GENERALIZED BI-CIRCULAR PROJECTIONS ON SPACES OF OPERATORS

FERNANDA BOTELHO AND JAMES JAMISON

Abstract. We provide a characterization of generalized bi-circular projections on spaces of operators supporting only a particular type of surjective isometries. We also characterize the hermitian operators and hermitian projections on this class of spaces.

1. Introduction

Fosner, Illisevic, and Li in [7] have introduced an interesting class of projections on Banach spaces. Their work generalizes earlier results by Stacho and Zalar on bi-circular projections, see [17] and [18]. A projection P on a Banach space X is said to be a bi-circular projection if $e^{ia}P + e^{ib}(I - P)$ is an isometry for all choices of real numbers a and b. These projections are in fact norm hermitian, see [11]. Fosner, Illisevic, and Li in [7] only require that $P + \lambda(I - P)$ be an isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$. In [7], the authors obtained nice results in the finite dimensional setting. In this paper, we study such projections for spaces of bounded operators between pairs of Banach spaces. We call these operators generalized bi-circular projections. Operators of the form $\mathcal{J}(T) = UTV$ on $\mathcal{B}(X, Y)$ with U and V surjective isometries on Y and X, respectively, are clearly surjective isometries on the $\mathcal{B}(X, Y)$. Isometries of this type are referred, throughout this paper, as isometries of type I. The isometry group of $\mathcal{B}(X, Y)$ is known to be particularly simple for certain pairs of Banach spaces. It has been shown that several spaces of operators support only isometries of type I, see e.g. [9] and [14]. In this paper, we provide a characterization of generalized bi-circular projections on spaces of operators supporting only isometries of type I.

2. Generalized bi-circular projections on some Functional Banach Spaces

The next theorem is our main result for generalized bi-circular projections on spaces of operators supporting only isometries of type I.

Theorem 2.1. *If X and Y are complex Banach spaces so that $\mathcal{B}(X, Y)$ supports only isometries of type I, then a projection P on $\mathcal{B}(X, Y)$ is a generalized bi-circular projection if and only if $P(T) = P_Y\, T$ or $P(T) = T\, P_X$ with P_X and P_Y generalized bi-circular projections on X and Y respectively.*

Date: May, 2007.

2000 *Mathematics Subject Classification.* Primary 30D55; Secondary 30D05.

Key words and phrases. Isometry, Isometries.

Proof. If the projection P is as stated in the proposition then clearly it is a generalized bi-circular projection on $\mathcal{B}(X, Y)$. Conversely, we assume that the operator P is a generalized bi-circular projection, then the isometry $\mathcal{J} = P + \lambda(Id - P)$ satisfies

$$\mathcal{J}^2 - (\lambda + 1)\mathcal{J} + \lambda Id = 0. \tag{2.1}$$

This last equation is equivalent to

$$U^2TV^2 - (\lambda + 1)UTV + \lambda T = 0, \tag{2.2}$$

for all $T \in \mathcal{B}(X, Y)$. Given a nontrivial $v \in Y$ we consider the rank one operator T of the form $T(x) = \varphi(x)v$, with $\varphi \in X^*$. We first observe that every x we must have that $\{x,\, V(x),\, V^2(x)\}$ is linearly dependent. If for every x $\{x,\, V(x)\}$ is linearly dependent then there exists a modulus one constant a so that $V = aId_X$. The equation 2.2 implies that

$$a^2U^2 - (\lambda + 1)aU + \lambda Id = 0. \tag{2.3}$$

A theorem due to Taylor, see [13] pg. 317, asserts the existence of two projections P_1 and P_2 on Y so that $P_1 + P_2 = Id$, $P_1\, P_2 = P_2\, P_1 = 0$ and $U = \overline{a}\lambda P_1 + \overline{a}P_2$. Consequently $\mathcal{J}(T) = (\lambda P_1 + P_2)\, T$ and $P(T) = P_2\, T$. We observe that P_2 is a generalized bi-circular projection on Y. If there exists x_0 so that $\{x_0,\, V(x_0)\}$ is linearly independent then $V^2(x_0) = ax_0 + b\, V(x_0)$. A convenient choice of $\varphi \in X^*$ so that $\varphi(x_0) = 0$ and $\varphi(V(x_0)) = 1$ reduces the equation 2.2 to $b\, U^2v - (\lambda+1)Uv = 0$, for all $v \in Y$. Hence $U = \gamma Id_Y$ for a constant γ of modulus 1. In this case, the equation 2.2 becomes $\gamma^2V^2 - \gamma(\lambda + 1)V + \lambda Id_X = 0$ and Taylor's theorem asserts the existence of projections Q_1 and Q_2 on X such that $V = \overline{\gamma}\lambda Q_1 + \overline{\gamma}Q_2$. This implies that $\mathcal{J}(T) = T\,[\lambda Q_1 + Q_2]$ and thus $P(T) = T\, Q_2$, with Q_2 a generalized bi-circular projection on X. $\square$

Remark 2.2. Grząślewicz, in [8], showed that the surjective isometries on $\mathcal{B}(l^p,\, l^r)$, with $\frac{1}{p} + \frac{1}{r} \neq 1$ and $p, r \in (1,\, \infty)$, are of type I. For $r \neq 2$, the bi-circular projections on l^r are just the average of the identity and an isometric reflection. This holds true for symmetric sequence spaces with 1-symmetric basis, as a consequence of Arazy's characterization of isometries on such spaces, see [1].

We now turn our attention to the problem of characterizing bi-circular projections on $\mathcal{B}(X, Y)$. Since bi-circular projections are hermitian, we start by characterizing hermitian operators on $\mathcal{B}(X, Y)$. We observe that whenever X and Y are the same Hilbert space $\mathcal{H}$, then $\mathcal{B}(\mathcal{H})$ is a C^* algebra. In this situation, the surjective isometries are not only of type I, hence the forthcoming Theorem 2.3 does not include this case. In [16], the structure of hermitian operators on $\mathcal{B}(\mathcal{H})$ was derived relying upon intrinsic algebraic properties of $\mathcal{B}(\mathcal{H})$. Here, we derive a similar result using techniques depending on basic Banach space properties of $\mathcal{B}(X, Y)$, given that $\mathcal{B}(X, Y)$ supports only surjective isometries of type I. The characterization of bi-circular projections on $\mathcal{B}(X, Y)$ will follow as a corollary.

Theorem 2.3. *If X and Y are complex Banach spaces so that $\mathcal{B}(X, Y)$ supports only isometries of type I, then S is a hermitian operator on $\mathcal{B}(X, Y)$ if and only if $\mathcal{S}$ is a real multiple of the Id or there exist hermitian operators $\mathbb{L}$ and $\mathbb{R}$ on Y and X, respectively, so that $\mathcal{S}T = \mathbb{L}T + T\mathbb{R}$.*

Proof. We first show that $\mathcal{S}$ is a hermitian operator, if $\mathcal{S}$ is of the form stated. It is sufficient to prove that $\mathcal{T}_t = e^{it\mathcal{S}}$ is a uniformly continuous one-parameter group of isometries. Indeed, since

$$e^{it\mathcal{S}}(T) = T + it(\mathbb{L}T + T\mathbb{R}) + \cdots + (it)^k \frac{1}{k!} \sum_{i=1}^{k} \binom{k}{i} \mathbb{L}^i T \mathbb{R}^{k-i} + \cdots ,$$

we have that $e^{it\mathcal{S}}(T) = e^{it\mathbb{L}} T e^{it\mathbb{R}}$. By assumption, $\{e^{it\mathbb{L}}\}$ and $\{e^{it\mathbb{R}}\}$ are uniformly continuous one parameter groups of isometries, hence so is $e^{it\mathcal{S}}$, and $\mathcal{S}$ is hermitian.

Conversely, if $\mathcal{S}$ is a hermitian operator on $\mathcal{B}(X, Y)$, then $\mathcal{T}_t = e^{it\mathcal{S}}$ is a uniformly continuous one-parameter group of isometries. For each $t \in R$, there exist isometries on X and Y, denoted by $\mathbb{R}_t$ and $\mathbb{L}_t$ respectively, so that $e^{it\mathcal{S}}T = \mathbb{L}_t T \mathbb{R}_t$. Without loss of generality, we assume that $\mathbb{R}_0 = Id_X$ and $\mathbb{L}_0 = Id_Y$. We show that both families are uniformly continuous functions of t. We first assume that $\mathcal{S}$ is not a multiple of Id. For every positive ϵ, there exists $\delta > 0$, so that for $|t - t_0| < \delta$ we have that $\|e^{it\mathcal{S}} - e^{it_0\mathcal{S}}\| < \epsilon$. We also have that

$$\|e^{it\mathcal{S}} - e^{it_0\mathcal{S}}\| \geq \sup\{\|\varphi(\mathbb{R}_t(x))\mathbb{L}_t(y) - \varphi(\mathbb{R}_{t_0}(x))\mathbb{L}_{t_0}(y)\|_Y : \|y\| = \|\varphi\| = 1\} =$$

$$\sup\{\|\varphi(\mathbb{R}_t(x))[\mathbb{L}_t(y) - \mathbb{L}_{t_0}(y)] + \varphi[\mathbb{R}_t(x) - \mathbb{R}_{t_0}(x)]\mathbb{L}_{t_0}(y)\|_Y : \|y\| = \|\varphi\| = 1\}.$$

If for every t, such that $|t - t_0| < \delta$, we can find unit vectors $x_t \in X$ for which $\{\mathbb{R}_t(x_t), \mathbb{R}_t(x_t) - \mathbb{R}_{t_0}(x_t)\}$ is linearly independent, then there exist norm 1 functionals $\varphi_t \in X^*$ so that $\varphi_t(\mathbb{R}_t(x_t)) = 1$ and $\varphi_t(\mathbb{R}_t(x_t) - \mathbb{R}_{t_0}(x_t)) = 0$. Consequently we have

$$\epsilon > \|e^{it\mathcal{S}} - e^{it_0\mathcal{S}}\| \geq \sup\{\|\mathbb{L}_t(y) - \mathbb{L}_{t_0}(y)\| : \|y\| = 1\} = \|\mathbb{L}_t - \mathbb{L}_{t_0}\|$$

and therefore $\|\mathbb{L}_t - \mathbb{L}_{t_0}\| < \epsilon$.

On the other hand, we also have that

$$\|\varphi(\mathbb{R}_t(x))[\mathbb{L}_t(y) - \mathbb{L}_{t_0}(y)] + \varphi[\mathbb{R}_t(x) - \mathbb{R}_{t_0}(x)]\mathbb{L}_{t_0}(y)\| \geq |\varphi(\mathbb{R}_t(x) - \mathbb{R}_{t_0}(x))| - \|\mathbb{L}_t - \mathbb{L}_{t_0}\|$$

and hence

$$\|\mathbb{R}_t - \mathbb{R}_{t_0}\| = \sup\{|\varphi(\mathbb{R}_t - \mathbb{R}_{t_0})(x))| : \|\varphi\| = 1\} \leq 2\epsilon.$$

Similarly, the inequality

$$\begin{aligned} &\|e^{it\mathcal{S}} - e^{it_0\mathcal{S}}\| \\ &\geq \sup_{\|y\|=\|\varphi\|=\|\psi\|=1}\{|\varphi(\mathbb{R}_t(x))\psi[\mathbb{L}_t(y) - \mathbb{L}_{t_0}(y)] + \varphi[\mathbb{R}_t(x) - \mathbb{R}_{t_0}(x)]\psi(\mathbb{L}_{t_0}(y)|\} \end{aligned}$$

implies that $\|\mathbb{R}_t - \mathbb{R}_{t_0}\| < \epsilon$, if for every t there exists y_t (of norm 1) so that $\{\mathbb{L}_{t_0}(y_t), \mathbb{L}_t(y_t) - \mathbb{L}_{t_0}(y_t)\}$ is linearly independent. Therefore it follows that $\|\mathbb{L}_t - \mathbb{L}_{t_0}\| \leq 2\epsilon$.

It remains to assume that there exists a sequence $\{t_n\}$ converging to t_0 so that for every $x \in X$ and $y \in Y$ we have that $\{\mathbb{L}_{t_0}(y), \mathbb{L}_{t_n}(y) - \mathbb{L}_{t_0}(y)\}$ and $\{\mathbb{R}_{t_n}(x), \mathbb{R}_{t_n}(x) - \mathbb{R}_{t_0}(x)\}$ are linearly dependent. In such case, we have that $e^{it_n\mathcal{S}} = c\, e^{it_0\mathcal{S}}$. The Theorem 6, in [15], implies that $\mathcal{S}$ is a multiple of the Id, contradicting our initial assumption. We have established the continuity of both families.

Now, we show that $\mathbb{L}_t$ and $\mathbb{R}_t$ are weakly differentiable functions of t. Since

$$\lim_{t \to t_0} \frac{\mathbb{L}_t T \mathbb{R}_t - \mathbb{L}_{t_0} T \mathbb{R}_{t_0}}{t - t_0} \text{ exists,}$$

then the following limit also exists

$$\psi(\mathbb{L}_{t_0}(y)) \lim_{t\to t_0} \varphi\left(\frac{\mathbb{R}_t(x)-\mathbb{R}_{t_0}(x)}{t-t_0}\right) + \varphi(\mathbb{R}_{t_0}(x)) \lim_{t\to t_0} \psi\left(\frac{\mathbb{L}_t(y)-\mathbb{L}_{t_0}(y)}{t-t_0}\right),$$

for every $\varphi \in X^*$ and $\psi \in Y^*$.

We assume that for every t near t_0 there exists x_t so that $\left\{\frac{\mathbb{R}_t(x_t)-\mathbb{R}_{t_0}(x_t)}{t-t_0}, \mathbb{R}_{t_0}(x_t)\right\}$ is linearly independent, then Hahn-Banach Theorem asserts the existence of $\varphi \in X^*$, attaining the value 0 at $\frac{\mathbb{R}_t(x_t)-\mathbb{R}_{t_0}(x_t)}{t-t_0}$ and the value of 1 at $\mathbb{R}_{t_0}(x_t)$. Therefore we have that

$$\lim_{t\to t_0} \frac{\mathbb{L}_t(y)-\mathbb{L}_{t_0}(y)}{t-t_0} \text{ exists for all } y \in Y.$$

Similarly, if we assume that for every t near t_0 there exists y_t so that

$$\left\{\frac{\mathbb{L}_t(y_t)-\mathbb{L}_{t_0}(y_t)}{t-t_0}, \mathbb{L}_{t_0}(y_t)\right\} \text{ is linearly independent.}$$

There exists $\psi \in Y^*$ attaining the value 0 on $\mathbb{L}_{t_0}(y_t)$ and the value 1 on $\frac{\mathbb{L}_t(y_t)-\mathbb{L}_{t_0}(y_t)}{t-t_0}$. This implies that

$$\lim_{t\to t_0} \phi\left(\frac{\mathbb{R}_t(x)-\mathbb{R}_{t_0}(x)}{t-t_0}\right) \text{ exists for all } x \in X.$$

If there exists a sequence $\{t_n\}$ converging to t_0 so that for every $x \in X$

$$\left\{\frac{\mathbb{R}_{t_n}(x)-\mathbb{R}_{t_0}(x)}{t_n-t_0}, \mathbb{R}_{t_0}(x)\right\}$$

is linearly dependent and if, in addition, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ so that $\left\{\frac{\mathbb{L}_{t_{n_k}}(y)-\mathbb{L}_{t_0}(y)}{t_{n_k}-t_0}, \mathbb{L}_{t_0}(y)\right\}$ is linearly dependent for every $y \in Y$. Without loss of generality, we may assume that both sets $\left\{\frac{\mathbb{R}_{t_n}(x)-\mathbb{R}_{t_0}(x)}{t_n-t_0}, \mathbb{R}_{t_0}(x)\right\}$ and $\left\{\frac{\mathbb{L}_{t_n}(y)-\mathbb{L}_{t_0}(y)}{t_n-t_0}, \mathbb{L}_{t_0}(y)\right\}$ are linearly dependent. Hence there are sequences of scalars a_n and b_n, complex numbers of modulus 1, for which $\mathbb{R}_{t_n} = a_n R_{t_0}$ and $\mathbb{L}_{t_n} = b_n \mathbb{L}_{t_0}$. Therefore $e^{it_n\mathcal{S}} = a_n b_n e^{it_0\mathcal{S}}$ or $e^{i(t_n-t_0)\mathcal{S}} = e^{\ln(a_n\, b_n)} Id$. Since the operator $\mathcal{S}$ is hermitian, it has real spectrum $(\sigma(\mathcal{S}))$, the spectrum of $\ln(a_n\, b_n)Id$ is clearly $\ln(a_n\, b_n)$. Theorem 6, in [15], implies that $a_n\, b_n = 1$ or $\mathcal{S} - \ln(a_n\, b_n)Id = (2k_n\pi i)Id$, for some integers k_n. If for every n, $a_n\, b_n = 1$ then $e^{i(t_n-t_m)\mathcal{S}} = Id$, which is impossible. Therefore $\mathcal{S}$ is a multiple of the Id.

We show that both families define one-parameter groups of isometries.

The group condition $\mathcal{T}_{t_1+t_2} = \mathcal{T}_{t_1}\mathcal{T}_{t_2}$ implies that $\mathbb{L}_{t_1+t_2} = \lambda(t_1,t_2)\, \mathbb{L}_{t_1} L_{t_2}$ and $R_{t_1+t_2} = \overline{\lambda}(t_1,t_2)\, R_{t_1} R_{t_2}$, for some modulus 1 scalars. We prove that $\lambda(t_1,t_2) = 1$, for every t_1 and t_2. Since we have assumed that $\mathbb{L}_0 = Id_Y$ and $R_0 = Id_X$, then $\mathbb{L}_0 = Id_Y = \lambda(t_1,-t_1) L_{t_1}\mathbb{L}_{-t_1} = \lambda(-t_1,t_1)\mathbb{L}_{-t_1}\mathbb{L}_{t_1}$ and $L_{t_1} = \overline{\lambda}(t_1,-t_1)\mathbb{L}_{-t_1}^{-1}$. This implies that $Id_X = \lambda(-t_1,t_1) L_{-t_1}\mathbb{L}_{t_1} = \lambda(-t_1,t_1)\overline{\lambda}(t_1,-t_1)\mathbb{L}_{-t_1}\mathbb{L}_{-t_1}^{-1}$. Therefore $\lambda(-t_1,t_1) = \lambda(t_1,-t_1)$ and $\mathbb{L}_{-t_1}\mathbb{L}_{t_1} = \mathbb{L}_{t_1}\mathbb{L}_{-t_1}$.

We clearly have $\lambda(0,t) = \lambda(t,0) = 1$, for all t.

First, we observe that $\lambda(t_1,t_2) = \lambda(t_2,t_1)$ if and only if $\mathbb{L}_{t_1}\mathbb{L}_{t_2} = \mathbb{L}_{t_2}\mathbb{L}_{t_1}$. In order to prove this last statement we proceed as follows:

$$\mathbb{L}_{3t} = \lambda(2t,t)\mathbb{L}_t\mathbb{L}_{2t} = \lambda(2t,t)\lambda(t,t)\mathbb{L}_t\mathbb{L}_t\mathbb{L}_t = \lambda(2t,t)\mathbb{L}_t\mathbb{L}_{2t}$$

and

$$\mathbb{L}_t\mathbb{L}_{2t} = \mathbb{L}_{2t}\mathbb{L}_t.$$

This last statement is equivalent to $\lambda(2t,t) = \lambda(t,2t)$. Inductively we show that $\mathbb{L}_{mt}\mathbb{L}_{nt} = \mathbb{L}_{nt}\mathbb{L}_{mt}$ and $\lambda(nt,mt) = \lambda(mt,nt)$, for n, m integers and t a real number. Therefore we have $\mathbb{L}_{r_1}\mathbb{L}_{r_2} = \mathbb{L}_{r_2}\mathbb{L}_{r_1}$ for rational values r_1 and r_2 and continuity implies that $\mathbb{L}_{t_1}\mathbb{L}_{t_2} = \mathbb{L}_{t_2}\mathbb{L}_{t_1}$ and $\lambda(t_1,t_2) = \lambda(t_2,t_1)$.

Furthermore, for arbitrary values of t, say t, t_1, t_2 we have that $\lambda(t+t_1,t_2)\lambda(t,t_1) = \lambda(t_1+t_2,t)\lambda(t_1,t_2)$. The weak differentiability previously established implies the differentiability of λ, then we have

$$\partial_t\lambda(t+t_1,t_2)\lambda(t,t_1) + \lambda(t+t_1,t_2)\partial_t\lambda(t,t_1) = \partial_t\lambda(t,t_1+t_2)\lambda(t_1,t_2).$$

Hence, for $t = t_2$, the equation above implies that $\partial_t\lambda(t_2,t_1) = 0$ and $\lambda(t_2,t_1) = C(t_1)$, a constant depending on t_1. For $t_2 = 0$, we have that $1 = \lambda(0,t_1) = C(t_1)$ and we have established that $\lambda = 1$.

The families $\{\mathbb{L}_t\}$ and $\{\mathbb{R}_t\}$ are one-parameter groups of uniformly continuous families of isometries, hence there exist hermitian operators $\mathbb{L}$ and $\mathbb{R}$ so that $\mathbb{L}_t = e^{it\mathbb{L}}$ and $\mathbb{R}_t = e^{it\mathbb{R}}$. Therefore we have that $\mathcal{T}_t = e^{it\mathcal{S}}T = e^{it\mathbb{L}}\,T\,e^{it\mathbb{R}}$ and the corresponding generator satisfies

$$\mathcal{S} = -i\left(\frac{d}{dt}e^{it\mathcal{S}}T\right)_{t=0} = \mathbb{L}T + T\mathbb{L}.$$

This completes the proof of the statement, provided that $\{\mathcal{T}_t\}$ is a nontrivial family.

If we assume that, for some t_0, $\mathcal{T}_{t_0}$ is a multiple of the Id, then Theorem 6, in [15], implies that $\lambda = 1$ or $\mathcal{S} - \ln(\lambda)Id = (2k\pi i)Id$, for some integer k. In either case $\mathcal{S}$ is a multiple of the identity. This completes the proof of the theorem. □

Corollary 2.4. *$\mathcal{P}$ is a hermitian projection on $\mathcal{B}(X,Y)$ if and only if, for every $T \in \mathcal{B}(X,Y)$, $\mathcal{P}T = T\mathbb{R}_1$ or $\mathcal{P}T = \mathbb{L}_1T$, where $\mathbb{R}_1$ and $\mathbb{L}_1$ are hermitian projections on X and Y, respectively.*

Proof. $\mathcal{P}$ is a hermitian operator on $\mathcal{B}(X,Y)$ then the previous theorem asserts the existence of $\mathbb{L}$ and $\mathbb{R}$, hermitian operators on the Y and X respectively, so that

$$\mathcal{P}T = \mathbb{L}T + T\mathbb{R},$$

for every $T \in \mathcal{B}(X,Y)$. Since $\mathcal{P}$ is a projection, we have that

$$\mathbb{L}^2T + 2\mathbb{L}T\mathbb{R} + T\mathbb{R}^2 = \mathbb{L}T + T\mathbb{R} \tag{2.4}$$

and

$$\varphi(x)\,\psi(\mathbb{L}^2(y)) + \varphi((2\mathbb{R} - Id)(x))\psi(\mathbb{L}(y)) + \varphi((\mathbb{R}^2 - \mathbb{R})(x))\psi(y) = 0, \tag{2.5}$$

for every $\varphi \in X^*$ and $\psi \in Y^*$. We first observe that for every $x \in X$, $\{x, \mathbb{R}(x), \mathbb{R}^2(x)\}$ is linearly dependent. If we assume that, for every $x \in X$, $\{x, \mathbb{R}(x)\}$ is also linearly dependent, then $\mathbb{R} = a\,Id_X$ for some scalar a of modulus 1. In this case, the equation 2.4 reduces to $\mathbb{L}^2 + (2a-1)\mathbb{L} + (a^2 - a)Id = 0$ and Taylor's theorem implies the existence of projections $\mathbb{L}_1$ and $\mathbb{L}_2$ so that $\mathbb{L}_1\mathbb{L}_2 = \mathbb{L}_2\mathbb{L}_1 = 0$ and $\mathbb{L} = (1-a)\mathbb{L}_1 - a\mathbb{L}_2$. Since $\mathbb{L}_1 + \mathbb{L}_2 = Id$, we have that $\mathbb{L} = \mathbb{L}_1 - aId$. Consequently $\mathcal{P}T = \mathbb{L}_1T$, with $\mathbb{L}_1$ and hermitian projection on Y.

If, there exists $x \in X$ so that $\{x, \mathbb{R}(x)\}$ is linearly independent, then $\mathbb{R}^2(x) = ax + b\,\mathbb{R}(x)$, for some scalars a and b. The equation 2.5 implies that $2\mathbb{L} + (b-1)Id = 0$. Hence $\mathbb{L}$ is a multiple of the Id and a similar argument proves the existence of a hermitian projection $\mathbb{R}_1$ on X so that $\mathcal{P}T = T\mathbb{R}_1$. □

Remark 2.5. The bi-circular projections on $\mathcal{B}(X,Y)$, if $\mathcal{B}(X,Y)$ supports only isometries of type I, are of the form: $\mathcal{P}T = T\mathbb{R}_1$ or $\mathcal{P}T = \mathbb{L}_1 T$, where $\mathbb{R}_1$ and $\mathbb{L}_1$ are bi-circular projections on X and Y, respectively.

Remark 2.6. Pairs of Banach spaces (X, Y) for which $\mathcal{B}(X,Y)$ supports only surjective isometries of type I are:

(1) $X = l^p$ and $Y = l^r$, with $p, r \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{r} \neq 1$, see [8]
(2) Khalil and Saleh ideal pairs, see [9].

References

[1] Arazy,J., *Isometries of Complex Symmetric Sequence Spaces*, Mathematische Zeitschrift **188**(1985),427–431.
[2] Behrends,E., *M-Structure and the Banach-Stone Theorem*, Lecture Notes in Mathematics **736**(1979),Springer-Verlag.
[3] Berkson,E., *Hermitian projections and orthogonality in Banach spaces*, Proc. London Math. Soc. **24:3**(1972),101–118.
[4] Botelho,F. and Jamison,J.E., *Generalized circular projections*, preprint (2006).
[5] Botelho,F. and Jamison,J.E., *Generalized Bi-circular projections on Spaces of Analytic Functions*, preprint (2006).
[6] Fleming,R. and Jamison,J., *Isometries on Banach Spaces*, (2003) Chapman & Hall.
[7] Fosner,M.,Ilisevic,D. and Li,C., *G-invariant norms and bicircular projections*, Linear Algebra and Its Applications **420**(2007), 596-608.
[8] Grząślewicz,R., *The isometries of* $\mathcal{L}(l^p, l^r)$, Funct. Approx. Comment. Math. **26** (1998), 287-291.
[9] Khalil,R. and Saleh,A., *Isometries on Certain Operator Spaces*, Proceedings AMS, **132:5**(2003), 1473–1481.
[10] Khalil,R., *Isometries on* $L^p \hat{\otimes} L^p$, Tamkang Journal of Mathematics, **16:2**(1985), 77–85.
[11] Jamison,J.E., *Bicircular projections on some Banach spaces*, Linear Algebra and Applications, to appear (2006).
[12] Lin, P., *Generalized Bi-circular Projections*, preprint (2006).
[13] Taylor,A.E. *Introduction to Functional Analysis*(1957), John Wiley & Sons Inc.
[14] Rao,T.S.S.R.K., *Remarks on a Result of Khalil and Saleh*, Proc. Amer. Math. Soc. **133:6**(1970), 209–214.
[15] Schmoeger, C., *Remarks on Commuting Exponentials in Banach Algebras, II*, Proceedings AMS. **128:11**(2004), 1721–1722.
[16] Sinclair,A.M., *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24**(1970), 209–214.
[17] Stachó,L.L. and Zalar,B. *Bicircular projections on some matrix and operator spaces*, Linear Algebra and Applications **384**(2004), 9–20.
[18] Stachó,L.L. and Zalar,B., *Bicircular projections and characterization of Hilbert spaces*, Proc. Amer. Math. Soc. **132**(2004),3019–3025.

Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152
E-mail address: mbotelho@memphis.edu

Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152
E-mail address: jjamison@memphis.edu