A Stone type theorem for one-parameter groups of non-linear automorphisms in Hilbert space

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Abstract. A commplete parametric algebraic description is given in terms of the inner product for the strongly continuous one-parameter subgroups of the group all biholomorphic automorphisms of the unit ball of a Hilbert space with emphasis to non Jordan type infinitesimal generators.

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1. Introduction, main results

Throughout this work $\mathbf{H}$ denotes an arbitrarily fixed infinite dimensional complex Hilbert space with the scalar product $\langle x|y\rangle$ which is linear in x and conjugate linear in y, giving rise to the norm $\|x\| = \langle x|x\rangle^{1/2}$. We denote the open unit ball $\{e \in \mathbf{H} : \ \|e\| < 1\}$ with $\mathbf{B}$ and for any vector $a \in \mathbf{H}$ we shall write $a^* := [x \mapsto \langle x|a\rangle]$ for its dual functional.
Recall that the group $\mathrm{Aut}(\mathbf{B})$ of all holomorphic automorphisms of $\mathbf{B}$ consists the biholomorphic maps $\mathbf{B} \leftrightarrow \mathbf{B}$, and the $\mathbf{H}$-unitary operators restricted to $\mathbf{B}$ form the isotropy subgroup of the origin of $\mathrm{Aut}(\mathbf{B})$. Stone's classical theorem on strongly continuous one-parameter groups of unitary operators can be reformulated in terms of $\mathrm{Aut}(\mathbf{B})$ as a statement that the infinitesimal generator of a strongly continuous one-parameter subgroup of $\mathrm{Aut}(\mathbf{B})$ leaving fixed the origin can be identified canonically with the restriction $iA|\mathbf{B}$ where A is a possibly unbounded self-adjoint linear operator with dense domain in $\mathbf{H}$. Our purpose will be an analogous description for the strongly continuous one-parameter subgroups of $\mathrm{Aut}(\mathbf{B})$ formed by possibly non-linear maps, including approximations with norm-continuous one-parameter subgroups. We establish the following main result.

Theorem 1.1. *Assume $\big[\Psi^t : \ t \in \mathbb{R}\big]$ is a strongly continuous one-parameter group of holomorphic automorphisms of* $\mathbf{B}$.[1] *Then there exists a vector $\overline{x}$ with $\|\overline{x}\| \leq 1$ along with a constant $\lambda \in \mathbb{R}$ and a densely defined possibly unbounded self-adjoint operator $A : \mathbf{Z} \to \mathbf{H}$ with dense domain such that*

$$(1.2) \qquad \mathbf{B} \cap (\overline{x} + \mathbf{Z}) = \mathbf{D} \quad \textit{where} \quad \mathbf{D} := \big\{x \in \mathbf{D} : \ t \mapsto \Psi^t(x) \ \textit{is differentiable on } \mathbb{R}\big\},$$

$$(1.3) \qquad \frac{d}{dt}\Big|_{t=0} \Psi^t(x) = -\big\langle (iA - \lambda)(x - \overline{x})\big|\overline{x}\big\rangle x + (iA + \lambda)(x - \overline{x}) \qquad (x \in \mathbf{D}).$$

Given any tuple $(A, \overline{x}, \lambda)$ consisting of a densely defined self-adjoint operator $A : \mathbf{Z} \to \mathbf{H}$, a vector $\overline{x}$ with $\|\overline{x}\| \leq 1$ and a real number λ, there exists (a necessarily unique) strongly

[1] That is $\Psi^{t+s} = \Psi^t \circ \Psi^s \in \mathrm{Aut}(\mathbf{B})$ for all couples $t, s \in \mathbb{R}$ and the functions $[t \mapsto \Psi^t(x)]$ are continuous $\mathbb{R} \to \mathbf{H}$ for any fixed vector $x \in \mathbf{B}$.

continuous one-parameter group $[\Psi^t :\ t \in \mathbb{R}]$ *satisfying* (1.2) *and* (1.3) *if and only if one of the following alternatives holds:* (1) $\|\overline{x}\|=1$; (2) $\|\overline{x}\|<1$, $\lambda=0$.

It is tempting to conjecture that all strongly continuous one-parameter groups of holomorphic automorphisms of bounded circular domains in Banach spaces can be reconstructed similarly from a polynomial of second degree defined on some dense algebraic submanifold. However, the techniques we use here depend heavily on Hilbert space structure and the presence of a linear model in the lifting space $\mathcal{H} := \mathbf{H} \oplus \mathbb{C} = \big\{ \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] :\ x \in \mathbf{H},\ \xi \in \mathbb{C} \big\}$. We can formulate the background in terms of this linear representation in $\mathcal{H}$ for the group of the non non-linear maps Ψ^t as follows.

Theorem 1.4. *For any triple* $\mathfrak{A} = (A, \overline{x}, \lambda) \in \mathcal{L}^{\text{unbded}}_{\text{selfadj}}(\mathbf{H}) \times \overline{\mathbf{B}} \times \mathbb{R}$, *the operator*

$$\mathcal{A}_{\mathfrak{A}} \begin{bmatrix} x \\ \xi \end{bmatrix} := \begin{bmatrix} (iA + \lambda I)(x - \xi \overline{x}) \\ \langle (iA - \lambda I)(x - \overline{x}) | \overline{x} \rangle \end{bmatrix} \quad \textit{with} \quad \mathrm{dom}(\mathcal{A}_{\mathfrak{A}}) = \begin{bmatrix} \mathrm{dom}(A) \\ 0 \end{bmatrix} + \mathbb{C} \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}$$

is the infinitesimal generator of a strongly continuous one parameter subgroup $[\mathcal{U}^t_{\mathfrak{A}} :\ t \in \mathbb{R}]$ *of* $\mathcal{L}(\mathbf{H})$. *A family* $[\Psi^t :\ t \in \mathbb{R}]$ *of mappings* $\mathbf{B} \to \mathbf{B}$ *is a strongly continuous one-parameter subgroup of* $\mathrm{Aut}(\mathbf{B})$ *if and only if, in terms of the coordinates* $[\,.\,]_{\mathbf{H}}$ *resp.* $[\,.\,]_{\mathbb{C}}$ *in* $\mathcal{H}$,

$$\Psi^t(x) = \Big[\mathcal{U}^t_{\mathfrak{A}} \begin{bmatrix} x \\ 1 \end{bmatrix} \Big]_{\mathbb{C}}^{-1} \Big[\mathcal{U}^t_{\mathfrak{A}} \begin{bmatrix} x \\ 1 \end{bmatrix} \Big]_{\mathbf{H}} \qquad (x \in \mathbf{B},\ t \in \mathbb{R})$$

for some triple $\mathfrak{A} = (A, \overline{x}, \lambda)$ *satisfying* (1) *or* (2) *in Theorem 1.1.*

As an immediate consequence, with the *Yosida approximants* [4,p.205] we get the following.

Corollary 1.5. *Any strongly continuous one parameter subgroup* $[\Psi^t :\ t \in \mathbb{R}]$ *of* $\mathrm{Aut}(\mathbf{B})$ *can be approximated pointwise with a sequence* $[\Psi^t_n :\ t \in \mathbb{R}]$ $(n = 1, 2, \ldots)$ *of uniformly continuous one-parameter groups of fractional linear maps of the form*

$$\Psi^t_n(x) = \Big[\exp(t\mathcal{A}_{\mathfrak{A},n}) \begin{bmatrix} x \\ 1 \end{bmatrix} \Big]_{\mathbb{C}}^{-1} \Big[\exp(t\mathcal{A}_{\mathfrak{A},n}) \begin{bmatrix} x \\ 1 \end{bmatrix} \Big]_{\mathbf{H}}$$

by the aid of the bounded linear operators $\mathcal{A}_{\mathfrak{A},n} := n\mathcal{A}_{\mathfrak{A}} \big(n\mathrm{Id}_{\mathcal{H}} - \mathcal{A}_{\mathfrak{A}} \big)^{-1}$.

2. Preliminaries: linear model with joint fixed points for one-parameter groups

Recall [5, Ch. VI] that the group of $\mathrm{Aut}(\mathbf{B})$ all holomorphic automorphisms of $\mathbf{B}$ admits a matrix representation. Namely each element Ψ of $\mathrm{Aut}(\mathbf{B})$ has the fractional linear form

$$\Psi(x) = \frac{Ax + b}{\langle x | c \rangle + d}\,, \qquad A \in \mathcal{L}(\mathbf{H}), \quad b, c \in \mathbf{H}, \quad d \in \mathbb{C} \tag{2.1}$$

and we have

$$\Psi_1 \circ \Psi_2(x) = \Psi_1\big(\Psi_2(x)\big) = \frac{Ax+b}{\langle x|c\rangle + d} \quad \text{whenever} \quad \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} = \begin{bmatrix} A_1 & b_1 \\ c_1^* & d_1 \end{bmatrix} \begin{bmatrix} A_2 & b_2 \\ c_2^* & d_2 \end{bmatrix}.$$

This representation is unique up to a constant, since in (2.1) we necessarily have

$$\begin{bmatrix} A & b \\ c^* & d \end{bmatrix} = d \begin{bmatrix} \mathrm{Q}_a & a \\ a^* & 1 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{where} \quad a := \Psi(0),\ U = (\beta_a^2 \mathrm{P}_a + \beta_a \overline{\mathrm{P}}_a)^{-1} \Psi'(0)$$

in terms of the the Fréchet derivative Ψ' and the standard notations

$$\mathrm{P}_a := \big[\text{orthogonal projection } \mathbf{H} \to \mathbb{C}a\big], \quad \beta_a = \sqrt{1 - \|a\|^2}, \quad \mathrm{Q}_a := \mathrm{P}_a + \beta_a(I - \mathrm{P}_a).$$

We call the matrix

$$\widetilde{\Psi} := \begin{bmatrix} \mathrm{Q}_a & a \\ a^* & 1 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{Q}_a U & a \\ (U^* a)^* & 1 \end{bmatrix}$$

corresponding to the case with constant $d = 1$ the *canonical representation* of Ψ. In the sequel we shall write

$$\mathcal{H} := \mathbf{H} \oplus \mathbb{C} = \big\{\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] : x \in \mathbf{H},\ \xi \in \mathbb{C}\big\}$$

and identify the matrix $M := \big[m_{ij}\big]_{i,j=1}^2$ where $m_{11} \in \mathcal{L}(\mathbf{H})$, $m_{12} \in \mathbf{H}$, $m_{2,1} \in \mathbf{H}^*$ and $m_{22} \in \mathbb{C}$ with the linear operator $\big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big] \mapsto M \big[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\big]$ on $\mathcal{H}$. Notice that, by (2.1) we have

$$\Psi(x) = \left[\widetilde{\Psi} \begin{bmatrix} x \\ 1 \end{bmatrix}\right]_{\mathbb{C}}^{-1} \left[\widetilde{\Psi} \begin{bmatrix} x \\ 1 \end{bmatrix}\right]_{\mathbf{H}} \qquad (x \in \mathbf{B}) \tag{2.2}$$

where $[\,\cdot\,]_{\mathbb{C}}$ resp. $[\,\cdot\,]_{\mathbf{H}}$ are the standard notations for the canonical projections $\mathcal{H} \to \mathbb{C}$ resp. $\mathcal{H} \to \mathbf{H}$. It is immediate that any $\Psi \in \mathrm{Aut}(\mathbf{B})$ extends holomorphically to the ball $(1 - \|\Psi(0)\|)^{-1}\mathbf{B}$. Hence we can define the group of all automorphisms of the closed unit ball $\overline{\mathbf{B}} := \{x \in \mathbf{H} :\ \|x\| \leq 1\}$ as

$$\mathrm{Aut}(\overline{\mathbf{B}}) := \big\{\overline{\Psi} :\ \Psi \in \mathrm{Aut}\,(\mathbf{B})\big\} \quad \text{where} \quad \overline{\Psi} := \big[\text{continuous extension of } \Psi \text{ to } \overline{\mathbf{B}}\big].$$

It is also well-known [5, Ch.VI] that any mapping $\overline{\Psi} \in \mathrm{Aut}(\overline{\mathbf{B}})$ is weakly continuous and preserves the Grassmann family $\mathrm{Aff}(\overline{\mathbf{B}})$ of all complex affine closed subspaces intersected with $\overline{\mathbf{B}}$.[1] By Schauder's fixed point theorem, $\mathrm{Fix}(\overline{\Psi}) \neq \emptyset$ since $\overline{\mathbf{B}}$ is weakly compact. Moreover we have the following alternatives:

(1) $\mathrm{Fix}(\overline{\Psi}) \in \mathrm{Aff}(\overline{\mathbf{B}})$, (2) $\mathrm{Fix}(\overline{\Psi})$ consists of two boundary points.

In case (2) from the proof of [5, Thm.VI.4.8] we see even that $\overline{\Psi} = \overline{\Phi} \circ \overline{\Theta}_a \circ \overline{\Phi}^{\,-1}$ with a suitable automorphism $\Phi \in \mathrm{Aut}(\overline{\mathbf{B}})$ and a *Möbius shift*

$$\Theta_a : x \mapsto \frac{\mathrm{Q}_a x + a}{1 + \langle x|a\rangle} \tag{2.3}$$

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[1] If $x = \sum_{k=1}^2 \lambda_k x_k$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\sum_{k=1}^2 \lambda_k = 1$ then $\overline{\Psi}(x) = \sum_{k=1}^2 \alpha_k \overline{\Psi}(x_k)$ for some $\alpha_1, \alpha_2 \in \mathbb{C}$ with $\sum_{k=1}^2 \alpha_k = 1$ $\big($namely $\alpha_k = \lambda_k[1 + \langle x_k|U^* a\rangle]/[1 + \langle \lambda_1 x_1 + \lambda_2 x_2|U^* a\rangle]\big)$.

for some $0 \neq a \in \mathbf{B}$ such that $\mathrm{Fix}(\overline{\Theta}_a) = \{-e, e\}$ where $e := a/\|a\|$.

The next result is an infinite dimensional extension for a simple special case of a far reaching theorem of Abate [1] established for finite dimensional uniformly convex domains. It seems that Dineen's bidual embedding method [3] is suitable in proving a complete infinite dimensional analogy with uniformly convex domains in Banach spaces. Instead, below we give a short direct proof on the basis of the special algebraic form (2.1).

Lemma 2.4. *Abelian subsets of* $\mathrm{Aut}(\overline{\mathbf{B}})$*, in particular one-parameter subgroups, admit common fixed points.*

Proof. Assume $\{\overline{\Psi}_j : j \in \mathcal{J}\} \subset \mathrm{Aut}(\overline{\mathbf{B}})$ with $\overline{\Psi}_j \circ \overline{\Psi}_k = \overline{\Psi}_k \circ \overline{\Psi}_j$ $(j, k \in \mathcal{J})$. By weak continuity, for any index family $\mathcal{K} \subset \mathcal{J}$, the set $\bigcap_{k\in\mathcal{K}} \mathrm{Fix}(\overline{\Psi}_k)$ of common fixed points is weakly compact. Thus, according to Riesz' intersection theorem, it suffices to see that $\bigcap_{k\in\mathcal{K}} \mathrm{Fix}(\overline{\Psi}_k) \neq \emptyset$ for finite index families $\mathcal{K}$. By proceeding to contradiction, let $\mathcal{K} = \{k_1, \ldots, k_N\}$ be a subset of $\mathcal{J}$ with minimal cardinality such that $\bigcap_{k\in\mathcal{K}} \mathrm{Fix}(\overline{\Psi}_k) = \emptyset$. Necessarily $N > 1$ and $\mathbf{S} := \bigcap_{n=1}^{N-1} \mathrm{Fix}(\overline{\Psi}_{k_n}) \neq \emptyset$ is either a weakly compact convex subset of $\overline{\mathbf{B}}$ or it consists of two boundary points. Since $\overline{\Psi}_{k_N}$ commutes with all the maps $\overline{\Psi}_{k_n}$ $(n < N)$, we have $\overline{\Psi}_{k_N}(\mathbf{S}) \subset \mathbf{S}$. Indeed, if $x \in \mathbf{S}$ then $\overline{\Psi}_{k_n}\big(\overline{\Psi}_{k_N}(x)\big) = \overline{\Psi}_{k_N}\big(\overline{\Psi}_{k_n}(x)\big) = \overline{\Psi}_{k_N}(x) \Rightarrow \overline{\Psi}_{k_N}(x) \in \mathrm{Fix}(\overline{\Psi}_{k_n})$ $(n < N)$. Hence Schauder's fixed point theorem excludes the case of $\mathbf{S}$ being convex. Suppose $\mathbf{S} = \{p, q\} \subset \partial\mathbf{B}$. Then necessarily $\mathrm{Fix}(\overline{\Psi}_{k_M}) = \{p, q\}$ for some index $M < N$ and $\overline{\Psi}_{k_N} : p \leftrightarrow q$. However, in this case we can write $\overline{\Psi}_{k_N} = \overline{\Phi} \circ \overline{\Theta}_a \circ \overline{\Phi}^{-1}$ with a suitable automorphism $\overline{\Phi} \in \mathrm{Aut}(\overline{\mathbf{B}})$ and a Möbius shift $\overline{\Theta}_a$ of the form (2.3) where $0 \neq a \in \mathbf{B}$. Then, by setting $\overline{\Omega} := \overline{\Phi}^{-1} \circ \overline{\Psi}_{k_M} \circ \overline{\Phi}$ and $e := a/|a\|$, we have $\{\pm e\} = \mathrm{Fix}(\overline{\Theta}_a)$ and $\overline{\Omega} : e \leftrightarrow -e$. On the other hand, it is immediate that $e = \lim_{n\to\infty} \overline{\Theta}_a^n(x)$ for every point $x \in \overline{\mathbf{B}} \setminus \{e\}$. Taking any point $f \in \mathrm{Fix}(\overline{\Omega})$ we get the contradiction $e = \lim_{n\to\infty} \overline{\Theta}_a^n(f) = \lim_{n\to\infty} \overline{\Theta}_a^n \circ \overline{\Omega}(f) = \lim_{n\to\infty} \overline{\Omega} \circ \overline{\Theta}_a^n(f) = \overline{\Omega}(e) = -e$.

Remark 2.5. In finite dimensions, it is customary to normalize (2.1) by requiring $\det \left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] = 1$. Thus, in case of $\dim(\mathbf{H}) = N$, in this manner one can establish a canonical identification of $\mathrm{Aut}(\mathbf{B})$ with a subgroup of the classical matrix group $\mathrm{SL}(N+1)$. Though in infinite dimensions such a normalization is not available, for one-parameter groups with common fixed point there is an alternative way as follows.

Definition 2.6. Let $\big([\Psi^t : t \in \mathbb{R}], \overline{x}\big)$ be a couple of a one-parameter subgroup of $\mathrm{Aut}(\mathbf{B})$ with common fixed point $\overline{x}$ for the continuous extensions of its members to $\overline{\mathbf{B}}$: $\overline{x} \in \overline{\mathbf{B}}$, $\overline{\Psi}^t(\overline{x}) = \overline{x}$ $(t \in \mathbb{R})$. In terms of the canonical representations define

$$\widehat{\Psi}^t_{\overline{x}} := \left[\widetilde{\Psi}^t \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}\right]_{\mathbb{C}}^{-1} \widetilde{\Psi}^t = \frac{1}{1 + \langle U_t \overline{x} | a_t \rangle} \begin{bmatrix} Q_t & a_t \\ a_t^* & 1 \end{bmatrix} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix} \qquad (t \in \mathbb{R}).$$

where $a_t = \Psi_t(0) \in \mathbf{B}$, $U_t \in \mathcal{L}(\mathbf{H})$ is a suitable unitary operator and $Q_t := \mathrm{Q}_{a_t} = P_t + \beta_t(I - P_t)$ with $P_t x := \mathrm{P}_{a_t} x = \|a_t\|^{-2} \langle x | a_t \rangle a_t$, $\beta_t = \sqrt{1 - \|a_t\|^2}$.
Later on, conveniently we shall simply write $\widehat{\Psi}^t$ instead of $\widehat{\Psi}^t_{\overline{x}}$ without danger of confusion.

Remark 2.7. As we have seen $\Psi^t(x) = (1 + \langle U_t x | a_t\rangle)^{-1}[Q_t U^t x + a_t] = \big[\Theta_{a_t} \circ U_t\big](x)$. Thus, by construction we have

$$\overline{\Psi^t}(x) = \Big[\widehat{\Psi}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbb{C}}^{-1} \Big[\widehat{\Psi}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbf{H}} \quad (x \in \overline{\mathbf{B}}), \qquad \widehat{\Psi}^t \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}.$$

It is worth to notice that the term $\langle U_t \overline{x} | a_t\rangle$ is actually independent of U_t as

$$\langle U_t \overline{x} | a_t\rangle = \frac{\langle \overline{x} - a_t | a_t\rangle}{1 - \langle \overline{x} | a_t\rangle}, \qquad \widehat{\Psi}^t = \frac{1 - \langle \overline{x} | a_t\rangle}{1 - \langle a_t | a_t\rangle} \begin{bmatrix} Q_t & a_t \\ a_t^* & 1 \end{bmatrix} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.8}$$

Proof: In general we have $P_t y = \langle y | a_t\rangle \langle a_t | a_t\rangle^{-2} a_t$ $(0 \neq a_t, y \in \mathbf{H})$. It follows $\langle P_t U_t \overline{x} | a_t\rangle = \langle U_t \overline{x} | a_t\rangle$ with $\langle \overline{P}_t U_t \overline{x} | a_t\rangle = 0$ for any $t \in \mathbb{R}$. Thus multiplying the fixed point equation $\overline{x} = \overline{\Psi}^t(\overline{x}) = (1 + \langle U_t \overline{x} | a_t\rangle)^{-1}(P_t + \beta_t \overline{P}_t) U_t \overline{x}$ with $|a_t\rangle$, we get $(1 + \langle U_t \overline{x} | a_t\rangle)^{-1} \langle U_t \overline{x} + a_t | a_t\rangle = \langle \overline{x} | a_t\rangle$ whence the relations (2.8) are immediate.

The power style indexing of $\widehat{\Psi}^t$ in t is justified by the proposition below.

Proposition 2.9. *Given a strongly continuous one-parameter group* $[\Psi^t : t \in \mathbb{R}]$ *in* $\mathrm{Aut}(\mathbf{B})$ *with common fixed point* $\overline{x} \in \overline{\mathbf{B}}$*, the family* $[\widehat{\Psi}^{t}_{\overline{x}} : \ t \in \mathbb{R}]$ *is a strongly continuous one-parameter group of operators in* $\mathcal{H}$.

Proof. Since $\Psi^t \circ \Psi^s = \Psi^{t+s}$ $(t, s \in \mathbb{R})$, for the representation matrices we have $\widehat{\Psi}^t \widehat{\Psi}^s = d_{t,s} \widehat{\Psi}^{t+s}$ with suitable constants $d_{t,s} \in \mathbb{C}_*$. The fixed point property $\overline{\Psi}^t(\overline{x}) = \overline{x}$ implies

$$\widehat{\Psi}^t \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} \qquad (t \in \mathbb{R}).$$

Hence necessarily $d_{t,s} = 1$ $(t, s \in \mathbb{R})$, thus $\big[\widehat{\Psi}^t : t \in \mathbb{R}\big]$ is a one-parameter matrix group. By assumption, the function $t \mapsto a_t = \Psi^t(0)$ is norm-continuous $\mathbb{R} \to \mathbf{B}$. Hence we can deduce the strong continuity of the $\mathbf{H}$-unitary operator valued function $t \mapsto U_t$. Namely consider any vector $x \in \mathbf{H}$. To establish the norm-continuity of the function $t \mapsto U_t$, we may assume without loss of generality that $x \in \mathbf{B}$. Then, by the aid of the Möbius shifts (2.3) we can write

$$U_t x = \big[\Theta_{a_t}^{-1} \circ \Psi_t\big](x) = \Theta_{-a_t}\big(\Psi(x)\big) \qquad (t \in \mathbb{R}).$$

Observe that the norm continuity of $t \mapsto a_t$ implies the continuity of $t \mapsto \langle x | a_t\rangle$ and $t \mapsto \beta_t \in [0, 1)$ entailing the norm-continuity of $t \mapsto P_t + \beta_t \overline{P}_t \in \overline{\mathrm{Ball}}\big(\mathcal{L}(\mathbf{H})\big)$. Hence the required norm-continuity of $t \mapsto U_t x = (1 - \langle x | a_t\rangle)^{-1}[(P_t x - a_t + \beta_t \overline{P}_t x]$ is immediate. In general, the product of two bounded strongly continuous linear operator valued functions $\mathbb{R} \to \mathcal{L}(\mathbf{X})$ over a normed space $\mathbf{X}$ is strongly continuous. Hence we conclude that the entries $(1,1), (1,2), (2,1)$ resp. $(2,2)$ of the matrices $\widehat{\Psi}^t$ are strongly continuous functions $\mathbb{R} \to \mathcal{L}(\mathbf{H})$, $\mathbb{R} \to \mathbf{H}$, $\mathbb{R} \to \mathbf{H}^* \simeq \mathbf{H}$ resp. $\mathbb{R} \to \mathbb{R}$ which completes the proof. Q.e.d.

Corollary 2.10. *Given a strongly continuous one-parameter group $[\mathcal{T}^t:\ t\in\mathbb{R}]$ in $\mathcal{L}(\mathcal{H})$, the following statements are equivalent*

(i) *for all $t\in\mathbb{R}$, the maps $x\mapsto \big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbb{C}}^{-1}\big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbf{H}}$ belong to* $\mathrm{Aut}(\mathbf{B})$*;*

(ii) *we have $\mathcal{T}^t=e^{\mu t}\widehat{\Psi}^t$ $(t\in\mathbb{R})$ for some strongly continuous one-parameter subgroup $[\Psi^t:\ t\in\mathbb{R}]$ of $\mathrm{Aut}(\mathbf{B})$ and a constant $\mu\in\mathbb{C}$;*

(iii) *each operator $\mathcal{T}^t$ maps the cone $\mathcal{K}:=\big\{\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]:\ |\xi|^2>\|x\|^2\big\}$ onto itself;*

(iv) *each operator $\mathcal{T}^t$ maps $\partial\mathcal{K}:=\big\{\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]:\ |\xi|^2=\|x\|^2\big\}$ onto itself.*

Proof. The implication (ii)⇒(i) is trivial by (2.2).

Proof of (i)⇒(ii): By assumption the maps $\Psi^t(x):=\big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbb{C}}^{-1}\big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbf{H}}$ $(t\in\mathbb{R},\ x\in\mathbf{B})$. are well-defined holomorphic automorphisms of the unit ball $\mathbf{B}$. By (2.2) we have $\mathcal{T}^t=d_t\widehat{\Psi}^t$ $(t\in\mathbb{R})$ with suitable constants $d_t\in\mathbb{C}_*$. Fixing any point $x\in\mathbf{B}$, the strong continuity of the group $[\mathcal{T}^t:\ t\in\mathbb{R}]$ implies the continuity of the function $\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]$ whence we deduce also the continuity of $t\mapsto\Psi^t(x)$ which entails the continuity of $t\mapsto\widehat{\Psi}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]=d_t^{-1}\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]$ and hence the continuity of $t\mapsto d_t$. By the one-parameter group property, all the relations $\mathcal{T}^{t+s}=\mathcal{T}^t\mathcal{T}^s$, $\widehat{\Psi}^{t+s}=\widehat{\Psi}^t\widehat{\Psi}^s$ $(t,s\in\mathbb{R})$ hold. Therefore $d_{t+s}=d_td_s$ $(t,s\in\mathbb{R})$ and the continuity of $t\mapsto d_t$ establishes the existence of a constant $\mu\in\mathbb{C}$ with $d_t=e^{\mu t}$ $(t\in\mathbb{R})$.

Proof of (i)⇔(iii)⇔(iv): Consider the projective Hilbert space $\mathcal{H}_*/\!\approx$ associated with $\mathcal{H}$ regarded as the set of all nontrivial punctured complex rays $\mathbb{C}_*\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]$ with the factor topology.[2] By homogeneity, any injective linear operator $\mathcal{T}\in\mathcal{L}(\mathcal{H})$ acts holomorphically on $\mathcal{H}_*/\!\approx$ by its factorization $\mathcal{T}_\approx:\mathbb{C}_*\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]\mapsto\mathbb{C}_*\mathcal{T}\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]$. In particular, as admitting a continuous inverse, each map $\mathcal{T}_\approx^t$ is a holomorphic automorphism of $\mathcal{H}/\approx$. Hence the equivalences (i)⇔(iii)⇔(iv) are straightforward consequences of the facts that, with the embedding $\Pi:x\mapsto\mathbb{C}\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]$ and its inverse $\pi\big(\mathbb{C}\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]\big):=x/\xi$ $(\xi\neq0)$, we have $\Pi\mathbf{B}:=\mathcal{K}$, $\pi\mathcal{K}=\mathbf{B}$ and $\big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbb{C}}^{-1}\big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbf{H}}=\pi\circ\mathcal{T}_\approx^t\circ\Pi\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]$ whenever $\big[\mathcal{T}^t\big[\begin{smallmatrix}x\\1\end{smallmatrix}\big]\big]_{\mathbb{C}}\neq0$.

Corollary 2.11. *Given any $\Theta\in\mathrm{Aut}(\overline{\mathbf{B}})$, the Θ-shifted automorphisms $\Phi^t:=\Theta\circ\Psi^t\circ\Theta^{-1}$ form strongly continuous one-parameter group with common fixed point $\overline{y}:=\Theta(\overline{x})$ when extended continuously to $\overline{\mathbf{B}}$ and $\widehat{\Phi}_{\overline{y}}^t=e^{\mu t}\widetilde{\Theta}^{-1}\widehat{\Psi}_{\overline{x}}^t\widetilde{\Theta}$ $(t\in\mathbb{R})$ for some $\mu\in\mathbb{C}$.*

3. Infinitesimal generators

Throughout this section, let $\big([\Psi^t:\ t\in\mathbb{R}],\overline{x}\big)$ be an arbitrarily fixed couple of a strongly continuous one-parameter group in $\mathrm{Aut}(\mathbf{B})$ with a common fixed point for the continuous extensions in $\overline{\mathbf{B}}$. Recalling the Hille–Yosida theorem [9, Kap.10], Proposition 2.9 ensures that the differential

$$(3.1)\quad \mathcal{A}:\mathfrak{h}\mapsto\frac{d}{dt}\widehat{\Psi}^t\mathfrak{h}\quad\text{with}\quad\mathcal{D}:=\mathrm{dom}(\mathcal{A})=\big\{\mathfrak{h}\in\mathcal{H}:t\mapsto\widehat{\Psi}^t\mathfrak{h}\ \text{is differentable on}\ \mathbb{R}\big\}$$

[2] As usually, $\mathcal{H}_*:=\mathcal{H}\setminus\{0\}$ with the equivalence relation $\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]\sim\big[\begin{smallmatrix}y\\\eta\end{smallmatrix}\big]:\iff\ \mathbb{C}_*\big[\begin{smallmatrix}y\\\eta\end{smallmatrix}\big]=\mathbb{C}_*\big[\begin{smallmatrix}x\\\xi\end{smallmatrix}\big]$ where $\mathbb{C}_*:=\mathbb{C}\setminus\{0\}$. A subset of $\mathcal{H}/\approx$ is open if the union of it members (rays in $\mathcal{H}_*$) is open in $\mathcal{H}$.

(called the *infinitesimal generator* of the linear model $\big[\widehat{\Psi}^t\colon t\in\mathbb{R}\big]$ where $\widehat{\Psi}^t\equiv\widehat{\Psi}^t_{\overline{x}}$ for short) is a not necessarily bounded linear map with closed graph and $[\widehat{\Psi}^t : t\in\mathbb{R}]$-invariant domain being dense in $\mathcal{H}$. Instead of the differential $\mathcal{A} = \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t$ of the representations we are primarily interested in the differential

$$\Omega := \frac{d}{dt}\Big|_{t=0}\Psi^t : \mathbf{D}\to\mathbf{H} \quad \text{where} \quad \mathbf{D} = \mathrm{dom}(\Omega) = \Big\{x\in\mathbf{B} :\ \frac{d}{dt}\Big|_{t=0}\Psi^t(x) \text{ exists}\Big\}.$$

In order that we could regard the vector field Ω as a non-linear infinitesimal generator for $[\Psi^t :\ t\in\mathbb{R}]$, we should see the density of $\mathbf{D}$ in $\overline{\mathbf{B}}$. In order to establish a non-linear Stone-type theorem, we should determine precise links to self-adjoint linear operators.

Lemma 3.2. $\mathbf{D}$ *is* $\big[\overline{\Psi}^t : t\in\mathbb{R}\big]$*-invariant. We have* $\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\in\mathcal{D} \iff x\in\mathbf{D}$ *whenever* $x\in\mathbf{B}$.

Proof. The $\big[\Psi^t : t\in\mathbb{R}\big]$-invariance of $\mathbf{D}$ is clear from the group property $\Psi^{t+s} = \Psi^t\circ\Psi^s$ $(t,s\in\mathbb{R})$. Moreover even $\mathrm{dom}\big(\frac{d}{dt}\big|_{t=0}\Theta\circ\Psi^t\circ\Theta^{-1}\big) = \Theta(\mathbf{D})$ whenever Θ is any holomorphic automorphism of $\mathbf{B}$. Hence, given any point $x\in\mathbf{B}$, we have $x\in\mathbf{D}$ if and only if $0 = \Theta_{-x}(x)\in\mathrm{dom}\big(\frac{d}{dt}\big|_{t=0}\Phi^t\big)$ with the one-parameter group of the maps $\Phi^t := \Theta_{-x}\circ\Psi^t\circ\Theta_x$ in terms of the Möbius transformations (2.3). That is, without loss of generality, it suffices only to see the equivalence $0\in\mathrm{dom}\big(\frac{d}{dt}\big|_{t=0}\Phi^t\big) \iff \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\in\mathcal{D}$. According to (2.8), by setting $a_t := \Psi^t(0)$ and $\mathfrak{a}_t := \widehat{\Psi}^t\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ we have $\mathfrak{a}_t = (1-\langle a_t|\overline{x}\rangle(1-\|a_t\|^2)^{-1}\left[\begin{smallmatrix}a_t\\1\end{smallmatrix}\right]$ $(t\in\mathbb{R})$. Hence the curves $t\mapsto a_t$ resp. $t\mapsto\mathfrak{a}_t$ are differentiable in the same time, which completes the proof.

For later use we also introduce the notations

$$\mathbf{Z} := \Big\{z\in\mathbf{H} :\ \begin{bmatrix}z\\0\end{bmatrix}\in\mathcal{D}\Big\}; \qquad Bz := \Big[\mathcal{A}\begin{bmatrix}z\\0\end{bmatrix}\Big]_{\mathbf{H}},\ \Lambda z := \Big[\mathcal{A}\begin{bmatrix}z\\0\end{bmatrix}\Big]_{\mathbb{C}} \quad (z\in\mathbf{Z}).$$

Lemma 3.3. $\mathbf{Z}$ *is a dense linear submanifold in* $\mathbf{H}$ *with* $\mathbf{D} = \big(\overline{x}+\mathbf{Z}\big)\cap\mathbf{B}$ *and* $\mathcal{D} = \left[\begin{smallmatrix}\mathbf{Z}\\0\end{smallmatrix}\right]+\mathbb{C}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right]$. *The set* $\mathbf{D}$ *is dense in* $\mathbf{B}$ *and*

$$\frac{d}{dt}\Big|_{t=0}\overline{\Psi}^t(x) = [\Lambda(\overline{x}-x)]x + B(x-\overline{x}) \qquad (x\in\mathbf{D}).$$

Proof. By definition, $\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right]\in\mathcal{D}$ with $\mathcal{A}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = 0$ since $\widehat{\Psi}^t\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right]$ $(t\in\mathbb{R})$. Since $\mathcal{D}$ is closed for linear combinations, it follows that $\left[\begin{smallmatrix}\mathbf{Z}\\0\end{smallmatrix}\right]+\mathbb{C}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = \mathcal{D}$ and that $\mathbf{Z}$ is the image of $\mathcal{D}$ by the bounded linear operator $\Pi\left[\begin{smallmatrix}x\\\xi\end{smallmatrix}\right] := x-\xi\overline{x}$. Since $\Pi\mathcal{H} = \mathbf{H}$ and since $\mathcal{D}$ is dense in $\mathcal{H}$, $\mathbf{Z} = \Pi\mathcal{D}$ is also dense in $\mathbf{H} = \Pi\mathcal{H}$. From Lemma 3.2 we know that $\mathbf{D} = \mathbf{B}\cap\big\{x :\ \left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\in\mathcal{D}\big\}$. Hence the relation $\mathbf{D} = \big(\overline{x}+\mathbf{Z}\big)\cap\mathbf{B}$ along with the density of $\mathbf{D}$ in $\mathbf{B}$ is immediate. Given any $x\in\mathcal{D}$, the relation $\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\in\mathcal{D}$ implies $\mathcal{A}\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right] = \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]$. Since $\overline{\Psi}^t(x) = \big\{\widehat{\Psi}^t\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\big\}_{\mathbb{C}}^{-1}\big\{\widehat{\Psi}^t\left[\begin{smallmatrix}x\\1\end{smallmatrix}\right]\big\}_{\mathbf{H}}$ along with $\widehat{\Psi}^0 = \mathrm{Id}$ and $\mathcal{A}\left[\begin{smallmatrix}\overline{x}\\1\end{smallmatrix}\right] = 0$, we get

$$\frac{d}{dt}\Big|_{t=0}\overline{\Psi}^t(x) = -\Big[\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbb{C}}^{-2}\Big(\frac{d}{dt}\Big|_{t=0}\Big\{\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big\}_{\mathbb{C}}\Big)\Big[\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbf{H}} + \Big[\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbb{C}}^{-1}\frac{d}{dt}\Big|_{t=0}\Big\{\widehat{\Psi}^0\begin{bmatrix}x\\1\end{bmatrix}\Big\}_{\mathbf{H}} =$$

$$= -\Big[\mathcal{A}\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbb{C}}x + \Big[\mathcal{A}\begin{bmatrix}x\\1\end{bmatrix}\Big]_{\mathbf{H}} = -\Big[\mathcal{A}\begin{bmatrix}x-\overline{x}\\0\end{bmatrix}\Big]_{\mathbb{C}}x + \Big[\mathcal{A}\begin{bmatrix}x-\overline{x}\\0\end{bmatrix}\Big]_{\mathbf{H}}.$$

Lemma 3.4. *Suppose a Hilbert space* $\mathbf{W}$ *is the orthogonal sum of the subspaces* $\mathbf{W}_1, \mathbf{W}_2$ *and* $\mathcal{C}$ *is the infinitesimal generator of a strongly continuous one-parameter subgroup* $[T^t : t\in\mathbb{R}]$ *of* $\mathcal{L}(\mathbf{W})$. *Then, for the cone* $\mathbf{K} := \{w_1\oplus w_2 : \|w_1\| > \|w_2\|\}$, *we have* $T^t\mathbf{K} = \mathbf{K}$ $(t\in\mathbb{R})$ *if and only if* $\mathcal{C}$ *is tangent to the boundary of* $\mathbf{K}$ *that is if*

$$\text{(3.5)}\qquad \mathrm{Re}\langle \mathcal{C}(w_1\oplus w_2)\big|w_1\rangle = \mathrm{Re}\langle \mathcal{C}(w_1\oplus w_2)\big|w_2\rangle \quad \big(w_1\oplus w_2\in \mathrm{dom}(\mathcal{C}),\ \|w_1\| = \|w_2\|\big).$$

Proof. It is immediate that $T^t\mathbf{K}\subset \mathbf{K}$ $(t\in\mathbb{R}) \Rightarrow T^t\partial\mathbf{K}\subset\partial\mathbf{K}$ $(t\in\mathbb{R}) \Rightarrow \frac{d}{dt}|_{t=0}T^t(w_1\oplus w_2) \in \mathrm{Tan}_{w_1\oplus w_2}(\mathbf{K})$ for $w_1\oplus w_2 \in \mathrm{dom}(\mathcal{C}) \Rightarrow$ (3.5). Assume (3.5) and let P denote the canonical projection of $\mathbf{W}$ onto $\mathbf{W}_1$ and define $T^{t,s} := \exp(t\mathcal{C} + sP)$ $(s,t\in\mathbb{R})$. By the theorem of bounded perturbations [4, p.158] the operators $T^{t,s}$ are all well-defined. Moreover, by [4, Corollary 1.7 p. 161] (applied with $B := sP$ and $A := \mathcal{C}$ there) we have $\lim_{s\to 0} T^{t,s}w = T^t w$ $(w\in\mathrm{dom}(\mathcal{C}), t\in\mathbb{R})$. Therefore, to establish that $T^t\mathbf{K}\subset\mathbf{K}$ $(t\in\mathbb{R})$, it suffices to see only that $T^{t,s}w\in\mathbf{K}$ whenever $w\in\mathrm{dom}(\mathcal{C})\cap\mathbf{K}$ and $t,s>0$. To proceed to contradiction, let $s,t>0$ and $w := w_1\oplus w_2 \in \mathrm{dom}(\mathcal{C})$ with $\|w_1\| > \|w_2\|$ but $\|[T^{t,s}w]_1\| \leq \|[T^{t,s}w]_2\|$. The function $\delta(\tau) := \|[T^{\tau,s}w]_1\|^2 - \|[T^{\tau,s}w]_2\|^2$ is differentiable in τ on the whole $\mathbb{R}$ and $\delta(0) > 0 \geq \delta(t)$. Thus there exist a point $t_*\in(0,t]$ such that $\delta(\tau) > 0 = \delta(t_*)$ $(0\leq\tau<t_*)$. Since $\delta(t_*) = 0$, the vector $w_* := T^{t_*,s}w$ belongs to $\partial\mathbf{K}$ and hence $\mathrm{Re}\langle\mathcal{C}w_*|[w_*]_1\rangle = \mathrm{Re}\langle\mathcal{C}w_*|[w_*]_2\rangle$. We get the contradiction $0 \geq \delta'(t_*) = 2\mathrm{Re}\langle(\mathcal{C}+sP)w_*|[w_*]_1\rangle - 2\mathrm{Re}\langle(\mathcal{C}+sP)w_*|[w_*]_2\rangle = 2s\|[w_*]_1\|^2 > 0$.

Corollary 3.6. $\mathrm{Re}\Big(-\Lambda v + \langle Bv|\overline{x}\rangle + \langle Bv|v\rangle\Big) = 0$ *whenever* $v\in\mathbf{Z}$ *with* $\|\overline{x}+v\| = 1$.

Proof. By Corollary 2.10, we have $\widehat{\Psi}^t\mathcal{K} = \mathcal{K}$ $(t\in\mathbb{R})$ where $\mathcal{K} := \big\{\left[\begin{smallmatrix}x\\ \xi\end{smallmatrix}\right] : |\xi| > \|x\|\big\}\subset\mathcal{H}$. An application of Lemma 3.4 with $\mathbf{W}_1 := \mathbb{C}$, $\mathbf{W}_2 := \mathbf{H}$, $\mathbf{K} := \mathcal{K}$, $T^t := \widehat{\Psi}^t$, $\mathcal{C} := \mathcal{A}$ establishes that $\mathrm{Re}\,[(\Lambda x)\overline{\xi}] = \mathrm{Re}\,\langle Bx|x\rangle$ whenever $\left[\begin{smallmatrix}x\\ \xi\end{smallmatrix}\right]\in\mathcal{D}$ and $\|x\| = |\xi|$. We obtain the statement with the choice $x := v+\overline{x}$ and $\xi := 1$ if $v\in\mathbf{Z}$ with $\|v+\overline{x}\| = 1$ because then, by Lemma 3.3, we have $\left[\begin{smallmatrix}x\\ 1\end{smallmatrix}\right]\in\mathcal{D}$.

Proposition 3.7. *For some symmetric linear operator* $A : \mathbf{Z}\to\mathbf{H}$ *and a suitable constant* $\lambda\in\mathbb{R}$ *which is necessarily* $= 0$ *if* $\|\overline{x}\|\neq 1$, *we have* (1.3) *as*

$$B = iA + \lambda I, \qquad \Lambda z = \big\langle (iA-\lambda I)z\big|\overline{x}\big\rangle \quad (z\in\mathbf{Z}).$$

Proof. Consider any vector $0\neq z\in\mathbf{Z}$. Let $\zeta\in\mathbb{C}$ be the (unique) constant such that $\overline{x}+\zeta z\perp z$ and define $\varrho := \sqrt{1-\|\overline{x}+\zeta z\|^2}$. Actually we have $\zeta = -\langle\overline{x}|z\rangle/\langle z|z\rangle$ and $1\geq\|\overline{x}\|^2 = \|\overline{x}+\zeta z\|^2 + \|-\zeta z\|^2$ showing that both ζ and ϱ are well-defined. Consider the unit vectors

$$v_\varphi := \overline{x} + \zeta z + e^{i\varphi}\varrho z \qquad (\varphi\in\mathbb{R}).$$

According to Corollary 3.6,

$$\mathrm{Re}\Big((\zeta+e^{i\varphi}\varrho)\big[-\Lambda z + \langle Bz\,|\,\overline{x}\rangle\big] + |\zeta+e^{i\varphi}\varrho|^2\langle Bz\,|\,z\rangle\Big) = 0 \qquad (\varphi\in\mathbb{R}).$$

Thus the identity $\operatorname{Re}\bigl(\alpha+\beta e^{i\varphi}+\gamma e^{-i\varphi}\bigr)=0$ for all $\varphi\in\mathbb{R}$ holds with the constants $\alpha:=\zeta\bigl[-\Lambda z+\bigl\langle Bz\,\big|\,\overline{x}\bigr\rangle\bigr]+(|\zeta|^2+\varrho^2)\bigl\langle Bz\,\big|\,z\bigr\rangle$, $\beta:=\varrho\bigl[-\Lambda z+\bigl\langle Bz\,\big|\,\overline{x}\bigr\rangle+\overline{\zeta}\bigl\langle Bz\,\big|\,z\bigr\rangle\bigr]$ and $\gamma:=\varrho\zeta\bigl\langle Bz\,\big|\,z\bigr\rangle$. Since $2\operatorname{Re}\bigl(\alpha+\beta e^{i\varphi}+\gamma e^{-i\varphi}\bigr)=2\operatorname{Re}(\alpha)+(\beta+\overline{\gamma})e^{i\varphi}+(\overline{\beta}+\gamma)e^{-i\varphi}$, we have necessarily $\operatorname{Re}(\alpha)=\beta+\overline{\gamma}=0$. From the relation $\beta+\overline{\gamma}=0$, it follows

$$\Lambda z-\langle Bz\,|\,\overline{x}\rangle=2\,\overline{\zeta}\operatorname{Re}\langle Bz\,|\,z\rangle=-2\,\langle z\,|\,\overline{x}\rangle\,\frac{\operatorname{Re}\langle Bz\,|\,z\rangle}{\langle z\,|\,z\rangle}\ , \tag{3.8}$$

and substituting this into the relation $0=\operatorname{Re}(\alpha)$, we get

$$0=(\varrho^2-|\zeta|^2)\operatorname{Re}\bigl\langle Bz\,\big|\,z\bigr\rangle=\bigl(1-\|\overline{x}\|^2\bigr)\operatorname{Re}\bigl\langle Bz\,\big|\,z\bigr\rangle\ . \tag{3.8$'$}$$

From (3.8) we see also that

$$z\mapsto\frac{\operatorname{Re}\bigl\langle Bz\,\big|\,z\bigr\rangle}{\bigl\langle z\,\big|\,z\bigr\rangle}=-\frac{1}{2}\,\frac{\Lambda z-\bigl\langle Bz\,\big|\,\overline{x}\bigr\rangle}{\bigl\langle z\,\big|\,\overline{x}\bigr\rangle}$$

is a real valued Gâteaux holomorphic function on the algebraically open and in $\mathbf{Z}$ algebraically dense domain $\{z\in\mathbf{Z}:\,z\not\perp\overline{x}\}$ which is possible only if being constant on $\mathbf{Z}$. By writing λ for this constant value, from (3.7) and (3.8$'$) we conclude that

$$\Lambda z=\bigl\langle(B-2\lambda I)z\,\big|\,\overline{x}\bigr\rangle,\quad \operatorname{Re}\bigl\langle\bigl(B-\lambda I\bigr)z\,\big|\,z\bigr\rangle=0\qquad(z\in\mathbf{Z}),$$

and, in particular, if $\|\overline{x}\|<1$ then necessarily $\lambda=0$ above. By setting $A:=-i(B+\lambda I)$, hence the statement including the symmetry of A is immediate.

The following geometric converse of 3.7 is elementary:

Remark 3.9. Given a dense linear submanifold $\mathbf{Z}$ in $\mathbf{H}$, a symmetric linear operator $A:\mathbf{Z}\to\mathbf{H}$, a vector $\overline{x}\in\overline{\mathbf{B}}$, the vector field $\Omega_\lambda(x):=-\langle(iA-\lambda I)(x-\overline{x}|\overline{x}\rangle x+(iA+\lambda I)(x-\overline{x})$ define for $\overline{x}+\mathbf{Z}$ is tangent to the unit sphere $\partial\mathbf{B}$ at the points $x\in\overline{x}+\mathbf{Z}$ with $\|x\|=1$ whenever either $\|\overline{x}\|=1$ and $\lambda\in\mathbb{R}$ or $\|\overline{x}\|<1$ and $\lambda=0$.

In the sequel we proceed to the problem if the operator A in Proposition 3.7 arising from the differential $\frac{d}{dt}\big|_{t=0}\Psi^t$ of a strongly continuous one-parameter semigroup of $\operatorname{Aut}(\mathbf{B})$ is necessarily self-adjoint and conversely if every self-adjoint operator may appear in 3.7.

4. The Jordan case

We continue the previous investigations with unchanged notations but under the additional hypothesis that

$$0\in\mathbf{D}=\operatorname{dom}\Bigl(\frac{d}{dt}\Big|_{t=0}\Psi^t\Bigr)\ . \tag{4.1}$$

As we know, $\mathbf{D} = \{x \in \mathbf{B} : \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathcal{D}\} = \mathbf{B} \cap (\overline{x} + \mathbf{Z})$ where $\mathcal{D} = \mathrm{dom}(\mathcal{A})$ with $\mathcal{A} = \frac{d}{dt}\big|_{t=0} \widehat{\Psi}^t_{\overline{x}}$ and $\mathbf{Z} = \{x \in \mathbf{H} : \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{D}\}$ is a dense complex linear submanifold in $\mathbf{H}$. Thus, as a consequence of (4.1), for the distinguished common fixed point of the extended automorphisms $\overline{\Psi}^t$ we have $\overline{x} \in \mathbf{Z} = \mathrm{dom}(B) = \mathrm{dom}(\Lambda)$. Therefore also

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{D} = \begin{bmatrix} \mathbf{Z} \\ \mathbb{C} \end{bmatrix}, \quad \mathcal{A}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} B(x - \xi\overline{x}) \\ \Lambda(x - \xi\overline{x}) \end{bmatrix} = \begin{bmatrix} B & -B\overline{x} \\ \Lambda & -\Lambda\overline{x} \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} \quad (x \in \mathbf{Z},\ \xi \in \mathbb{C}).$$

Remark 4.2. Recall [8] that the complete holomorphic vector fields on $\mathbf{B}$ are the infinitesimal generators of the uniformly continuous one-parameter subgroups of $\mathrm{Aut}(\mathbf{B})$ and, with suitable $c \in \mathbf{H}$ and a bounded self-adjoint operator $C \in \mathcal{L}(\mathbf{H})$ they can be written in the form $x \mapsto a - \{xc^*x\} + iCx$ by means of the Jordan triple product.

$$\{xy^*z\} := \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x \tag{4.3}$$

In terms of the factorization $\Psi^t = \Theta_{a_t} \circ U_t|\mathbf{B}$ we introduce the following vector resp. not necessarily bounded symmetric linear operator:

$$b := \frac{d}{dt}\Big|_{t=0} \Psi^t(0) = \lim_{t\to 0} \frac{1}{t} a_t, \quad R := -i\frac{d}{dt}\Big|_{t=0} U_t : x \mapsto \lim_{t\to 0} \frac{1}{it}(U_t - I)x.$$

Proposition 4.4. *Under hypothesis* (4.1), *we have* $\mathbf{D} = \mathbf{B} \cap \mathbf{Z}$ *along with* $\mathrm{dom}(R) \supset \mathbf{Z}$ *and the vector field* $\Omega := \frac{d}{dt}\big|_{t=0}\Psi^t$ *admits the Jordan form*

$$\Omega(x) = b - \{xb^*x\} + iRx \qquad (x \in \mathbf{D}). \tag{4.5}$$

Proof. The relation $\mathbf{D} = \mathbf{B} \cap \mathbf{Z}$ is clear since $\overline{x} \in \mathbf{Z}$. By the definition of the generator $\mathcal{A}$,

$$\mathcal{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lim_{t\to 0} \frac{1}{t}\big(\widehat{\Psi}^t - \widehat{\Psi}^0\big)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lim_{t\to 0} \frac{1}{t}\Big\{\frac{1}{1 + \langle U_t\overline{x}|a_t\rangle}\begin{bmatrix} a_t \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big\}.$$

Since $\lim_{t\to 0} a_t = 0$ and $\|U_t\overline{x}\| \le 1$ $(t \in \mathbb{R})$, taking (2.8) into account, we see that the limit

$$b := \lim_{t\to 0} \frac{1}{t} a_t = \frac{d}{dt}\Big|_{t=0} \Psi^t(0)$$

is well-defined and

$$B\overline{x} = -\Big[\mathcal{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big]_{\mathbf{H}} = -b, \quad \Lambda\overline{x} = -\Big[\mathcal{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big]_{\mathbb{C}} = \langle \overline{x} \,|\, b\rangle.$$

As a consequence we also have

$$\beta_t = \sqrt{1 - \|a_t\|^2} = \sqrt{1 - \|tb + o(t)\|^2} = 1 - \frac{1}{2}\|b\|^2 t^2 + o(t^2),$$
$$Q_t = P_t + \beta_t(I - P_t) = I + (1 - \beta_t)P_t = I + o(t) \quad \text{(in operator norm)}.$$

Since also $U_t z = z + o(1)$ in norm, hence we deduce that for any vector $z \in \mathbf{Z}$, and $\zeta \in \mathbb{C}$,

$$(4.6) \qquad \widehat{\Psi}^t \begin{bmatrix} z \\ \zeta \end{bmatrix} = \frac{1-\langle \overline{x}|a_t\rangle}{1-\|a_t\|^2} \begin{bmatrix} Q_t U_t & a_t \\ a_t^* U_t z & 1 \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} U_t z - t\langle \overline{x}|b\rangle z \\ t\langle U_t z|b\rangle + \zeta \end{bmatrix} + o(t) \quad \text{in norm.}$$

By definition, $\left[\begin{smallmatrix} Bz \\ \Lambda z \end{smallmatrix}\right] = \lim_{t\to 0} \frac{1}{t}\big(\widehat{\Psi}^t - I\big)\left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right]$. Hence with well-defined limits we conclude that

$$(4.7) \qquad Bz = \lim_{t\to 0} \frac{1}{it}(U_t - I)z - \big\langle \overline{x}\,\big|b\big\rangle z \qquad (z \in \mathbf{Z}).$$

The strong limit of $t^{-1}(U_t - I)|\mathbf{Z}$ is necessarily symmetric due to the fact that each U_t is unitary. Thus comparing (4.7) with Proposition 3.7 stating that the operator B has the form $iA + \lambda I$ with some $\lambda \in \mathbb{R}$ and a symmetric operator A with $\mathrm{dom}(A) = \mathbf{Z}$, we get

$$(4.8) \qquad \lambda = -\mathrm{Re}\big\langle \overline{x}\,\big|\,b\big\rangle\,, \quad A = \lim_{t\to 0}^{\text{strong}} \big(U^t - I\big) - \mathrm{Im}\big\langle \overline{x}\,\big|\,b\big\rangle I \Big|\mathbf{Z} \quad \text{in 3.7.}$$

We calculate Ω by substituting (4.7-8) into its form $\Omega(x) = \big[\Lambda(\overline{x}-x)\big]x + B(x-\overline{x})$ applying also the relations $B\overline{x} = -b$, $\Lambda\overline{x} = \langle x|b\rangle$, $B = iA+\lambda I$, $\Lambda x = \langle (iA-\lambda I)x|\overline{x}\rangle = \langle (B-2\lambda I)x|\overline{x}\rangle$. Namely, given any vector $x \in \mathbf{D}$, taking into account the antisymmetry of the operator $iA = B-\lambda I$, we can write

$$\begin{aligned}
\Omega(x) &= \big[\Lambda\overline{x}\big]x - \big[\Lambda x\big]x + Bx - B\overline{x} = \\
&= \langle \overline{x}|b\rangle x - \langle (B-2\lambda I)x|\overline{x}\rangle x + Bx + b = \\
&= b + \big[\langle \overline{x}|b\rangle I + B\big]x - \langle (B-\lambda I)x|\overline{x}\rangle x + \lambda\langle x|\overline{x}\rangle x = \\
&= b + iRx + \langle x|(B-\lambda I)\overline{x}\rangle x + \lambda\langle x|\overline{x}\rangle x = \\
&= b + iRx + \langle x|B\overline{x}\rangle x = b + iRx - \langle x|b\rangle x.
\end{aligned}$$

Corollary 4.9. $\mathbf{Z} = \mathrm{dom}(R)$ *that is* $x \in \mathbf{Z}$ *if and only if the limit* $\lim_{t\to 0} t^{-1}(U^t x - x)$ *exists.*

Proof. Recall that $\mathbf{Z} = \big\{x \in \mathbf{H} : \frac{d}{dt}\big|_{t=0}\widehat{\Psi}^t\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \text{ is well-defined}\big\}$. From Proposition 4.4 we know that $iRx = \lim_{t\to 0} t^{-1}(U_t - I)x$ is well-defined for every vector $x \in \mathbf{Z}$. Conversely, suppose $u = \lim_{t\to 0} t^{-1}(U_t x - x)$ is well-defined. Then $U_t x = x + tu + o^{\mathrm{norm}}(t)$ and (4.6) establishes that $\widehat{\Psi}^t\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] + t\left[\begin{smallmatrix} u-\langle \overline{x}|a_t\rangle x \\ \langle x|b\rangle \end{smallmatrix}\right] + o^{\mathrm{norm}}(t)$.

Lemma 4.10. *We have* $U_{-t} = U_t^{-1} = U_t^*$, $a_{-t} = -U_t^* a_t \quad (t \in \mathbb{R})$.

Proof. Given any $t \in \mathbb{R}$, we have $\Psi^{-t} = \Psi_t^{-1}$ that is $\Theta_{a_{-t}}U_{-t} = \big[\Theta_{a_t}U_t\big]^{-1} = U_t^{-1}\Theta_{a_t}^{-1} = U_t^{-1}\Theta_{-a_t} = \big[U_t^{-1}\Theta_{-a_t}U_t\big]U_t^{-1} = \Theta_{U_t^{-1}(-a_t)}U_t^{-1}$. By the unambiguous decomposability of holomorphic automorphisms of circular domains into Möbius and unitary parts [2], hence we deduce that $\Theta_{a_{-t}} = \Theta_{-U_t^{-1}a_t}$ and $U_{-t} = U_t^{-1}$.

Lemma 4.11. *The operator R is self-adjoint with* $\mathrm{dom}(R) = \mathbf{Z}$.

Proof. In view of (4.6), and since $U_t x = x + itRx + o^{\mathrm{norm}}(t)$ for any $x \in \mathbf{Z} = \mathrm{dom}(R) = \mathrm{dom}(\frac{d}{dt}\big|_{t=0} U_t)$, we conclude that

$$\frac{d}{dt}\Big|_{t=0} \widehat{\Psi}^t \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} iR-\langle \overline{x}|b\rangle I & b \\ b^* & -\langle \overline{x}|b\rangle \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \quad \text{for any} \quad \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathcal{D} = \begin{bmatrix} \mathbf{Z} \\ \mathbb{C} \end{bmatrix}.$$

The linear operator in $\mathcal{L}(\mathcal{H})$ with matrix $\left[\begin{smallmatrix} -\langle \overline{x}|b\rangle I & b \\ b^* & -\langle \overline{x}|b\rangle \end{smallmatrix}\right]$ is bounded. Since $\mathcal{A} = \frac{d}{dt}\big|_{t=0} \widehat{\Psi}^t$ with domain $\mathcal{D}$ is the generator of a strongly continuous semigroup in $\mathcal{L}(\mathcal{H})$, by the theorem of bounded perturbations [4], also the operator with matrix $\left[\begin{smallmatrix} iR & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with domain $\mathcal{D}$ is the generator of a strongly continuous one-parameter subgroup of $\mathcal{L}(\mathcal{H})$ entailing that iR is the generator of a strongly continuous group $\big[V_t : t \in \mathbb{R}\big]$ in $\mathcal{L}(\mathbf{H})$. Since $U_{-t} = U_t^{-1} = U_t^*$, the arguments on sun adjoint semigroups in [4, p. 69] show that $\lim\limits_{t\to 0} t^{-1}(U_t^* - I) = -iR$ is the generator of the sun adjoint group $\big[V_t^* : t \in \mathbb{R}\big] = \big[V_{-t} : t \in \mathbb{R}\big]$ and we have $-iR = (iR)^*$ which completes the proof.

Theorem 4.12. *Any vector field of the form* (4.5) *where R is a not necessarily bounded self-adjoint operator with dense domain $\mathbf{Z} \subset \mathbf{H}$, is the infinitesimal generator defined on $\mathbf{D} := \mathbf{Z} \cap \mathbf{B}$ of a pointwise continuous one-parameter group $[\Phi^t : t \in \mathbb{R}]$ of holomorphic automorphisms of $\mathbf{B}$.*[3]

Proof. It suffices to see that there is a strongly (i.e. pointwise) continuous one-parameter group $\big[\mathcal{V}^t : t \in \mathbb{R}\big]$ of bounded linear operators of the space $\mathcal{H}$ such that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{V}^t \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \quad (x \in \mathbf{Z},\ \xi \in \mathbb{C}), \qquad \mathcal{V}^t \mathcal{K} \subset \mathcal{K} := \Big\{ \begin{bmatrix} x \\ \xi \end{bmatrix} :\ \|x\|^2 \geq |\xi|^2 \Big\}.$$

Namely, in this case the maps

$$\Phi^t(x) := \Big[\mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbb{C}}^{-1} \Big[\mathcal{V}^t \begin{bmatrix} x \\ 1 \end{bmatrix}\Big]_{\mathbf{H}} \qquad (t \in \mathbb{R},\ x \in \mathbf{D})$$

suit the requirements of the theorem since $x \in \mathbf{D} \Rightarrow \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \in \mathcal{K} \Rightarrow \mathcal{V}^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \Rightarrow \Phi^t(x) \in \mathbf{D}$ and $x \in \mathbf{D} \Rightarrow \frac{d}{dt}\big|_{t=0} \Phi^t(x) = -\big[\mathcal{V}^0 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbb{C}}^{-2} \frac{d}{dt}\big|_{t=0} \big[\mathcal{V}^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbb{C}} \big[\mathcal{V}^0 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbf{H}} + \big\{\mathcal{V}^0 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big\}_{\mathbb{C}}^{-1} \frac{d}{dt}\big|_{t=0} \big[\mathcal{V}^t \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbf{H}}$ $= -\big[\left[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbb{C}} x + \big[\left[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbf{H}} = -\langle x|b\rangle x + iRx + b = \Omega(x)$. Notice that, by Corollary 2.10, a strongly continuous one parameter group of linear operator leaves the cone $\mathcal{K}$ invariant if all its members map the boundary $\partial\mathcal{K} = \big\{\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] : \|x\| = |\xi|\big\} = \big\{\left[\begin{smallmatrix} x \\ e^{i\tau}\|x\| \end{smallmatrix}\right] : x \in \mathbf{H},\ \tau \in \mathbb{R}\big\}$ into itself. Therefore it suffices to check that there is a (necessarily unique) strongly continuous one-parameter group in $\mathcal{L}(\mathcal{H})$ with domain $\mathbf{Z} \oplus \mathbb{C} = \left[\begin{smallmatrix} \mathbf{Z} \\ \mathbb{C} \end{smallmatrix}\right]$ such that

$$\frac{d}{dt} \mathcal{V}^t \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix} \mathcal{V}^t \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \Big\| \Big[\mathcal{V}^t \begin{bmatrix} x \\ \|x\| \end{bmatrix}\Big]_{\mathbf{H}} \Big\| = \Big| \Big[\mathcal{V}^t \begin{bmatrix} x \\ \|x\| \end{bmatrix}\Big]_{\mathbb{C}} \Big| \qquad (x \in \mathbf{Z},\ t \in \mathbb{R}).$$

[3] That is, for all $x \in \mathbf{D} := \mathbf{Z} \cap \mathbf{B}$, the functions $t \mapsto \Phi^t(x)$ range in $\mathbf{Z}$, they are differentiable and satisfy the identity $\frac{d}{dt} \Phi^t(x) = \mathcal{V}\big(\Phi^t(x)\big)$ $(t \in \mathbb{R})$.

By Stone's theorem, the $\mathcal{H}$-unitary operators $\mathcal{W}^t\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] := \left[\begin{smallmatrix} \exp(itR)x \\ \xi \end{smallmatrix}\right]$ form a strongly continuous one-parameter group whose infinitesimal generator is defined on $\mathrm{dom}(R)\oplus\mathbb{C} = \mathbf{Z}\oplus\mathbb{C}$ with the diagonal matrix $\left[\begin{smallmatrix} iR & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Since the matrix $\left[\begin{smallmatrix} 0 & b \\ b^* & 0 \end{smallmatrix}\right]$ represents a bounded linear operator in $\mathcal{H}$, by the theorem of bounded perturbations [4], there is a strongly continuous one-parameter group $\left[\mathcal{V}^t : t\in\mathbb{R}\right]$ whose generator is defined on $\mathbf{Z}\oplus\mathbb{C}$ with the matrix $\left[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\right]$. In particular $\frac{d}{dt}\mathcal{V}^t\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right] = \left[\begin{smallmatrix} iR & b \\ b^* & 0 \end{smallmatrix}\right]\mathcal{V}^t\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$ $(t\in\mathbb{R},\ x\in\mathbf{Z})$. To complete the proof, we show that necessarily

$$\frac{d}{dt}\left[\left\|\left[\mathcal{V}^t\left[\begin{matrix} x \\ \|x\| \end{matrix}\right]\right]_{\mathbf{H}}\right\|^2 - \left|\left[\mathcal{V}^t\left[\begin{matrix} x \\ \|x\| \end{matrix}\right]\right]_{\mathbb{C}}\right|^2\right] = 0 \qquad (t\in\mathbb{R},\ x\in\mathbf{Z}).$$

Consider any vector $x\in\mathbf{Z}$ and write $\left[\begin{smallmatrix} x_t \\ \xi_t \end{smallmatrix}\right] := \mathcal{V}^t\left[\begin{smallmatrix} x \\ \|x\| \end{smallmatrix}\right]$ for all parameters $t\in\mathbb{R}$. Then

$$\begin{aligned}
\frac{d}{dt}\left[\|x_t\|^2 - |\xi_t|^2\right] &= 2\,\mathrm{Re}\Big[\big\langle dx_t/dt \big| x_t\big\rangle - \big(d\xi_t/dt\big)\overline{\xi_t}\Big] = \\
&= 2\,\mathrm{Re}\left\{\Big\langle\left[\left[\begin{matrix} iR & b \\ b^* & 0 \end{matrix}\right]\left[\begin{matrix} x_t \\ \xi_t \end{matrix}\right]\right]_{\mathbf{H}}\Big| x_t\Big\rangle - \left[\left[\begin{matrix} iR & b \\ b^* & 0 \end{matrix}\right]\left[\begin{matrix} x_t \\ \xi_t \end{matrix}\right]\right]_{\mathbb{C}}\big(d\xi_t/dt\big)\overline{\xi_t}\right\} = \\
&= 2\,\mathrm{Re}\left[\big\langle iRx_t + \xi_t b\big| x_t\big\rangle - \big\langle x_t\big| b\big\rangle\overline{\xi_t}\right] = \\
&= -2\,\mathrm{Im}\big\langle Rx_t\big| x_t\big\rangle + 2\,\mathrm{Im}\big(\big\langle \xi b\big| x_t\big\rangle - \big\langle x_t\big|\xi b\big\rangle\big) = 0.
\end{aligned}$$

5. Proof of Theorems 1.1 and 1.4

Consider any pointwise continuous one-parameter subgroup $\left[\Psi^t :\ t\in\mathbb{R}\right]$ of $\mathrm{Aut}\big(\mathbf{B}\big)$ and let us fix any common fixed point $\overline{x}\in\overline{\mathbf{B}}$ of the continuous extensions of the maps Ψ^t $(t\in\mathbb{R})$ (guaranteed by Lemma 2.4). From Proposition 3.7, we know already that (1.2-3) hold for some dense linear complex submanifold $\mathbf{Z}$ of the underlying Hilbert space with a symmetric linear operator A with $\mathrm{dom}(A) = \mathbf{Z}$. We have to see that A is even self-adjoint in any case and, conversely every self-adjoint operator with domain $\mathbf{Z}$ may appear in (1.3) with any constant $\lambda\in\mathbb{R}$ if $\overline{x}\in\partial\mathbf{B}$ or $\lambda = 0$ if $\overline{x}\in\mathbf{B}$. In order to establish a link to the Jordan case, fix any point $c\in\mathbf{D} = \mathrm{dom}\big(\frac{d\Psi^t}{dt}\big|_{t=0}\big)$ and let

$$\Phi^t := \Theta_{-c}\Psi^t\Theta_c \quad (t\in\mathbb{R}), \quad \overline{y} := \Theta_c(\overline{x})$$

by means of the Möbius transformations (2.3). Clearly $\left[\overline{\Phi}^t :\ t\in\mathbb{R}\right]$ is a strongly continuous one-parameter subgroup of $\mathrm{Aut}\big(\mathbf{B}\big)$ such that

$$(5.1) \qquad 0 = \Theta_{-c}(c) \in \Theta_{-c}\left(\mathrm{dom}\Big(\frac{\partial\Psi^t}{dt}\Big|_{t=0}\Big)\right) = \mathrm{dom}\Big(\frac{\partial\Phi^t}{dt}\Big|_{t=0}\Big), \quad \overline{y}\in\bigcap_{t\in\mathbb{R}}\mathrm{Fix}\big(\overline{\Phi}^t\big).$$

Thus we can apply the results of Section 4 in particular Lemma 4.11 to $[\Phi^t :\ t\in\mathbb{R}]$ to conclude that there is a dense complex linear submanifold $\mathbf{Y}\subset\mathbf{H}$ along with a self-adjoint operator R with $\mathrm{dom}(R) = \mathbf{Y}$ and a vector $b\in\mathbf{H}$ such that

$$\widehat{\Phi}^t_{\overline{y}} = e^{-\langle\overline{x}|b\rangle t}\exp\left(t\left[\begin{matrix} iR & b \\ b^* & 0 \end{matrix}\right]\right) \qquad (t\in\mathbb{R}).$$

Hence Corollary 2.11 establishes the existence of a constant $\nu \in \mathbb{C}$ with

$$\widehat{\Psi}^t_{\overline{x}} = e^{\nu t}\beta^{-2}\widetilde{\Theta_c}\widehat{\Phi}^t_{\overline{y}}\widetilde{\Theta_{-c}} \qquad (t \in \mathbb{R})$$

due to the identity $\widetilde{\Theta_c}\widetilde{\Theta_{-c}} = \beta^2\mathcal{I} = (1-\|c\|^2)\left[\begin{smallmatrix} I & 0 \\ 0 & 1\end{smallmatrix}\right]$ for the canonical representations

$$\widetilde{\Theta_{\pm c}} = \begin{bmatrix} Q & \pm c \\ \pm c^* & 1 \end{bmatrix} \quad \text{where} \;\; Q := \beta I + (1-\beta)P,\; \beta := \sqrt{1-\|c\|^2},\; P := \mathrm{P}_c.$$

By passing to infinitesimal generators, with $\mu := \nu - \langle \overline{x}|b\rangle$, we get

$$\text{(5.2)} \qquad \frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\begin{bmatrix} x \\ \xi \end{bmatrix} = \mu\begin{bmatrix} x \\ \xi \end{bmatrix} + \beta^{-2}\widetilde{\Theta_c}\begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix}\widetilde{\Theta_{-c}}\begin{bmatrix} x \\ \xi \end{bmatrix} \quad \text{for} \;\; \begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathrm{dom}\Big(\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\Big).$$

From Lemma 3.3 and (5.1) we see that

$$\begin{bmatrix} \mathbf{Z} \\ 0 \end{bmatrix} + \mathbb{C}\begin{bmatrix} \overline{x} \\ \xi \end{bmatrix} = \mathrm{dom}\Big(\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\Big) = \widetilde{\Theta_c}\mathrm{dom}\Big(\frac{d\widehat{\Phi}^t_{\overline{y}}}{dt}\Big|_{t=0}\Big) = \widetilde{\Theta_c}\begin{bmatrix} \mathbf{Y} \\ \mathbb{C} \end{bmatrix}.$$

Thus, given any vector $z \in \mathbf{H}$, we have $z \in \mathbf{Z}$ if and only if $\left[\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right] = \widetilde{\Theta_c}\left[\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right]$ for some $y \in \mathbf{Y}$ and $\eta \in \mathbb{C}$ that is if $z = Qy - \langle y|c\rangle c = \big[Q - \|c\|^2P\big]y$ for some $y \in \mathbf{Y}$. It follows

$$\text{(5.3)} \qquad \mathbf{Z} = \big[Q - \|c\|^2P\big]\mathbf{Y} = Q^{-1}\mathbf{Y}$$

because the operators P, Q commute, we have $\beta^2 = 1 - \|c\|^2 > 0$ and $Q\big[Q - \|c\|^2P\big] = \big[\beta I + (1-\beta)P\big]\big[\beta I + (\beta^2-\beta)P\big] = \beta^2 I$. We are now ready to establish the self-adjointness of the operator A in (1.3). By (5.2) we have

$$(iA + \lambda I)z = \left[\frac{d\widehat{\Psi}^t_{\overline{x}}}{dt}\Big|_{t=0}\begin{bmatrix} z \\ 0 \end{bmatrix}\right]_{\mathbf{H}} = \mu z + \beta\big[iQRQz - Qbc^*z + cb^*Qz\big] \quad (z \in \mathbf{Z}).$$

That is, with the bounded self-adjoint operator $S := i\beta[Qbc^* - cb^*Qz] = [iQbc^*] + [iQbc^*]^*$ we have $A = \beta QRQ + S + i(\lambda - \mu)I$. We know the symmetry of A already entailing the relations $\mu = \lambda$ with $A = \beta QRQ + S$. Here the operator QRQ self-adjoint with $\mathrm{dom}(QRQ) = Q^{-1}\mathrm{dom}(R) = Q^{-1}\mathbf{Y} = \mathbf{Z} = \mathrm{dom}(A)$ since R is a self-adjoint operator with $\mathrm{dom}(R) = \mathbf{Y}$ while Q is an invertible bounded self-adjoint operator, Therefore, as being the bounded self-adjoint perturbation, the operator A is necessarily self-adjoint.
To see the converse, we need only to check the reversibility of some of our previous arguments. Assuming A to be self-adjoint in (1.2-3), it is the theorem of bounded perturbations [4] ensures that the operator $\mathcal{A}\left[\begin{smallmatrix} z+\xi\overline{x} \\ \xi \end{smallmatrix}\right] := \left[\begin{smallmatrix} (iA+\lambda I)z \\ \langle (iA-\lambda I)z|\overline{x}\rangle(z+\xi\overline{x}) \end{smallmatrix}\right]$ $(z \in \mathbf{Z},\ \xi \in \mathbb{C})$ is the infinitesimal generator of a strongly continuous one-parameter subgroup $[\mathcal{U}^t : t \in \mathbb{R}]$ of $\mathcal{L}(\mathbf{H}$ with graph being tangent to the boundary of cone $\mathcal{K}$ in Corollary 2.10(iii) and we have $\mathcal{K} = \mathcal{U}^t\mathcal{K}$ $(t \in \mathbb{R})$. Hence the holomorphic maps $\Psi^t(x) := \big[\mathcal{U}^t\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbb{C}}^{-1}\big[\mathcal{U}^t\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]\big]_{\mathbf{H}}$ are well-defined on the

unit ball **B** leaving it invariant, and form a strongly continuous one-parameter subgroup in Aut(**B**).

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