Strongly continuous one-parameter groups of automorphisms of multilinear functionals

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PROBLEM SETTING

J. Jamison - F. Botelho 2008

$X_1, \ldots, X_N$ complex Banach spaces

$\mathcal{B} := \big\{\text{bounded } N\text{-linear maps } X_1 \times \cdots \times X_N \to \mathbb{C}\big\}$

$\mathbf{U} : \mathbb{R} \to \mathfrak{A} := \big\{\text{surjective linear isometries } \ \mathcal{B} \to \mathcal{B}\big\}$

strongly cont. 1-parameter group of surj. lin. isom.

$\mathbf{U}(t+h) = \mathbf{U}(t)\mathbf{U}(h) \qquad (t, h \in \mathbb{R})$

$t \mapsto \mathbf{U}(t)\Phi$ continuous $\qquad (\Phi \in \mathcal{B})$

Conjecture:

$$\mathbf{U}(t) = \Big[\underbrace{U_1^t \otimes \cdots \otimes U_N^t}_{\phi \mapsto \phi(U_1^t x_1, \ldots, U_N^t x_N)}\Big]^*$$

$t \mapsto U_k^t$ str.cont. 1-par.grp. in $\mathfrak{A}_k := \{$surj.lin.isom. $X_k \to X_k\}$

NON-LINEAR ISSUE

$$\Big\{\text{Surj.lin.isom. of } X\Big\} = \Big\{F \in \underbrace{\mathrm{Aut}\big(\mathrm{Ball}(X)\big)}_{\textit{hol.aut. of unit ball}} \; : \; F(0) = 0\Big\}$$

$$\|\phi\| = \sup_{\|x_1\|=1,\dots,\|x_N\|=1} \big|\phi(x_1,\dots,x_N)\big| \qquad \text{in} \quad \mathcal{B}$$

Question: What about *strongly cont. 1-par. groups* in $\mathrm{Aut}(\mathcal{B})$?

Conjecture: $\mathrm{Aut}(\mathcal{B})$ LINEAR for $N > 2$ factors

$\uparrow$

Stachó 1982: TRUE with HILBERT spaces of $\dim > 1$

Hilbert cases $N = 1, 2$: JB*-triples $\mathbf{H}$ resp. $\mathcal{L}\big(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\big)$

HILBERT CASE $N > 2$; MAIN RESULT

$\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)}$ Hilbert spaces

$\mathrm{Aut}(\mathcal{B}) = \big\{ \big([U_1 \otimes \cdots \otimes U_N] \circ I_\pi \big)^* : \ U_k \in \mathcal{U}(\mathbf{H}^{(k)}), \ \pi \text{ adm.index-perm.} \big\}$

$\mathbf{U} : \mathbb{R} \to \mathrm{Aut}(\mathcal{B})$ **str***.cont. 1-par.grp:
$t \mapsto \big[\mathbf{U}^t \phi\big](\mathbf{x}_1, \ldots, \mathbf{x}_N)$ cont.

Lemma. $\mathbf{U}(t) = \big[U_{1,t} \otimes \cdots \otimes U_{N,t} \big]^* \qquad (t \in \mathbb{R})$

THEOREM. *There are possibly* **unbounded self-adjoint** $A_k : \mathrm{dom}(A_k) \to \mathbf{H}^{(k)}$ *(defined on dense linear submanifolds) with*

$$\mathbf{U}(t) = \Big(\big[\exp(itA_1) \big] \otimes \cdots \otimes \big[\exp(itA_N) \big] \Big)^* \qquad (t \in \mathbb{R}).$$

Corollary. *If* $\mathbf{W} : \mathbb{R} \to \mathrm{Aut}\big(\mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})\big)$ *is a* **str*.cont. lin.** *1-par.grp then* $\mathbf{W}(t)X = \exp(tA_1)X\exp(tA_2)$

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BACKGROUND OF TECH. DIFFICULTIES

$\mathbb{T} := \{\kappa : \ |\kappa| = 1\}$

Ambiguous representation for unitary ops

$$U_1 \otimes \cdots \otimes U_N = V_1 \otimes \cdots \otimes V_N$$
$$\Updownarrow$$
$$V_k = \kappa_k U_k, \quad \kappa_k \in \mathbb{T} \quad (k = 1, \ldots, N) \qquad \prod_{k=1}^{N} \kappa_k = 1$$

ELEMENTARY FUNCTIONALS

$$\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^* : \ (\mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto \prod_{k=1}^N \mathbf{h}_k^*(\mathbf{x}_k) = \prod_{k=1}^N \langle \mathbf{x}_k | \mathbf{h}_k \rangle$$

Lemma. $\Phi := \mathbf{g}_1^* \otimes \cdots \otimes \mathbf{g}_N^*,\ \Psi := \mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^*$ *(unit vectors)*

$$\big\|\Phi - \Psi\big\| \leq \varepsilon \quad \Longrightarrow \quad \underbrace{\mathrm{dist}(\mathbb{T}\mathbf{g}_k, \mathbb{T}\mathbf{h}_k)}_{\min_{\kappa,\mu\in\mathbb{T}} \|\kappa\mathbf{g}_k - \mu\mathbf{h}_k\|} \leq 2^{N-1}\varepsilon$$

$$\mathbb{P}(\mathbf{H}) := \{\mathbb{T}\mathbf{g} : \ \mathbf{g} \in \partial\mathrm{Ball}(\mathbf{H})\}$$

Lemma. $\mathbf{F} : \mathbb{R} \to \mathbb{P}(\mathbf{H})$ *cont.* $\Longrightarrow \exists\ t \mapsto \mathbf{h}_t \in \partial\mathrm{Ball}(\mathbf{H})$ *cont.*

$$\mathbf{F}(t) = \mathbb{T}\mathbf{h}_t \qquad (t \in \mathbb{R})$$

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ADJUSTED STRONG* CONTINUITY

Lemma*. $\Psi(t) = \mathbf{h}_{1,t}^* \otimes \cdots \otimes \mathbf{h}_{N,t}^*$ *(with unit vectors)*

$\Psi : \mathbb{R} \to \mathcal{B}$ *continuous,* $\Longrightarrow$ $\exists\, \kappa_1, \ldots, \kappa_N : \mathbb{R} \to \mathbb{T}$

$$\prod_{k=1}^{N} \kappa_k(t) \equiv 1 \;, \qquad t \mapsto \kappa_k(t)\mathbf{h}_{k,t} \;\; \textit{norm-continuous}$$

Proposition. $t \mapsto \big[\overbrace{U_{1,t}}^{unit.} \otimes \cdots \otimes \overbrace{U_{N,t}}^{unit.}\big]^*$ *str*.cont.* $\mathbb{R} \to \mathfrak{A},$ $\Longrightarrow$

$\exists\, \kappa_1, \ldots, \kappa_N : \mathbb{R} \to \mathbb{T}$ $\qquad t \mapsto \kappa_k(t)U_{k,t}$ *str.cont.* $(k = 1, \ldots, N)$

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EXTENSION: MORE BANACH SPACE GEOMETRY

$\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$ **uniformly smooth** Banach spaces

Recall: $\mathbf{X}$ is uniformly convex (rotond):

$$\inf \left\{ 1 - \|\mathbf{x} + \mathbf{y}\| / 2 : \ \mathbf{x}, \mathbf{y} \in \partial \mathrm{Ball}(\mathbf{X}) \right\} > 0$$

$\mathbf{X}$ is uniformly smooth:

$$\lim_{\tau \searrow 0} \frac{1}{\tau} \sup \left\{ \|\mathbf{x} + \mathbf{y}\| / 2 + \|\mathbf{x} - \mathbf{y}\| / 2 - 1 : \ \|\mathbf{x}\| = 1, \ \|\mathbf{y}\| \leq \tau \right\} = 0$$

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Lemma. *Let* $\phi^{(k)}, \psi^{(k)} \in \partial\mathrm{Ball}[\mathbf{X}^{(k)}]^*$. *T.f.a.e.:*
(1) $\phi^{(1)} \otimes \cdots \otimes \phi^{(N)} = \psi^{(1)} \otimes \cdots \otimes \psi^{(N)}$,
(2) $\exists\ \kappa_1, \ldots, \kappa_N \in \mathbb{T}$ *with* $\prod_k \kappa_k = 1$, $\psi^{(k)} = \kappa_k \phi_k$.

Lemma. *Let* $\phi_j^{(k)} \in \partial\mathrm{Ball}[\mathbf{X}^{(k)}]^*$ *be nets such that*

$$\phi_j^{(k)} \otimes \cdots \otimes \phi_j^{(N)} \to \phi^{(k)} \otimes \cdots \otimes \phi^{(N)} \quad \text{pointwise.}$$

Then $\exists \left[\kappa_j^{(k)}\right]$ *in* $\mathbb{T}$ *with* $\left\|\kappa_j^{(k)}\phi_j^{(k)} - \phi^{(k)}\right\| \to 0 \quad (k=1,\ldots,N)$.

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BANACH EXTENSIONS (continued)

$\mathbf{X}^{(1)}, \dots \mathbf{X}^{(N)}$ unif.smooth Banach spaces

Proposition. *Let* $t(\in \mathbb{R}) \mapsto \phi_t^{(k)} \otimes \cdots \otimes \phi_t^{(N)}(\mathbf{x}_1, \dots, \mathbf{x}_N)$ *cont. for any fixed* $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}$ *where* $\phi_t^{(k)} \in \partial \mathrm{Ball}[\mathbf{X}^{(k)}]^*$*. Then* $\exists$ *functions* $t(\in \mathbb{R}) \mapsto \kappa_t^{(k)} \in \mathbb{T}$ *such that*

$$\prod_{k=1}^{N} \kappa_t^{(k)} = 1 \quad (t \in \mathbb{R}), \qquad t \mapsto \kappa_t^{(k)} \phi_t^{(k)} \text{ is norm-cont.}$$

Corollary. *If* $U_{k,t} \in \mathcal{L}(\mathbf{X}^{(k)})$ $(t \in \mathbb{R},\ 1 \le k \le N)$ *are* **isometries** *and* $t \mapsto \left[U_{1,t} \otimes \cdots \otimes U_{N,t}\right]^*$ *is* **str***.*cont. then* $\exists$ *functions* $t(\in \mathbb{R}) \mapsto \kappa_t^{(k)} \in \mathbb{T}$ *such that each* $t \mapsto U_{k,t}$ *is* **str.***cont. (pointwise cont.).*

SEPARATE COMMUTATIVITY (smooth Banach setting)

$t \mapsto \mathbf{U}(t) := \left[U_{1,t} \otimes \cdots \otimes U_{N,t}\right]^*$ str.cont. 1-par.grp
Proposition $\Longrightarrow$ without loss of generality:

$$1) \qquad t \mapsto U_{k,t} \in \mathcal{U}(\mathbf{X}^{(k)}) \ \text{ strongly continuous;}$$
$$2) \qquad U_{k,0} = \mathrm{Id}, \quad U_{k,-t} = U_{k,t}^* \quad (t \in \mathbb{R}).$$

Lemma. $\{U_{k,t} : \ t \in \mathbb{R}\}$ $(k = 1, \ldots, N)$ *are Abelian.*
[A priory only $\quad U_{k,t}U_{k,s} \in \mathbb{T}U_{k,s}U_{k,t}$!]

Theorem. $\exists\ \kappa_1, \ldots, \kappa_N : \mathbb{R} \to \mathbb{T}, \quad \exists$ *1-par.grp-s* $\ t \mapsto U_k^t$

$$U_{k,t} = \kappa_k(t)U_k^t, \quad \kappa_k(0) = 1 \qquad (t \in \mathbb{R},\ 1 \le k \le N).$$

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$$\mathbf{U}(t) = \Big(\big[\kappa_1(t)U_1^t\big] \otimes \cdots \otimes \big[\kappa_N(t)U_N^t\big]\Big)^*$$

k fixed, $\mathbf{X} \equiv \mathbf{H} := \mathbf{H}^{(k)}$

$t \mapsto U^t := U_k^t \in \mathcal{U}(\mathbf{H})$ **non.cont.** 1-pgr., $\kappa = \kappa_n : \mathbb{R} \to \mathbb{T}$
$t \mapsto \kappa(t)U^t$ str.cont. ~~1-prg.~~, $\kappa(-t) = \overline{\kappa(t)}$, $\kappa(0) = 1$.

Cyclic decomp: $\mathbf{H} = \oplus_{\alpha\in\mathcal{A}}\mathbf{H}_\alpha$, $\mathbf{H}_\alpha := \overline{\mathrm{Span}}\{\mathbf{U^t x}_\alpha : \mathbf{t} \in \mathbb{R}\}$

Main idea: It suffices to see

$$\kappa(t)U^t|\mathbf{H}_\alpha = \chi_\alpha(t)\widetilde{U}_\alpha^t$$

$\exists\ \chi_\alpha \in \mathcal{C}(\mathbb{R},\mathbb{T})$, $t \mapsto \widetilde{U}_\alpha^t$ str.cont. 1-prg. in $\mathcal{U}(\mathbf{H}_\alpha)$

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LOCAL FUNCTION REPRESENTATION (Hilbert spaces)

Henceforth α **fixed**

Gelfand repr. $\quad \mathbf{T} : \mathcal{C}(\underbrace{\Omega}_{comp.}) \leftrightarrow \overline{\mathrm{Span}}_{t\in\mathbb{R}} U^t, \quad \underbrace{u^t}_{\Omega\to\mathbb{T}} \mapsto U^t$

$\sim$**Halmos's idea** $\quad \mathbf{H}_\alpha$ cyclic $\Rightarrow$

$\exists$ prob. Radon $(\Omega, \underbrace{\mu}_{=\mu_\alpha})$, $\mathbf{T} \hookrightarrow \mathbf{T}_\alpha : L^2(\Omega,\mu) \leftrightarrow \mathbf{H}_\alpha$ isometry

$$\left\langle U^t \mathbf{T}_\alpha f \,\middle|\, \mathbf{T}_\alpha g \right\rangle = \int_{\omega\in\Omega} u^t(\omega) f(\omega) \overline{g(\omega)}\ \mu(d\omega)$$

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LOCAL FUNCTION REPRESENTATION (Hilbert spaces)

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Notations, assumptions

(ω,μ) prob. Radon measure, Ω compact

$\mathbf{M}_a : L^2(\Omega,\mu) \ni f \mapsto af \quad (a \in L^\infty(\Omega,\mu)) \quad$ mult.op.

$u^t \in \mathcal{C}(\Omega,\mathbb{T}), \quad u^{t+s} = u^t u^s$

$\kappa : \mathbb{R} \to \mathbb{T}\ , \qquad \underline{t \mapsto \kappa(t)\mathbf{M}_{u^t}\ \text{str.cont.}}$

Lemma. $\{a_1, a_2, \ldots\}$ *bounded* $\subset L^\infty(\Omega,\mu)$, $\Longrightarrow$

$\mathbf{M}_{a_1}, \mathbf{M}_{a_2}, \ldots \to 0$ *strongly* $\iff$ $a_n \to 0$ **stochastically** *(wrt.* μ*):*

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- $$\Delta(h) := \sup_{|t|\leq h} \int_{\omega_1,\omega_2\in\Omega} \left|u^t(\omega_1) - u^t(\omega_2)\right|^2 \mu(d\omega_1)\mu(d\omega_2)$$

Proposition. $\Delta(h) \searrow 0$ *as* $h \searrow 0$

Proof. Use the fact:

$$\mathbf{M}_{\kappa(h)u^h} \overset{str}{\to} \mathbf{M}_{u^0} \quad \text{i.e.} \quad \lim_{h\to 0} \mu\{\omega :\ |\kappa(h)u^h(\omega) - 1| > \varepsilon\} = 0$$

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Arc length distance on $\mathbb{T}$: $\quad d(\kappa_1,\kappa_2) := 2\arcsin\frac{|\kappa_1-\kappa_2|}{2}$

- $\Omega^{(2)}_{t,r} := \big\{(\omega_1,\omega_2)\in\Omega^2 :\ d\big(u^t(\omega_1),u^t(\omega_2)\big) < r\big\}$

  $(\omega_1,\omega_2)\in\Omega^{(2)}_{t/n,\pi/n} \Longrightarrow$

  $\log_*\left[u^{kt/n}(\omega_1)/u^{kt/n}(\omega_2)\right] = \dfrac{k}{n}\log_*\left[u^t(\omega_1)/u^t(\omega_2)\right] \quad (|k|\le n),$

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Corollary. $\alpha(\omega_1,\omega_2) := \frac{1}{i/\ell}\log_*\left[u^{1/\ell}(\omega_1)/u^{1/\ell}(\omega_2)\right]$ *for* $(\omega_1,\omega_2)\in D_\ell$ *well-def.* $\mu\otimes\mu$ *a.e.*

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Theorem. $t\mapsto u^t\in\mathcal{C}(\Omega)$ *1-prg.,* $t\mapsto\kappa(t)\mathbf{M}_{u^t}$ *str.cont.* $\Longrightarrow$ $\exists\,\chi\in\mathcal{C}(\mathbb{R},\mathbb{T})\quad\exists\widetilde{\alpha}:\Omega\to\mathbb{R}$ *μ-mesurable*

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FINISH OF THE PROOF (for Hilbert case)

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1) Reducibility of surjective isometries by a topological direct sum of sparable subspaces.

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PROBLEMS CONCERNING GENERALIZATIONS

Remark. The proof works if

$$\textit{each}\;\; \mathbf{X}^{(k)} = \big[\mathbf{H}^{(k)} \;\; \textit{with an equivalent smooth norm}\big],$$

$+$ *the surjective linear isometries on each* $\mathbf{X}^{(k)}$ *are of scalar type.*

Conjecture. *If* $\mathbf{X} = \big[\textit{Hilbert space with an equivalent norm}\big]$ *then any surjective linear isometry of* $\mathbf{X}$ *is unitary with respect some scalar product.* (In finite dimensions this is TRUE)

Problem. *Let* $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$ *be Banach spaces and let* U *be a surj. lin. isometry of* $\mathcal{B}(\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)})$*. Under which geometric hypothesis can we write it in the form* $U = \big[U_1 \otimes \cdots \otimes U_N\big]^*$ *?*

CARTAN FACTORS

Cartan factors: Types 1–6

Every JB^*-triple embeds into an ℓ^∞-product of Cartan factors

Type 1: $\mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})$
Types 2,3: $\mathcal{L}^{(\mathrm{T},\pm)}(\mathbf{H}) = \{L \in \mathcal{L}(\mathbf{H}) : \ L = \pm L^{\mathrm{T}}\}$
Type 4: Spin factor on $\mathbf{H}$, $\quad \{\mathbf{xay}\} := \langle \mathbf{x}|\mathbf{a}\rangle \mathbf{y} + \langle \mathbf{y}|\mathbf{a}\rangle \mathbf{x} - \langle \mathbf{x}|\overline{\mathbf{y}}\rangle \overline{\mathbf{a}}$
Type 5: $\mathbf{O}^{1\times 2}$ $\quad$ 1×2 complex octonion matrices (dim=16)
Type 6: $\mathcal{H}_3(\mathbf{O})$ $\quad$ 3×3 complex Hermitian octonion matrices (dim=27)

$\mathbf{h} \mapsto \overline{\mathbf{h}}$ conjugation: $\overline{\mathbf{e}_\alpha} = \mathbf{e}_\alpha$, $\overline{i\mathbf{e}_\alpha} = -i\mathbf{e}_\alpha$ for a fixed ON basis
$L \mapsto L^{\mathrm{T}}$ transposition (wrt. conjugation $\overline{\cdot}$): $L^{\mathrm{T}}\mathbf{h} = \overline{L^*\overline{\mathbf{h}}}$

Factors of type > 3 are not interesting (str. cont. wrt. Hilbert norm)

Type 1: $\mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}) \simeq \{\text{2-lin. funct. } \mathbf{H}^{(1)} \times \mathbf{H}^{(2)} \to \mathbb{C}\} \longrightarrow$ Case $N = 2$.

Types $k = 2, 3$:

$$\mathfrak{A}_k := \left\{\text{surjective lin. isom. of } \mathcal{L}^{(\mathrm{T},(-1)^k)}(\mathbf{H})\right\}$$

$$\mathcal{L}^{(\mathrm{T},(-1)^k)}(\mathbf{H}) \simeq \left\{ \begin{array}{c} \text{symm.} \\ \text{antisymm.} \end{array} \ \text{2-lin. } \mathbf{H} \times \mathbf{H} \to \mathbb{C} \text{ funct.} \right\}$$

$$\mathbf{U} \in \mathfrak{A}_2 \iff \exists\, U \in \mathcal{U}(\mathbf{H}) \qquad \underline{\mathbf{U} : L \mapsto U L U^{\mathrm{T}}}$$

TYPES 2,3

Theorem. *Let* $k = 2, 3$, $\mathbf{U}_k : \mathbb{R} \to \mathfrak{A}_k$ *str. cont. 1prg. Then*

$\exists\, A$ unbded. self-adj. $\mathbf{H}$-op. $\mathbf{U}(t) = \big[L \mapsto \exp(itA) L \exp(it \underbrace{A^{\mathrm{T}}}_{\overline{A}})\big]$

Proof. $\mathbf{U}_k : \underbrace{U_t}_{unit.} \otimes U_t^{\mathrm{T}} \big| F_k$

$\exists \sigma : \mathbb{R} \to \{-1, 1\}$ $t \mapsto \sigma(t) U(t)$ str.cont.

SEP. CONT. + LOC. + PRB. $\longrightarrow$ Theorem.

GENERAL JB*-TRIPLES

E JB*-triple $\quad E \hookrightarrow \mathrm{Atomic}\big(E^{**}\big) = \bigoplus_{k\in\mathcal{K}} \underbrace{F_k}_{Cartan}$

$\mathbf{U} : \mathbb{R} \to \mathrm{Aut}(E)$ str. cont, 1prg.
$\Pi_k : E \to F_k$ canonical proj.

$t \mapsto \mathbf{U}(t)^{**} \circ \Pi_k$ $\underbrace{\text{str. cont.}}_{???}$ 1prg. $\mathbb{R} \to \mathrm{Aut}(F_k)$

If YES , Gelfand-Neumark description for str.cont. 1prg of $\mathrm{Aut}(E)$.

Problems. (1) *E w*-dense JB*-subtriple in F (Cartan factor),*
$t \mapsto \mathbf{U}(t)$ str.cont. 1prg. $\mathbb{R} \to \mathrm{Aut}(E) \Longrightarrow^{?} t \mapsto \mathbf{U}^{w*}(t)$ str.cont.?
(2) *Category description of w*-dense JB*-subtriples of Cartan factors*

Conjecture. [∼2003 Isidro-Stacho] E w*-dense JB*-subtrp in factor Type 1 $\iff \exists E_0 \subset E$ $w*-dense$ TRO (ternary ring of operators)

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$t \mapsto \Phi^t \in \mathrm{Aut}(B)$ str.cont 1-prg: $\quad t \mapsto \Phi^t(x)$ norm.cont. $\forall x$

$$\Phi^t = M_{\Phi^t(0)} \circ \underbrace{U_t}_{\mathrm{LIN} \in \mathrm{Aut(B)}} \qquad M_a(x) = a + \underbrace{\beta(a)}_{I - 2D(a) + Q(a)^2}{}^{1/2}[I - D(x,a)]^{-1}x$$

Lemma. *$t \mapsto \Phi^t$ str.cont. $\iff$ $t \mapsto \Phi^t(0)$ cont., $t \mapsto U_t$ str.cont.*

Question: $\Phi^t \stackrel{?}{=} \exp\left[t\big(a - \{aa^*x\} + Ax\big)\partial/\partial x\right]$ *A unbded.$\{\ldots\}$-der.*

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EXAMPLE IN HILBERT SPACE

$E := L^2(\mathbb{R})$ $\quad S^t : f \mapsto f(x+t)$ $\quad$ shifts $\qquad$ INF.GEN: $A := \frac{d}{dx}$

Consider: $\quad \Phi^t = \exp\left[t\big(a - \{aa^* f\} + Af\big)\frac{\partial}{\partial f}\right] \qquad \Phi^t(0) : x \mapsto f(t,x)$

$$\frac{\partial f}{\partial t} = a - \int_{-\pi}^{\pi} f(t,\xi)\overline{a(\xi)}\, d\xi \cdot f + \frac{\partial f}{\partial x} \qquad f(0,x) \equiv 0$$

$y := x + t,\ u := t,\ f(t,x) = g(y,u),\ b(u,y) = a(y-u) \ \Rightarrow$

$$\frac{\partial g}{\partial u} = b(\underbrace{x-u}_{retard.}) - \int_{-\pi}^{\pi} g(u,\eta)\overline{b(\eta)}\, d\eta \cdot g, \qquad g(0,y) \equiv 0$$

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