=2\,\mathrm{Re}\left[\langle iAx_t+\xi_t b|x_t\rangle-\langle x_t|b\rangle\overline{\xi_t}\right]=\\ &=-2\,\mathrm{Im}\langle Ax_t|x_t\rangle+2\,\mathrm{Im}\big(\langle\xi b|x_t\rangle-\langle x_t|\xi b\rangle\big)=0. \qquad \text{Q.e.d.}\end{aligned}$$

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* In general, if a Hilbert space $\mathbf{W}$ is the orthogonal sum of the subspaces $\mathbf{W_k}$ $(k=1,2)$ and C is the infinitesimal generator of a strongly continuous one-parameter group $[T^t : t\in\mathbb{R}]$ in $\mathcal{L}(\mathbf{W})$ then we have $T^t\mathbf{C}=\mathbf{C}$ $(t\in\mathbb{R})$ for the cone $\mathbf{C}:=\left\{\mathbf{w}_1\oplus\mathbf{w}_2 : \mathbf{w}_k\in\mathbf{W}_k,\ \|\mathbf{w}_1\|>\|\mathbf{w}_2\|\right\}$ if and only if C is tangent to the boundary of $\mathbf{C}$ that is if
$(*)$ $\mathrm{Re}\langle C(\mathbf{w}_1\oplus\mathbf{w}_2),\mathbf{w}_1\rangle=\mathrm{Re}\langle C(\mathbf{w}_1\oplus\mathbf{w}_2),\mathbf{w}_2\rangle$ whenever $\mathbf{w}_1\oplus\mathbf{w}_2\in\mathrm{dom}(C)$ with $\|\mathbf{w}_1\|=\|\mathbf{w}_2\|$.
Proof. It is immediate that $T^t\mathbf{C}\subset\mathbf{C}$ $(t\in\mathbb{R})$ $\Rightarrow$ $T^t\partial\mathbf{C}\subset\partial\mathbf{C}$ $(t\in\mathbb{R})$ $\Rightarrow$ $\frac{d}{dt}|_{t=0}T^t(\mathbf{w}_1\oplus\mathbf{w}_2)\in\mathrm{Tan}_{\mathbf{w}_1\oplus\mathbf{w}_2}(\mathbf{C})$ for $\mathbf{w}_1\oplus\mathbf{w}_2\in\mathrm{dom}(C)$ $\Rightarrow$ $(*)$. Assume $(*)$ and let P denote the canonical projection of $\mathbf{W}$ onto $\mathbf{W}_1$ and define $T^{t,s}:=\exp(tC+sP)$ $(s,t\in\mathbb{R})$. By the theorem of bounded perturbations [8, p.158] the operators $T^{t,s}$ are all well-defined. Moreover, by [8, Corollary 1.7 p. 161] (applied with $B:=sP$ and $A:=C$ there) we have $\lim_{s\to 0}T^{t,s}\mathbf{w}=T^t\mathbf{w}$ $(\mathbf{w}\in\mathrm{dom}(C),t\in\mathbb{R})$. Therefore, to establish that $T^t\mathbf{C}\subset\mathbf{C}$ $(t\in\mathbb{R})$, it suffices to see only that $(**)$ $T^{t,s}\mathbf{w}\in\mathbf{C}$ whenever $\mathbf{w}\in\mathrm{dom}(C)\cap\mathbf{C}$ and $t,s>0$. To proceed to contradiction, let $s,t>0$ and $\mathbf{w}:=\mathbf{w}_1\oplus\mathbf{w}_2\in\mathrm{dom}(C)$ with $\|\mathbf{w}_1\|>\|\mathbf{w}_2\|$ but $\|[T^{t,s}\mathbf{w}]_1\|\le\|[T^{t,s}\mathbf{w}]_2\|$. The function $\delta(\tau):=\|[T^{\tau,s}\mathbf{w}]_1\|^2-\|[T^{\tau,s}\mathbf{w}]_2\|^2$ is differentiable in τ on the whole $\mathbb{R}$ and $\delta(0)>0\ge\delta(t)$. Thus there exist a point $t_*\in(0,t]$ such that $\delta(\tau)>0=\delta(t_*)$ $(0\le\tau<t_*)$. Since $\delta(t_*)=0$, $\mathbf{w}_*:=T^{t_*,s}\mathbf{w}\in\partial\mathbf{C}$ and hence $\mathrm{Re}\langle C\mathbf{w}_*,[\mathbf{w}_*]_1\rangle=\mathrm{Re}\langle C\mathbf{w}_*,[\mathbf{w}_*]_2\rangle$. We get the contradiction $0\ge\delta'(t_*)=2\mathrm{Re}\langle(C+sP)\mathbf{w}_*,[\mathbf{w}_*]_1\rangle-2\mathrm{Re}\langle(C+sP)\mathbf{w}_*,[\mathbf{w}_*]_2\rangle=2s\|[\mathbf{w}_*]_1\|^2>0$.

Reduction to the Jordan case $0 \in \mathcal{B}$

In accordance with the previous standard notations, throughout this section $\big[\overline{\Psi}^t : t \in \mathbb{R}\big]$ denotes a pointwise continuous one-parameter subgroup of $\mathrm{Aut}\big(\overline{\mathrm{Ball}}(\mathbf{H})\big)$ with infinitesimal generator $\mathcal{X} : \mathcal{B} \to \mathbf{H}$ where $\mathcal{B} := \big\{x \in \overline{\mathrm{Ball}}(\mathbf{H}) :\ t \mapsto \overline{\Psi}^t(x)$ is differentiable on $\mathbb{R}\big\}$. We regard $\mathcal{X}$ as a vector field defined on the intersection of a dense complex-affine submanifold with the unit ball of $\mathbf{H}$.
Let us fix any point $a \in \mathcal{B}$ with $\|a\| < 1$ and, in terms of the Möbius transformations (M), consider the transformations

$$\overline{\Phi}^t := \Theta_{-a}^{\#}\overline{\Psi}^t = \Theta_{-a} \circ \overline{\Psi}^t \circ \Theta_a$$

given by the adjoint action of $\Theta_{-a} = \Theta_a^{-1}$. Clearly $\big[\overline{\Phi}^t :\ t \in \mathbb{R}\big]$ is a pointwise continuous one-parameter subgroup of $\mathrm{Aut}\big(\overline{\mathrm{Ball}}(\mathbf{H})\big)$ whose infinitesimal generator is the vector field

$$\mathcal{V} := \Theta_{-a}^{\#}\mathcal{X} : y \mapsto \frac{d}{dt}\Big|_{t=0} \Theta_{-a}\big(\overline{\Psi}^t(\Theta_a(y))\big)$$

defined on the domain

$$\mathcal{B}_1 := \Theta_{-a}\mathcal{B} = \mathcal{D}_1 \cap \overline{\mathrm{Ball}}(\mathbf{H}).$$

Recall that $0 \in \mathcal{B}_1$ and $\mathcal{D}_1$ is some dense complex-linear submanifold in $\mathbf{H}$ and, as a consequence of the results of the previous section, we can write $\mathcal{V}$ in the Jordan form (J) with some vector $b \in \mathbf{H}$ and a suitable not necessarily bounded self-adjoint linear operator $A : \mathcal{D}_1 \to \mathbf{H}$. By the aid of the inverse adjoint action $\Theta_a^{\#}$, hence we get $\mathcal{X}$ in the form

$$\begin{aligned}
\mathcal{X}(x) = \big[\Theta_a^{\#}\mathcal{V}\big](x) &= \frac{d}{dt}\Big|_{t=0}\Theta_a\Big(\overline{\Phi}^t\big(\Theta_{-a}(x)\big)\Big) = \\
&= \frac{d}{dt}\Big|_{t=0}\Theta_a\Big(\Theta_{-a}(x) + t\mathcal{V}\big(\Theta_{-a}(x)\big) + o(t)\Big) = \\
&= \frac{d}{dt}\Big|_{t=0}\big(1 + \langle y + tk|a\rangle\big)^{-1}\Big(\big[\mathrm{P}_a + \beta_a\overline{\mathrm{P}}_a\big](y + tk) + a\Big)\Big|_{\substack{y=\Theta_{-a}(x)\\ k=b-\langle y|b\rangle y+iAy}}
\end{aligned}$$

for the locations $x \in \mathcal{B}$. Next we calculate the a detailed algebraic form for $\mathcal{X}$. Notice that

$$\frac{d}{dt}\Big|_{t=0}\big(1 + \langle y + tk|a\rangle\big)^{-1} = -\langle k|a\rangle\big(1 + \langle y|a\rangle\big)^{-2}$$

!!

The classical case $\|\overline{x}\| < 1$**.** Since the Möbius transformation $\Theta_{\overline{x}}$ takes the origin into $\overline{x}$ and since its inverse coincides with $\Theta_{-\overline{x}}$, the automorphisms of the form $T^t := \Theta_{-\overline{x}} \circ \Psi^t \circ \Theta_{\overline{x}}$

leave the origin fixed and hence $\big[T^t : t \in \mathbb{R}\big]$ is a strongly continuous one-parameter group consisting of restrictions of linear unitary operators to Ball($\mathbf{H}$). By Stone's classical theorem, there is a not necessarily bounded self-adjoint operator A with dense domain in $\mathbf{H}$ such that $T^t = \exp(itA)$ $(t \in \mathbb{R})$. It follows $\Psi^t = \Theta_{\overline{x}} \circ \exp(itA) \circ \Theta_{-\overline{x}}$ $(t \in \mathbb{R})$ entailing that the (non-linear) infinitesimal generator of $\big[\overline{\Psi}^t : t \in \mathbb{R}\big]$ is

$$\frac{d}{dt}\Big|_{t=0} \overline{\Psi}^t(x) = i\Big[\Theta'_{\overline{x}}\big(\Theta_{-\overline{x}}(x)\big)\Big] A\Theta_{-\overline{x}}(x)$$

for x in the domain $\Theta_{\overline{x}}\big(\mathrm{dom}(A) \cap \overline{\mathrm{Ball}}(\mathbf{H})\big) = (\overline{x} + \mathcal{D}_0) \cap \overline{\mathrm{Ball}}(\mathbf{H}) = \mathcal{B}$.

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