On C_0-semigroups of holomorphic Carathéodory isometries in Hilbert space

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Abstract

We establish closed formulas for all strongly continuous one-parameter semigroups of holomorphic Carathéodory isometries of the unit ball of a Hilbert space in terms of spectral resolutions of skew self-adjoint dilations related to Vesentini's non-linear infinitesimal generator.

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1. Introduction

Throughout this work $\mathbf{H}$ denotes an arbitrarily fixed complex Hilbert space with scalar product $\langle x|y\rangle$ being antilinear in y and the canonical norm $\|x\| := \langle x|x\rangle^{1/2}$. We also keep fixed the standard notations $\mathbf{B} := \{x \in \mathbf{H} : \|x\| < 1\}$, $a^* := [x \mapsto \langle x|a\rangle]$ for the open unit ball, and the adjoint representation of bounded linear functionals, respectively. We regard the elements h, h^* $(h \in \mathbf{H})$ as column resp. row matrices and, given a linear map $A : \mathbf{S} \to \mathbf{H}$ on some linear submanifold of $\mathbf{H}$. We use the canonical matrix identifications $x \oplus \xi \equiv \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$ resp. $\left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] \equiv \big[x \oplus \xi \mapsto (Ax+b) \oplus (c^*x+d)\big]$ with $x, b, c \in \mathbf{H}$ and $\xi, d \in \mathbb{C}$. This gives rise to the familiar linear representation of fractional linear maps on $\mathbf{H}$:

$$\mathfrak{F}\Big(\begin{bmatrix} A & b \\ c^* & d \end{bmatrix}\Big) := \big[x \mapsto (c^*x+d)^{-1}(Ax+b)\big].$$

Our object of chief interest will be the semigroup $\mathrm{Iso}(d_{\mathbf{B}})$ of all holomorphic isometries of $\mathbf{B}$ with respect to the Carathéodory metric $d_{\mathbf{B}}$. Recall [3] that all its elements are fractional linear maps (restricted to $\mathbf{B}$), namely they are compositions of *Möbius transformations*[1] with linear isometries of $\mathbf{H}$.

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[1]Fractional linear transformations mapping $\mathbf{B}$ injectively onto itself.

In 1987, in his pioneering work [9] Vesentini studied subsemigroups of $\mathrm{Iso}(d_{\mathbf{B}})$ arising from strongly continuous one-parameter matrix-semigroups.[2] He established that the correspondence $\mathfrak{F}^{\#} : \big[\mathcal{U}^t : t \in \mathbb{R}_+\big] \mapsto \big[\mathfrak{F}(\mathcal{U}^t)|\mathbf{B} : \ t \in \mathbb{R}_+\big]$ maps the family $\mathfrak{S}$ of all strongly continuous one-parameter semigroups of $\mathbb{C}$-linear isometries of the indefinite norm $\|x\|^2 - |\xi|^2$ on $\mathbf{H} \oplus \mathbb{C}$ into the family $\mathfrak{B}$ of all strongly continuous one-parameter semigroups $[\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$. According to [9, Th.VII] , given $[\mathcal{U}^t : t \in \mathbb{R}_+] \in \mathfrak{S}$ with the infinitesimal generator $\mathcal{A} = \frac{d}{dt}\big|_{t=0+}\mathcal{U}^t$, for the corresponding non-linear objects $\Psi^t := \mathfrak{F}(\mathcal{U}^t)\big|\mathbf{B}$ we have $\big\{p \in \mathbf{B} : t \mapsto \Psi^t(p) \text{ is differentiable}\big\} = \big\{x \in \mathbf{B} : \ x\oplus 1 \in \mathrm{dom}(\mathcal{A})\big\}$ which is dense in the ball $\mathbf{B}$. It is well-known [9, 4] that here we can identify the in $\mathbf{H}\oplus\mathbb{C}$ densely defined linear operator $\mathcal{A}$ with an $\mathbf{H}\oplus\mathbb{C}$-split matrix if and only if the orbit $t \mapsto \Psi^t(0)$ is differentiable. This happens if and only if the generator $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{bmatrix} iA + \nu & b \\ b^* & \nu \end{bmatrix}, \qquad \nu\in\mathbb{C}, \ b\in\mathbf{H}, \ A\in\mathrm{Her_s}(\mathbf{H}) \tag{1.1}$$

with $\mathrm{dom}(\mathcal{A}) = \mathrm{dom}(A) \oplus \mathbb{C}$ where $\mathrm{Her_s}(\mathbf{H})$ stands for the family of all unbounded $\mathbf{H}$-hermitian operators (maximal symmetric, in $\mathbf{H}$ densely defined closed linear operators). Even the cases with non-differentiable 0-orbit can be treated by passing to a semigroup $[\Phi^t : t \in \mathbb{R}_+]$ of the form $\Theta^{-1} \circ \Psi^t \circ \Theta$ with any Möbius transformation Θ such that $\Theta(0) \in \mathrm{dom}(\Gamma)$. Indeed, since the Möbius group is transitive on $\mathbf{B}$, hence any strongly continuous one-parameter semigroup $[\Psi^t : t \in \mathbb{R}_+] \in \mathfrak{B}$ is equivalent up to a Möbius transformation (*Möbius equivalent* for short in the sequel) to a semigroup $[\Phi^t : t \in \mathbb{R}_+] \in \mathfrak{B}$ whose infinitesimal generator has the form

$$\Gamma(x) = \frac{d}{dt}\Big|_{t=0+}\Phi^t \ = b - \langle x|b\rangle x + iAx, \quad x \in \mathrm{dom}(R) \cap \mathbf{B} \tag{1.2}$$

with some maximal symmetric operator A defined densely on $\mathbf{H}$ and some vector $b \in \mathbf{H}$. Also conversely, if iA is the infinitesimal generator for some strongly continuous one-parameter subsemigroup of $\mathcal{L}(\mathbf{H})$ then, for any $b \in \mathbf{H}$, the vector field (1.2) is the infinitesimal generator of a strongly continuous one-parameter subsemigroup of $\mathrm{Iso}(d_{\mathbf{B}})$. In cite10,6 these considerations were extended to semigroups of fractional linear transformations arising from a strongly continuous one parameter semigroup of automorphisms of a Krein space, concluding with a description of the fractional-linear image of a strongly continuous one parameter group by means of the solutions of Ricatti type equations $\dot{x} = \Gamma(x)$ with analogous vector fields to (1.2) in Pontriagin spaces. It should be noted that as far no argument appeared in the literature concerning the seemingly plausible surjectivity of the map $\mathfrak{F}^{\#}$. The question is rather harmless in our setting: in the case of the unit ball of a Hilbert space an argument with joint fixed points (Proposition 3.5) furnishes positive answer. However, e.g. in the case of the unit ball of $\mathcal{L}(\mathbf{H})$ the surjectivity of the respective $\mathfrak{F}^{\#}$ seems to be open and highly not trivial.

2. Results

Henceforth, for short, C_0S [resp. C_0G] will abbreviate the terms *strongly continuous one-parameter semigroup* [*-group*]. We shall write $\mathrm{gen}[U^t : t \in \mathbb{R}_+]$ or $\mathrm{gen}[\widetilde{U}^t : t \in \mathbb{R}]$ for

[2]Given a topological space $\mathbf{X}$, a family $[T^t : t\in\mathbb{R}_+]$ $\big[$resp. $[T^t : t\in\mathbb{R}]\big]$ of self-maps $T^t : \mathbf{X} \to \mathbf{X}$ is a *strongly continuous one-parameter semigroup* [resp. *group*] if $T^{s+t} = T^s \circ T^t$ $\big(s, t \in \mathbb{R}_+$ [resp. $\mathbb{R}]\big)$ and all the orbits $t \mapsto T^t(x)$ $(x \in \mathbf{X})$ are continuous.

the infinitesimal generator of the C_0S $[U^t : t \in \mathbb{R}_+]$ or C_0G $[\widetilde{U}^t : t \in \mathbb{R}]$, respectively. Given a closed subspace $\mathbf{K}$ in the Hilbert spaces $\mathbf{H}$ or $\mathbb{C}e \oplus \mathbf{H}$ we shall write $\mathrm{P}_{\mathbf{K}}$ the orthogonal projection onto $\mathbf{K}$ without danger of confusion.

In this paper we develop a triangularization method leading to explicit algebraic formulas for a C_0S generated by a vector field (1.2) in terms of fixed points of Γ and quadratures of a C_0S formed by complex linear isometries of a 1-codimensional subspace of $\mathbf{H}$. As a consequence we conclude that any C_0S of holomorphic Carathéodory isometries of $\mathbf{B}$ admits a *dilation* to a C_0G of surjective holomorphic Carathéodory isometries of the unit ball of some covering Hilbert space. Our fixed point approach seems to be new even in finite dimensions (with uniformly continuous one-parameter groups).

Recall [3] that any Carathéodory isometry $\Psi \in \mathrm{Iso}(d_{\mathbf{B}})$ admits a continuous extension $\overline{\Psi}$ to the closed unit ball $\overline{\mathbf{B}}$. Given a C_0S $\mathbf{\Psi} = [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$, the extensions $\overline{\mathbf{\Psi}} := [\overline{\Psi^t} : t \in \mathbb{R}_+]$ form also a C_0S (see [7] in more general setting). According to [9, Section 7] $\overline{\mathbf{\Psi}}$ admits common fixed points whose family $\mathrm{Fix}(\overline{\mathbf{\Psi}})$ consists of one or two boundary points or it is the intersection of $\overline{\mathbf{B}}$ with some closed complex-affine submanifold containing points from $\mathbf{B}$ in which case $\mathbf{\Psi}$ is simply Möbius equivalent to a C_0S of linear isometries of $\mathbf{H}$ restricted to $\mathbf{B}$.

Our main goal is the following classification of the remaining cases with explicit formulas up to Möbius equivalence.

Theorem 2.1. *Suppose the vector field* (1.2) *is the infinitesimal generator of a C_0S $\mathbf{\Phi} := [\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ having a common boundary fixed point $e \in \mathrm{Fix}(\overline{\mathbf{\Phi}}) \cap \partial\mathbf{B}$. Then we have*

$$\begin{aligned} &\mathrm{P}_{\mathbb{C}e}\, \Phi^t(\xi e + x_0) = \Big[1 - (1-\xi)e^{-2\lambda t}/\varphi_{\lambda,\mu}(t, x_0, \xi)\Big] e \\ &\mathrm{P}_{\mathbf{H}_0} \Phi^t(\xi e + x_0) = \Big[(1-\xi)e^{-2\lambda t}\big(\textstyle\int_0^t e^{\lambda s} V_0^s\, ds\big) b_0 + e^{-\lambda t} V_0^t x_0\Big] \Big/ \varphi_{\lambda,\mu}(t, x_0, \xi) \end{aligned} \tag{2.2}$$

for all points $x_0 + \xi e \in \mathbf{B}$ with $x_0 \perp e$ where $\mathbf{H}_0 := \mathbf{H} \ominus \mathbb{C}e$, $\lambda := \mathrm{Re}\langle e|b\rangle$, $\mu := \mathrm{Im}\langle e|b\rangle$, $b_0 := \mathrm{P}_{\mathbf{H}_0} b$, $[V_0^t : t \in \mathbb{R}_+]$ is the C_0S of linear $\mathbf{H}_0$-isometries generated by the skew-$\mathbf{H}_0$-hermitian operator $iP_{\mathbf{H}_0}(A - \mu)\big|\mathbf{H}_0$ and

$$\begin{aligned} \varphi_{\lambda,\mu}(t, x_0, \xi) := 1 &+ (1-\xi)\Big\langle \big(\textstyle\int_0^t e^{-2\lambda s} \int_0^s e^{\lambda r} V_0^r\, dr\, ds\big) b_0 \Big|\, b_0 \Big\rangle - \\ &- (1-\xi)(\lambda + i\mu)\textstyle\int_0^t e^{-2\lambda s} ds + \Big\langle \big(\textstyle\int_0^t e^{-\lambda s} V_0^s\, ds\big) x_0 \Big|\, b_0 \Big\rangle. \end{aligned} \tag{2.3}$$

Remark 2.4. The following converse can be discovered from the proofs later on (see Remark 3.13). Given any couple of vectors $e, b_0 \in \mathbf{H}$ such that $\|e\| = 1$ and $b_0 \perp e$ along with any C_0S $[V_0^t : t \in \mathbb{R}_+]$ of linear isometries of $\mathbf{H}_0 = \mathbf{H} \ominus (\mathbb{C}e)$ and two real constants λ, μ, the maps (0.0) form a C_0S in $\mathrm{Iso}(d_{\mathbf{B}})$.

Remark 2.5. In case of $\lambda \neq 0$, one can express the integrated operators in (0.0) in terms of the resolvent $R(\pm\lambda, iS_0)$ of the $\mathbf{H}_0$-hermitian operator $S_0 := i^{-1}\mathrm{gen}[V_0^t : t \in \mathbb{R}_+]$. Namely we have $\int_0^t e^{-\lambda\tau} V_0^\tau\, d\tau = \big(1 - e^{-\lambda t} V_0^t\big) R(\lambda, iS_0)$, $\int_0^t e^{-2\lambda\tau} \int_0^\tau e^{\lambda\sigma} V_0^\sigma\, d\sigma\, d\tau = \frac{1}{2\lambda}(1 - e^{-2\lambda t}) R(-\lambda, iS_0) - \big(1 - e^{-\lambda t} V_0^t\big) R(\lambda, iS_0) R(-\lambda, iS_0)$.

Theorem 2.6. *Let $\mathbf{\Psi} := [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ be a C_0S with $e \in \mathrm{Fix}(\overline{\mathbf{\Psi}}) \subset \partial\mathbf{B}$. Then, with the notations of Theorem* 2.1 *we have the alternatives*

(i) $\mathrm{Fix}(\overline{\mathbf{\Psi}})$ *consists of two points and* $\mathbf{\Psi}$ *is Möbius equivalent to some* C_0S $[\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ *of the form*

$$\Phi^t(\xi e + x_0) = \frac{\xi + \tanh(\lambda t)}{1 + \xi \tanh(\lambda t)}\, e + \frac{e^{-\lambda t}}{\cosh(\lambda t) + \xi \sinh(\lambda t)} V_0^t x_0; \tag{2.7}$$

(ii) $\{e\} = \mathrm{Fix}(\overline{\mathbf{\Psi}})$, *there is a* $\mathbf{\Psi}$*-invariant disc of the form* $\emptyset \neq (e + \mathbb{C}v) \cap \mathbf{B}$ *and* $\mathbf{\Psi}$ *is Möbius equivalent to a* C_0S $[\Phi^t : t \in \mathbb{R}_+]$ *of the form*

$$\Phi^t(\xi e + x_0) = \frac{1 + i\mu t}{1 - i\mu t}\, \frac{\xi - i\mu t/(1 + i\mu t)}{1 + i\mu t\xi/(1 - i\mu t)}\, e + \frac{1}{1 - i\mu t(1 - \xi)}\, V_0^t x_0; \tag{2.8}$$

(iii) *there is no* $\mathbf{\Psi}$*-invariant disc of the form* $\emptyset \neq (e+\mathbb{C}v) \cap \mathbf{B}$ *and* $\mathbf{\Psi}$ *is Möbius equivalent to a* C_0S $[\Phi^t : t \in \mathbb{R}_+]$ *of the form* (0.0) *with* $\lambda = 0$.

Remark 2.9. In the setting of Theorem 2.1 a non-empty disc $(e+\mathbb{C}v)\cap\mathbf{B}$ is $[\Phi^t : t \in \mathbb{R}_+]$-invariant if and only if $e \not\perp v \in \mathrm{dom}(A)$ and $(iA + \langle e|\, b\rangle)v \in \mathbb{C}e$ as established in Lemma 4.1. Hence, in finite dimensions only cases (i),(ii) may appear. Example 4.2 with possible independent interest for physics or stochastic processes shows that case (iii) is not void.

Recall that, as an implicit simple special case[3] of Deddens [1, Main Thm.], every C_0S $[U^t : t \in \mathbb{R}_+]$ of isometries of $\mathbf{H}$ admits a *unitary group dilation* in the following sense: there exists a Hilbert space $\widehat{\mathbf{H}}$ containing $\mathbf{H}$ as a subspace along with a C_0G $[\widehat{U}^t : t\in\mathbb{R}]$ of unitary operators of $\widehat{\mathbf{H}}$ such that $U^t = \widehat{U}^t\big|\mathbf{H}$ $(t \in \mathbb{R}_+)$. Applying a unitary dilation $[\widehat{V}_0^t : r\in\mathbb{R}]$ of the isometry semigroup $[V_0^t : t\in\mathbb{R}_+]$ in 2.2 we readily obtain the following result with non-linear dilations.

Corollary 2.10. *Given any* C_0S $[\Psi^t : t \in \mathbb{R}_+]$ *of holomorphic Caratéodory isometries of* $\mathbf{B}$, *there is a strongly continuous one parameter group* $[\widehat{\Psi}^t : t \in \mathbb{R}_+]$ *of surjective holomorphic Carathéodory isometries of the unit ball* $\widehat{\mathbf{B}}$ *of some Hilbert space* $\widehat{\mathbf{H}}$ *containing* $\mathbf{H}$ *as a subspace such that* $\Psi^t = \widehat{\Psi}^t\big|\mathbf{B}$ $(t \in \mathbb{R}_+)$.

By means of the functional calculus of the skew self-adjoint generator $i\widehat{S}_0$ of the dilation group $[\widehat{V}_0^t : t \in \mathbb{R}]$ of the C_0S $[V_0^t : t \in \mathbb{R}_+]$ in the setting of Theorem 2.1, we get the following.

Corollary 2.11. *In* (2.2) *we can write*

$$\varphi_{\lambda,\mu}(t,x_0,\xi) = \Big\langle x_0 \Big| f_1(t, \lambda, \widehat{S}_0)b_0 \Big\rangle + (1-\xi)\Big[\Big\langle f_2(t, \lambda, \widehat{S}_0)b_0 \Big| b_0\Big\rangle - (\lambda + i\mu)\textstyle\int_0^t e^{-2\lambda s} ds\Big] + 1,$$

$$P_{\mathbf{H}_0}\Phi^t(x) = \varphi_{\lambda,\mu}(t, x_0, \xi)^{-1}\Big[e^{-\lambda t}\exp(it\widehat{S}_0)x_0 + (1 - \xi)e^{-2\lambda t} f_1(t, \lambda, \widehat{S}_0)b_0\Big]$$

with the bounded analytic functions $f_j(t, \lambda, \cdot) : \mathbb{R} \to \mathbb{C}$ $(j = 1, 2;\ \lambda, t \in \mathbb{R})$

$$f_1(t, \lambda, \sigma) := \frac{1 - e^{-t(\lambda+i\sigma)}}{\lambda + i\sigma} = \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda + i\sigma)^n}{(n + 1)!} t^{n+1}, \qquad f_2(t, \lambda, \sigma) :=$$

$$:= \frac{e^{-2\lambda t}}{2\lambda(\lambda+i\sigma)} + \frac{1}{2\lambda(\lambda-i\sigma)} - \frac{e^{-t(\lambda-i\sigma)}}{\lambda^2+\sigma^2} = \sum_{n=2}^{\infty}\Big[\frac{(-2\lambda)^n}{2\lambda(\lambda + i\sigma)} - \frac{(-\lambda + i\sigma)^n}{\lambda^2 + \sigma^2}\Big]\frac{t^n}{n!}.$$

[3] We begin Section 4 with an elementary proof in Banach space setting.

3. Triangularization with boundary fixed points

Lemma 3.1. *Assume* $\mathbf{\Psi} = [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ *is a* C_0S *where* $\Psi^t = \mathfrak{F}\mathcal{U}_t\big|\mathbf{B}$ *with* $\mathcal{U}_t \in \mathcal{L}(\mathbf{H} \oplus \mathbb{C})$ $(t \in \mathbb{R}_+)$. *Then there is a family* $[\mu_t : t \in \mathbb{R}_+] \subset \mathbb{C} \setminus \{0\}$ *such that* $\big[\mu_t\mathcal{U}_t : t \in \mathbb{R}_+\big]$ *is a* C_0S *in* $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$.

Proof. Let e be a common fixed point of the the transformations $\overline{\Psi^t} = \mathfrak{F}\mathcal{U}_t|\overline{\mathbf{B}}$ $(t \in \mathbb{R}_+)$. We are going to show that the choice $\mu_t := \big[\mathcal{U}_t(e \oplus 1)\big]_{\mathbb{C}}^{-1}$ entailing $\mu_t\mathcal{U}_t(e \oplus 1) = e \oplus 1$ suits our requirements. Consider the matrices $\mathcal{V}^t := \mu_t\mathcal{U}_t$. Clearly $\mathfrak{F}\mathcal{V}^t = \mathfrak{F}\mathcal{U}_t$ $(t \in \mathbb{R}_+)$. Since the map $\mathcal{U} \mapsto \mathfrak{F}\mathcal{U}|\mathbf{B}$ is a homomorphism with respect to compositions, and since its preimages are unique up to non-zero factors, we have $\mathfrak{F}\mathcal{V}^{t+s}|\mathbf{B} = \Psi^{t+s} = \Psi^t \circ \Psi^s = \mathfrak{F}(\mathcal{V}^t\mathcal{V}^s)|\mathbf{B}$ and hence $\mathcal{V}^{t+s} = d_{t,s}\mathcal{V}^t\mathcal{V}^s$ $(t, s \in \mathbb{R})$ with suitable constants $d_{t,s} \neq 0$. The fixed point property

$$\mathcal{V}^t(e \oplus 1) = e \oplus 1 \quad (t \in \mathbb{R}_+) \tag{3.2}$$

ensures that $d_{t,s} \equiv 1$ that is the family $[\mathcal{V}^t : t \in \mathbb{R}_+]$ is a one-parameter matrix semigroup. To see its strong continuity, recall [3, Ch. VI] that the *Möbius shifts*

$$\Theta_a := \mathfrak{F}\mathcal{M}_a, \ \ \mathcal{M}_a := \begin{bmatrix} Q_a & a \\ a^* & 1 \end{bmatrix}, \ \ Q_a := \mathrm{P}_{\mathbb{C}a} + \sqrt{1 - \|a\|^2}(1 - \mathrm{P}_{\mathbb{C}a}) \quad (a \in \mathbf{B}) \tag{3.3}$$

act transitively on $\mathbf{B}$. Thus, since every element of $\mathrm{Iso}(d_{\mathbf{B}})$ keeping the origin fixed is a restriction of a linear isometry of $\mathbf{H}$, we can write $\Psi^t = \Theta_{a_t} \circ U_t$ where $a_t := \Psi^t(0)$ and U_t is a suitable linear isometry of $\mathbf{H}$. Since $U_t = \mathfrak{F}\big[\begin{smallmatrix} U_t & 0 \\ 0 & 1 \end{smallmatrix}\big]$, with suitable constants $\delta_t \neq 0$ we can write

$$\mathcal{V}_t := \delta_t \mathcal{M}_{a_t} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix} = \delta_t \begin{bmatrix} Q_{a_t}U_t & a_t \\ [U_t^* a_t]^* & 1 \end{bmatrix} \qquad (t \in \mathbb{R}_+).$$

The value of δ_t is determined unambiguously by (3.2): $\delta_t = \big[1 + \langle U_t e | a_t \rangle\big]^{-1}$. Thus to complete the proof, it suffices to see the continuity of the functions $t \mapsto a_t$, $t \mapsto [U_t x, Q_{a_t} x]$ $(x \in \mathbf{H})$. It is an immediate consequence of [2,App. A6] that the product $t \mapsto A_t B_t$ is strongly continuous for any couple of uniformly bounded strongly continuous operator valued functions $t \mapsto A_t \in \mathcal{L}(\mathbf{X}_1, \mathbf{X}_2)$, $t \mapsto B_t \in \mathcal{L}(\mathbf{X}_2, \mathbf{X}_3$ in case of normed spaces $\mathbf{X}_k$. By assumption, the orbit $t \mapsto a_t = \Psi^t(0)$ is a norm-continuous map $\mathbb{R}_+ \to \mathbf{B}$ entailing the norm continuity of the function $t \mapsto Q_{a_t}$. We deduce the strong continuity of the $\mathbf{H}$-isometry valued function $t \mapsto U_t$ as follows. Consider any vector $x \in \mathbf{H}$. We may assume $x \in \mathbf{B}$ without loss of generality. Then, by the aid of the Möbius shifts (3.3) we can write

$$U_t x = \big[\Theta_{a_t}^{-1} \circ \Psi_t\big](x) = \Theta_{-a_t}\big(\Psi(x)\big) \qquad (t \in \mathbb{R}_+)$$

whence the continuity of $t \mapsto U_t x = (1 - \langle x | a_t \rangle)^{-1}[Q_{a_t} x - a_t]$ is immediate.

3.4. Standard notations, assumptions. Henceforth, for the proofs for Section 2, we assume without loss of generality the following facts.

(i) $\mathbf{\Psi} := [\Psi^t : t \in \mathbb{R}_+]$ is an arbitrarily given C_0S of holomorphic Carathéodory isometries of $\mathbf{B}$ having no common fixed point within $\mathbf{B}$.

(ii) $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ $(t \in \mathbb{R}_+)$ with a suitable Möbius transformation Θ;

(iii) the orbit $t \mapsto \Phi^t(0)$ is differentiable and $\Phi^t = \mathfrak{F}^{\#}\mathcal{U}^t\big|\mathbf{B}$ with some C_0S $[U^t : t\in\mathbb{R}_+]$ of linear $\mathbf{H}$-isometries,

$$\mathcal{A} := \mathrm{gen}[\mathcal{U}^t : t\in\mathbb{R}_+] = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix}, \qquad b\in\mathbf{H},\ iA = \mathrm{gen}[U^t : t \in \mathbb{R}_+];$$

(iv) $e \in \partial\mathbf{B}$ is a joint boundary fixed point of the maps $\overline{\Phi^t}$, we write

$$\mathbf{H}_0 := \mathbf{H} \ominus \mathbb{C}e, \quad P := \mathrm{P}_{\mathbb{C}e}, \quad P_0 := \mathrm{P}_{\mathbf{H}_0} = 1-P, \quad T : x \mapsto x+e, \quad \mathcal{T} := \begin{bmatrix} \mathrm{id}_{\mathbf{H}} & e \\ 0 & 1 \end{bmatrix}.$$

Proposition 3.5. *We have $e\in\mathrm{dom}(A)$ with $\mathcal{A}(e\oplus 1) = \nu(e\oplus 1)$ and $b = (\nu - iA)e$ for some $\nu\in\mathbb{C}$. The possibly unbounded operator $A_0 := P_0A\big|\mathbf{H}_0 \cap \mathrm{dom}(A)$ is $\mathbf{H}_0$-hermitian and, in terms of $(\mathbb{C}e \oplus \mathbf{H}_0 \oplus \mathbb{C})$-matrices, we have*

$$\mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{bmatrix} -\overline{\nu} & 0 & 0 \\ -b_0 & iA_0 & 0 \\ \nu & b_0^* & \nu \end{bmatrix} \quad \text{where} \quad b_0 := P_0 b, \quad \nu = \langle e|\, b\rangle. \tag{3.6}$$

Proof. By assumption 3.2(ii), $e \oplus 1$ is a joint eigenvector of the linear operators $\mathcal{U}^t$. Hence $\mathcal{U}^t(e\oplus 1) = \zeta_t(e\oplus 1)$ $(t\in\mathbb{R}_+)$ with a continuous solution $[t \mapsto \zeta_t]$ of the Cauchy equation $\zeta_{s+t} = \zeta_s\zeta_t$. Thus for some $\nu \in \mathbb{C}$, $\zeta_t = e^{\nu t}$ and we have

$$e\oplus 1 \in \mathrm{dom}(\mathcal{A}) = \{\mathfrak{z} : t \mapsto \mathcal{U}^t\mathfrak{z} \text{ is differentiable}\}, \quad \mathcal{A}(e\oplus 1) = \nu(e\oplus 1).$$

As a consequence, $e \in \mathrm{dom}(A) = \mathrm{P}_{\mathbf{H}}\mathrm{dom}(\mathcal{A})$ and the operator

$$\widetilde{A}_0 := A - PA - AP + PAP = (1-P)A(1-P) = P_0AP_0$$

is a bounded perturbation ranging in $\mathbf{H}_0$ of $A \in \mathrm{Her}_s(\mathbf{H})$ with a self-adjoint operator of finite rank. Hence its restriction A_0 to $\mathbf{H}_0$ is a well-defined unbounded $\mathbf{H}_0$-hermitian operator. Since A is a $(\mathbb{C}e \oplus \mathbf{H}_0)$-matrix operator, we can write

$$\mathcal{A} = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} i\alpha & ia_0^* & \beta \\ ia_0 & iA_0 & b_0 \\ \overline{\beta} & b_0^* & 0 \end{bmatrix}, \quad b_0 := P_0 b,\ \beta := \langle b|e\rangle,\ a_0 := P_0Ae,\ \alpha v := \langle Ae|e\rangle$$

in terms of $(\mathbb{C}e \oplus \mathbf{H}_0 \oplus \mathbb{C})$-matrices. The eigenvector equation $\mathcal{A}(e\oplus 1) = \nu(e\oplus 1)$ means $iAe + b = \nu e$ with $\langle e|\, b\rangle = \nu$ entailing $i\alpha + \beta = \nu$, $ia_0 + b_0 = 0$, $\overline{\beta} = \nu$. Since

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{T}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

in $(\mathbb{C}e\oplus\mathbf{H}_0\oplus\mathbb{C})$-matrix form, hence (0.0) is immediate.

Notation 3.7. Henceforth $[U_0^t : t\in\mathbb{R}_+]$ denotes the C_0S of $\mathbf{H}_0$-isometries generated by the operator $iA_0 := P_0A\big|\mathbf{H}_0 \cap \mathrm{dom}(A)$.

Lemma 3.8. *Let $\mathbf{E}_1, \mathbf{E}_2$ be Banach spaces, $\mathcal{G} := \left[\begin{smallmatrix} G_1 & 0 \\ H & G_2 \end{smallmatrix}\right]$ with $H \in \mathcal{L}(\mathbf{E}_1, \mathbf{E}_2)$ and $G_k = \mathrm{gen}[W_k^t : t \in \mathbb{R}_+]$ for some C_0S $[W_k^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{E}_k)$. Then the family*

$$\mathcal{S}^t := \begin{bmatrix} W_1^t & 0 \\ \int_0^t W_2^{t-s} H W_1^s ds & W_2^t \end{bmatrix} \qquad (t \in \mathbb{R}_+)$$

is a C_0S in $\mathcal{L}(\mathbf{E}_1 \oplus \mathbf{E}_2)$ such that $\mathrm{gen}[\mathcal{S}^t : t \in \mathbb{R}_+] = \mathcal{G}$.

Proof. The family $\mathbf{W} := [\mathcal{W}^t : t \in \mathbb{R}_+]$ where $\mathcal{W}^t := W_1^t \oplus W_2^t$ is a C_0S in $\mathbf{E}_1 \oplus \mathbf{E}_2$ and $\mathcal{G}$ is a bounded perturbation of $\mathrm{gen}(\mathbf{W}) = G_1 \oplus G_2 (\equiv \left[\begin{smallmatrix} G_1 & 0 \\ 0 & G_2 \end{smallmatrix}\right])$ with $\mathrm{dom}(\mathcal{G}) = \mathrm{dom}(G_1) \oplus \mathrm{dom}(G_2)$ by the operator $\mathcal{H} := \left[\begin{smallmatrix} 0 & 0 \\ H & 0 \end{smallmatrix}\right]$. According to [2, Thm.III.1.10], for every fixed $\mathfrak{z} = x \oplus y \in \mathrm{dom}(\mathcal{G})$ we have

$$\mathcal{S}^t \mathfrak{z} = \sum_{n=0}^{\infty} S_n(t) \quad \text{where} \quad S_0(t) := \mathcal{W}^t \mathfrak{z}, \quad S_{n+1}(t) = \int_0^t \mathcal{W}^{t-s} \mathcal{H} S_n^{(k)}(s)\, ds.$$

Since $\mathcal{H}$ is an off-diagonal 2×2 triangular operator matrix, $S_n(t) = 0$ for $n > 1$.

3.9. Proof of Theorem 2.1

Since $\mathcal{T}$ a bounded invertible $\mathbf{H}\oplus\mathbb{C}$-operator and $\mathcal{A} = \mathrm{gen}[\mathcal{U}^t : t \in \mathbb{R}_+]$, we have

$$\mathcal{T}^{-1} \mathcal{A} \mathcal{T} = \mathrm{gen}\big[\mathcal{V}^t : t \in \mathbb{R}_+\big] \quad \text{for} \quad \mathcal{V}^t := \mathcal{T}^{-1} \mathcal{U}^t \mathcal{T}.$$

Since $\Phi^t = \mathfrak{F}(\mathcal{U}^t)\big|\mathbf{B}$ $(t \in \mathbb{R}_+)$, in terms of the translation $Tx := x + e$ we can interpret the C_0S $[\mathcal{V}^t : t \in \mathbb{R}^+]$ as the linear representation by means of $\mathfrak{F}$ of the semigroup $[T^{-1} \circ \Phi^t \circ T : t \in \mathbb{R}_+]$ formed by holomorphic isometries of the shifted ball $\mathbf{B} - e$ whose continuous extensions leave the origin fixed. Due to the projective identities $\mathfrak{F}\big(\mathcal{T}^{-1} \mathcal{V} \mathcal{T}\big) = T^{-1} \circ \mathfrak{F}(\mathcal{V}) \circ T$ $\big(\mathcal{V} \in \mathcal{L}(\mathbf{H} \oplus \mathbb{C})\big)$, for the points $x \in T^{-1}\mathbf{B} = \mathbf{B} - e$ we have

$$\mathfrak{F}\mathcal{V}^t(x) = \big[\mathfrak{F}(\mathcal{T}^{-1} \mathcal{U}^t \mathcal{T})\big](x) = [T^{-1} \circ \Phi \circ T](x) = \Phi(x + e) - e.$$

Therefore

$$\Phi^t(x) = \mathfrak{F}\mathcal{V}^t(x - e) + e \quad (x \in \mathbf{B}).$$

By the aid of Lemma 3.8 and (3.6) we calculate a quadrature form for $\mathcal{V}^t$ as follows. Regarding the top left 2×2-corner of the matrix $\mathcal{T}^{-1}\mathcal{A}\mathcal{T}$ we get

$$\begin{bmatrix} -\overline{\nu} & 0 \\ -b_0 & iA_0 \end{bmatrix} = \mathrm{gen}\big[V^t : t \in \mathbb{R}_+\big], \quad V^t = \begin{bmatrix} e^{-\overline{\nu} t} & 0 \\ \int_0^t U_0^{t-s} e^{-\overline{\nu} s}(-b_0) ds & U_0^t \end{bmatrix}. \tag{3.10}$$

Another application of Lemma 3.8 to $\mathcal{T}^{-1}\mathcal{A}\mathcal{T}$ yields

$$\mathcal{V}^t = \begin{bmatrix} V^t & 0 \\ \int_0^t e^{\nu(t-s)} b^* V^s ds & e^{\nu t} \end{bmatrix} \qquad (t \in \mathbb{R}_+). \tag{3.11}$$

As a consequence of (3.11), since $\mathcal{V}^t(x\oplus 1)=\big[V^t x\big]\oplus e^{\nu t}\big[\int\limits_0^t\langle e^{-\nu s}V^\tau x|\,b\rangle d\tau+1\big]$, we get

$$\Phi^t(x)=\frac{e^{-\nu t}V^t(x-e)}{\big\langle\int_0^t e^{-\nu s}V^s(x-e)\,ds\big|\,b\big\rangle+1}+e \qquad (x\in\mathbf{B},\ t\in\mathbb{R}_+). \tag{3.12}$$

We substitute (3.10) into (3.12) in terms of the new parametrization

$$\lambda=\operatorname{Re}\nu,\quad \mu:=\operatorname{Im}\nu,\quad V_0^t:=e^{-i\mu t}U_0^t.$$

Given any vector $z=z_0+\zeta e, z_0\in\mathbf{H}_0$, and recalling the commutativity of convolutions,

$$e^{-\nu t}V^t z=\zeta e^{-2\lambda t}\Big[e-\int\limits_0^t e^{\lambda s}V_0^s b_0\,ds\Big]+e^{-\lambda t}V_0^t z_0,\qquad \int\limits_0^t\big\langle e^{-\nu s}V^s z\big|b\big\rangle\,ds=$$

$$=\zeta(\lambda+i\mu)\frac{1-e^{-2\lambda t}}{2\lambda}-\zeta\int\limits_0^t e^{-2\lambda s}\int\limits_0^s e^{\lambda r}\big\langle V_0^r b_0\big|b_0\big\rangle\,dr+\int\limits_0^t e^{-\lambda s}\big\langle V_0^s z_0\big|b_0\big\rangle\,ds\,ds.$$

The statement of Theorem 2.1 is immediate from (3.9) with $z\coloneqq x-e=x_0+(\xi-1)e$.

Remark 3.12. It is discovered from the above proof that any tuple

$$\mathfrak{a}:=\big(\mathbf{H},e,[V_0^t:t\in\mathbb{R}_+],b_0,\lambda,\mu\big)$$

with a Hilbert space $\mathbf{H}$, a unit vector $e\in\mathbf{H}$, a C_0S $[V_0^t:t\in\mathbb{R}_+]$ of $\mathbf{H}_0(:=\mathbf{H}\ominus\mathbb{C}e)$-isometries, a vector $b_0\in\mathbf{H}_0$ and two real constants gives rise to a C_0S $[\Phi_{\mathfrak{a}}^t:t\in\mathbb{R}_+]$ of holomorphic Carathéodory isometries of the open unit ball $\mathbf{B}$ of $\mathbf{H}\equiv\mathbb{C}e\oplus\mathbf{H}_0$ whose generator $\Gamma(x)=\frac{d}{dt}\big|_{t=0+}\Phi_{\mathfrak{a}}^t(x)=\frac{d}{dt}\big|_{t=0+}\mathfrak{F}(\mathcal{T}\mathcal{V}^t\mathcal{T}^{-1})x$ has the form (1.2) with

$$b=\begin{bmatrix}\lambda-i\mu\\ b_0\end{bmatrix},\quad A=\begin{bmatrix}2\mu & -ib_0^*\\ ib_0 & A_0\end{bmatrix},\quad iA_0=\operatorname{gen}[V_0^t:t\in\mathbb{R}_+]. \tag{3.13}$$

In particular we can extend $[\Phi_{\mathfrak{a}}^t:t\in\mathbb{R}_+]$ to a C_0G $[\Phi_{\mathfrak{a}}^t:t\in\mathbb{R}]\subset\operatorname{Iso}(d_{\mathbf{B}})$ if and only if $[V_0^t:t\in\mathbb{R}_+]$ consists of $\mathbf{H}_0$-unitary operators (cf. [9, Thm.II]). Furthermore, given any tuple $\mathfrak{b}:=(\mathbf{H},A,e,\lambda)$ with a densely defined maximal symmetric linear $\mathbf{H}$-operator A, there is a unique C_0S $[\Psi_b^t:t\in\mathbb{R}_+]\subset\operatorname{Iso}(d_{\mathbf{B}})$ whose infinitesimal generator is of the form (1.2) with $b:=(\nu-iA)e$ where $\nu:=\lambda+i\mu$ and $\mu=\langle Ae|\,e\rangle$.

4. Invariant discs

Lemma 4.1. *The C_0S $[\Phi^t:t\in\mathbb{R}_+]\subset\operatorname{Iso}(d_{\mathbf{B}})$ with generator (1.2) and joint boundary fixed point $e\in\partial\mathbf{B}$ admits no invariant disc of the form $\mathbf{B}\cap(e+\mathbb{C}e)\neq\emptyset$ if and only if the operator $iA+\langle e|b\rangle$ is not injective or $e\in\operatorname{range}(iA-\langle e|b\rangle)$.*

Proof. Consider any vector $v\in\mathbf{H}$ such that $e+v\in\mathbf{B}$. The disc $\Delta_e^v:=\mathbf{B}\cap(e+\mathbb{C}v)$ is $[\Phi^t:t\in\mathbb{R}_+]$-invariant if and only if the vector field (1.2) is tangent to it that is if $b-\langle e+\tau v|b\rangle(e+\tau v)+iA(e+\tau v)\in\mathbb{C}v$ whenever $e+\tau v\in\mathbf{B}$. This happens if and only if $-\langle v|b\rangle e+iAv=\zeta v$ for some $\zeta\in\mathbb{C}$ because we have $e\in\operatorname{dom}(\Gamma)=\operatorname{dom}(A)$ and $\Gamma(e)=b-\langle e|b\rangle+iAe=0$ (due to the fact that the point e is $[\Phi^t:t\in\mathbb{R}_+]$-invariant). According to Proposition 3.5, here we have $b=(\nu-iA)e$ where $\nu=\langle e|b\rangle$.

Therefore $\zeta v = -\langle v|(\nu - iA)e\rangle e + iAe = \langle(-\overline{\nu} - iA)v|e\rangle e + iAe$. Notice that, in general, $rmP_{\mathbb{C}e}x = \langle x|e\rangle e = x - \mathrm{P}_{\mathbf{H}_0}x$ $(x \in \mathbf{H})$. Thus the disc Δ_e^v is $[\Phi^t : t \in \mathbb{R}_+]$-invariant if and only if $-\overline{\nu}\mathrm{P}_{\mathbb{C}e}v + \mathrm{P}_{\mathbf{H}_0}(iAv) - \zeta v = 0$ i.e. $\mathrm{P}_{\mathbf{H}_0}(iAv - \zeta v) = 0$ and $\mathrm{P}_{\mathbb{C}e}(-\overline{\nu} - \zeta)e = 0$ for some $\zeta \in \mathbb{C}$. By assumption $\Delta_e^v \neq \emptyset$ which is possible if and only if $\mathrm{P}_{\mathbb{C}e}v \neq 0$ implying $\zeta = -\overline{\nu}$. Hence we conclude that the $[\Phi^t : t \in \mathbb{R}_+]$-invariance of Δ_e^v is equivalent to the relation $\mathrm{P}_{\mathbf{H}_0}(iAv + \overline{\nu})v = 0$ i.e. to $(iA + \overline{\nu})v \in \mathbb{C}e$ which completes the proof.

Example 4.2. The C_0S of the type $[\Psi^t_{\mathfrak{b}} : t \in \mathbb{R}_+]$ in Remark 3.12 with $\mathbf{H} := L^2(\mathbb{R})$, $Af := [x \mapsto xf(x)]$ $\big(\mathrm{dom}(A) := \{f : \int_{-\infty}^{\infty} |xf(x)|^2\, dx < \infty\}\big)$, $e := (2\pi)^{-1/2}\exp\big(-(x-1)^2/2\big)$ and $\lambda := 0$ admits no invariant 1-dimensional disc. Proof: We have $\langle Ae|\, e\rangle = (2\pi)^{-1}\int_{-\infty}^{\infty} x\exp(-(x-1)^2)\, dx = (2\pi)^{1/2} \neq 0$. Thus, according to the construction of the C_0S $\mathbf{\Psi}_{\mathfrak{b}}$, $\nu = \langle e|\, b\rangle = i\mu = i\langle Ae|\, e\rangle/2 \in i\mathbb{R}\setminus\{0\}$. The relation $(iA + \overline{\nu})v = \zeta e$ would imply $v = -i\zeta\exp(-(x-1)^2/2)/(x-\mu) \in L^2(\mathbb{R})$ which is possible only if $v = 0$.

4.3. Proof of Theorem 2.6

Recall [3] that the 1-dimensional complex affine discs the form $\Delta_{p,q} := \big(p + \mathbb{C}(q-p)\big) \cap \mathbf{B}$ $(q \neq p, q \in \partial\mathbf{B})$ are the ranges of complex geodesics for the Carathéodory distance $d_{\mathbf{B}}$, and $d_{\mathbf{B}}$-isometries preserve their family. In particular, in the case when $p \neq q \in \partial\mathbf{B}$ are joint fixed points of the continuous extensions $\overline{\Psi^t}$ the disc $\Delta_{p,q}$ is automatically $[\Psi^t : t \in \mathbb{R}_+]$-invariant. Suppose $\Psi^t(\Delta_{p,q}) = \Delta_{p,q}$ $(t \in \mathbb{R}_+)$. Then the restricted maps $\psi^t_{p,q} := \Psi^t\big|D_{p,q}$ form a C_0S of holomorphic automorphisms of a 1-dimensional Hilbert ball, thus their continuous extensions $\overline{\psi^t_{p,q}}$ to $\overline{D_{p,q}}$ admit at least one fixed point which is necessarily a joint fixed point for the maps $\overline{\Psi^t}$. A 1-dimensional application of Theorem 2.1 shows that all the orbits $t \mapsto \psi^t_{p,q}(x) = \Psi^t(x)$ $(x \in \Delta_{p,q})$ are automatically real analytic. Hence, given any Möbius transformation Θ, the C_0S $[\Phi^t : t \in \mathbb{R}_+]$ with $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ leaves the $d_{\mathbf{B}}$-geodesic $D_{\Theta(p),\Theta(q)}$ invariant with differentiable 0-orbit $t \mapsto \Phi^t(0)$. Also conversely: if $[\Phi^t : t \in \mathbb{R}_+]$ is a C_0S leaving the disc $\Delta_{e,-e}(= \{\zeta e : |\zeta| < 1\})$ invariant and $\Phi^t(e) = e$, $\Psi^t = \Theta^{-1} \circ \Phi^t \circ \Theta$ $(t \in \mathbb{R}_+)$ as in 3, then the image $\Theta(\overline{\Delta_{e,-e}})$ is a $[\overline{\Psi^t} : t \in \mathbb{R}_+]$-invariant 1-dimensional affine section of $\overline{\mathbf{B}}$ containing a joint fixed point (the point $\Theta^{-1}(e)$) of $[\overline{\Psi^t} : t \in \mathbb{R}_+]$.

Proof of (i),(ii). It remains only to verify the possibility of the simplified representations (2.7),(2.8) by means of an appropriate choice for the coordinatizing Möbius transformation Θ in 3. By setting $x_0 := 0$ in 2.2, it is straightforward to check that a C_0S $[\Phi^t : t \in \mathbb{R}_+]$ of the form 2.2 leaves the disc $\Delta_{e,-e}$ invariant if and only if $b_0 = 0$ and $\Phi^t(\xi e) = \omega_{\lambda,\mu}(t,\xi)e$ $(|\xi| < 1)$ with the function

$$\omega_{\lambda,\mu}(t,\xi) := 1 - \frac{2\lambda(1-\xi)e^{-2\lambda t}}{2\lambda - (1-\xi)(\lambda + i\mu)(1 - e^{-\lambda t})}.$$

It is also easy to see that the constant 1 is a joint fixed point of all functions $\omega_{\lambda,\mu}(t,\cdot)$ and, for fixed $\lambda, \mu \in \mathbb{R}$, the family $\omega_{\lambda,\mu}(t,\cdot)$ $t \in \mathbb{R}_+)$ admits another fixed point namely the constant $\xi_{\lambda,\mu} := \frac{i\mu-\lambda}{i\mu+\lambda}$ with modulus 1 if and only if we have $\mu = 0$. Due to folklore 2-transitivity properties of the Möbius group (for direct proof see [7]), given any two couples $(e_1, e_2), (f_1, f_2) \in [\partial\mathbf{B}]^2$ of distinct boundary points there exists a Möbius transformation $\Theta^{(e_1,f_1,e_2,f_2)}$ with the effect $e_k \mapsto f_k$ $(k = 1, 2)$. Thus in case if $[\overline{\Psi^t} : t \in \mathbb{R}_+]$ has only a unique fixed point $p \in \partial\mathbf{B}$ but the disc $\Delta_{p,q}$ is $[\Psi^t : t \in \mathbb{R}_+]$-invariant, with any choice

$\Theta := \Theta^{(p,e,q,\kappa e)}$ where $|\kappa| = 1$ we get a formula for Φ^t by substituting $b_0 = 0$ and $\mu = 0$ in (2.2) which is (2.8).

If $[\Psi^t : t \in \mathbb{R}_+]$ admits two distinct fixed points $p, q \in \partial\mathbf{B}$ then, as we have shown, the disc $\Delta_{p,q}$ is automatically $[\Psi^t : t \in \mathbb{R}_+]$-invariant, and with the choice $\Theta := \Theta^{(p,e,q,-e)}$ we get a formula for Φ^t by substituting $b_0 = 0$ and $\mu = 0$ in (2.2) establishing (2.7).

Proof of (iii). Suppose indirectly that $0 \neq \lambda = \mathrm{Re}\langle e|b\rangle$. Then the skew symmetry of iA entails $\mathrm{range}(iA - \langle e|b\rangle) = \mathbf{H}$. By Lemma 4.1, we have a non-trivial Γ-invariant disc and we are in the settings of (i) or (iii). By assumption, (i) is not the case. However, in the case of (iii) we have $\langle e|b\rangle = i\mu \in i\mathbb{R}$ automatically.

4.4. Proof for Remark 2.5 The operator S_0 is closed with dense domain in $\mathbf{H}_0$. Since S_0 is also symmetric, both $\pm iS_0$ are dissipative (namely $\mathrm{Re}\langle \pm iSx_0|x_0\rangle = 0$ for $x_0 \in \mathrm{dom}(S)$) with the properties that both $\mathrm{range}(\pm iS_0 + \delta)$ are dense in $\mathbf{H}$ for any $\delta > 0$ and that the operators $(iS+\delta)^{-1} : \mathrm{range}(S) \to \mathbf{H}_0$ $(0 \neq \delta \in \mathbb{R})$ are all bounded and densely defined.[4] Given $\delta \in \mathbb{R}\setminus\{0\}$, by [2, II.Lemma 1.3], for any $x_0 \in \mathrm{range}(iS_0 - \delta)$ and $t > 0$ we have $\int_0^t e^{-\delta\tau} V_0^\tau x_0 \, d\tau = \int_0^t e^{-\delta\tau} V_0^\tau (iS_0 - \delta)[(iS_0 - \delta)^{-1} x_0]\, d\tau = \big(e^{-\delta t} V_0^t - 1\big)(iS_0 - \delta)^{-1} x_0$. The boundedness of both the operators V^t and the resolvent $R(\delta, iS) = \mathrm{closure}\big((\delta - iS_0)^{-1}\big)$ establishes 2.5 for $t \in \mathbb{R}_+$ and $0 \neq \lambda \in \mathbb{R}$ with integrals of strongly continuous bounded operator valued functions.

5. Dilation

Lemma 5.1. *Let $[U^t : t \in \mathbb{R}_+]$ be a C_0S of linear isometries of a Banach space $\mathbf{E}$. Suppose $\mathbf{E}$ is a subspace of another Banach space $\mathbf{F}$ and there is a surjective isometry $V \in \mathcal{L}(\mathbf{F})$ such that $U^1 = V\big|\mathbf{E}$. Then there is a subspace $\mathbf{E} \subset \widehat{\mathbf{E}} \subset \mathbf{F}$ along with a C_0G $[\widehat{U}^t : t \in \mathbb{R}]$ of surjective linear isometries of $\widehat{\mathbf{E}}$ such that $U^t = \widehat{U}^t\big|\mathbf{E}$ $(t \in \mathbb{R}_+)$ with $\mathrm{dom}\big(\mathrm{gen}[\widehat{U}^t : t \in \mathbb{R}]\big) \supset \mathrm{dom}\big(\mathrm{gen}[U^t : t \in \mathbb{R}]\big)$*

Proof. Let $\widehat{\mathbf{E}} := \mathrm{closure}\big(\mathbf{E}_\infty\big)$ in $\mathbf{F}$ where $\mathbf{E}_\infty := \bigcup_{n=0}^\infty$ with $\mathbf{E}_n := V^{-n}\mathbf{E}$. By assumption $V\mathbf{E} = U^1\mathbf{E} \subset \mathbf{E}$ whence, by induction we conclude that the subspaces $\mathbf{E}_n$ $(n \in \mathbb{Z}_+)$ form an increasing sequence. Therefore all the operators $U_n^t := V^{-2n}U^{t+n}V^n\big|\mathbf{E}_n$ $(t \geq -n,\ n \in \mathbb{Z}_+)$ are well-defined isometries $\mathbf{E}_n \to \mathbf{E}_{\lceil n-t \rceil}$. We have $U_n^t = U_{n+1}^t\big|\mathbf{E}_n$ for all indices $n \in \mathbb{Z}_+$. Indeed if $\widehat{x} \in \mathbf{E}_n$ and $t \geq -n$ then

$$\begin{aligned} U_{n+1}^t\widehat{x} &= V^{-2n-2}U^{t+n+1}V^{n+1}\widehat{x} = V^{-2n-2}U^{t+n+1}U^1V^n\widehat{x} = \\ &= V^{-2n-2}U^{t+n+2}V^n\widehat{x} = V^{-2n-2}V^2U^{t+n}V^n\widehat{x} = V^{-2n}U^{t+n}V^n\widehat{x} = U_n^t\widehat{x} \end{aligned}$$

since V extends U^1 and we have $V^{n+1}\widehat{x} \in \mathbf{E}$ implying $V^{n+1}\widehat{x} = U^1V^n\widehat{x}$. Hence

$$U_\infty^t\widehat{x} := \lim_{n\to\infty} U_n^t\widehat{x} = \big[U_n^t\widehat{x} :\ n \in \mathbb{Z}_+,\ n \geq t\big] \qquad (\widehat{x} \in \mathbf{E}_\infty)$$

[4] Indeed $y_0 \perp \mathrm{range}(\pm iS_0 + \delta)$ means $0 = \langle \pm iS_0x_0 - \delta x_0|y_0\rangle$ that is $0 = \langle x_0| \mp iS_0y_0 - \delta y_0\rangle$ for $(x_0 \in \mathrm{dom}(S))$ entailing $\mp iS_0y_0 + \delta y_0 = 0$ with $\delta\|y_0\|^2 = \pm i\langle S_0y_0|y_0\rangle \in i\mathbb{R}$ which is possible only if $y = 0$. Thus by the Lumer-Phillips theorem [2, II.Thm.3.15], also the operator $-iS_0$ generates a strongly continuous contraction (actually isometry) semigroup and all the values $0 \neq \delta \in \mathbb{R}$ belong to the resolvent set of iS.

is a well-defined linear isometry of the linear manifold $\mathbf{E}_\infty$ for any $t \in \mathbb{R}$. Since $\mathrm{range}(U_n^t) \supset V^{-2n}U^{\lceil t\rceil+n}\mathbf{E} = V^{\lceil t\rceil-n}\mathbf{E}$ for $t \geq -n$, we have $\mathrm{range}(U_\infty^t) = \mathbf{E}_\infty$ $(t \in \mathbb{R})$. Thus the operators $\widehat{U}^t := \mathrm{closure}(U_\infty^t)$ $(t \in \mathbb{R})$ are well-defined surjective linear $\widehat{\mathbf{E}}$-isometries, each of which extending the respective U^t. We check they form a C_0G as follows. Since $[\widehat{U}^t : t \in \mathbb{R}]$ is an equilipschitzian family, it suffices to see that its restriction $[\widehat{U}^t : t \in \mathbb{R}]$ to the dense submanifold $\mathbf{E}_\infty$ of $\widehat{\mathbf{E}}$ is a C_0G . Given $s,t \in \mathbb{R}$ and $\widehat{x} \in \widehat{E}_\ell$, we have $\widehat{U}^t\widehat{x} = V^{-2n}U^{t+n}V^n\widehat{x} \in \widehat{\mathbf{E}}_{2n}$ whenever $n \geq t\ell$ and $\widehat{U}^s(\widehat{U}^t\widehat{x}) = V^{-2m}U^{t+m}V^m\widehat{U}^t\widehat{x}$ whenever $m \geq \max\{-s, 2n\}$. It follows $\widehat{U}^s\widehat{U}^t\widehat{x} = \widehat{U}^{s+t}\widehat{x}$ because hence, with $k \geq 2(|s| + |t| + \ell)$ we have

$$\widehat{U}^s(\widehat{U}^t\widehat{x}) = V^{-4k}U^{s+2k}V^{2k}V^{-2k}U^{t+k}V^k\widehat{x} = \\ = V^{-4k}U^{s+t+3k}V^k\widehat{x} = V^{-4k}U^{s+t+2k}V^{2k}\widehat{x}V^k = \widehat{U}^{s+t}\widehat{x}.$$

To see strong continuity, consider any vector $\widehat{x} \in \widehat{E}_\ell$. Then for any integer $n \geq \ell$ the orbit $(-n,\infty) \ni t \mapsto \widehat{U}^t\widehat{x} = V^{-2n}U^{t+n}(V^n\widehat{x})$ is continuous since V^{-2n} is an isometry and $(V^n\widehat{x}) \in \mathbf{E}$. Hence we can see also the required generator domain inclusion property: with $\widehat{x} := x \in \mathrm{dom}\big(\mathrm{gen}[U^t : t \in \mathbb{R}_+]\big)$ we have $V^n x = U^n x \in \mathrm{dom}\big(\mathrm{gen}[U^t : t \in \mathbb{R}_+]\big)$ entailing even the differentiability of the orbits $(-n,\infty) \ni t \mapsto \widehat{U}^t x$.

In particular, since every linear isometry of a Hilbert space admits a unitary dilation [8], in our setting of interest we conclude the following.

Corollary 5.2. *If $[U^t : t \in \mathbb{R}_+]$ is a C_0S of linear $\mathbf{H}$-isometries, there exists a Hilbert space $\widehat{\mathbf{H}}$ containing $\mathbf{H}$ as a subspace along with a C_0G $[\widehat{U}^t : t \in \mathbb{R}]$ of $\widehat{\mathbf{H}}$-unitary operators such that $U^t = \widehat{U}^t\big|\mathbf{H}$ $(t \in \mathbb{R}_+)$ whose generator is an extension of* $\mathrm{gen}[U^t : t \in \mathbb{R}_+]$.

5.3. Proof of Corollaries 2.10-11

Given any Hilbert space $\widehat{\mathbf{H}}$ containing $\mathbf{H}$ as a subspace, every Möbius transformation of $\mathbf{H}$ extends to a Möbius transformation of $\widehat{\mathbf{H}}$. Hence it suffices to see only that any C_0S of the form of Theorem 2.1 admits a group a dilation of the same algebraic form in a larger Hilbert space. Let $[\Phi^t : t \in \mathbb{R}_+]$ be given as in Theorem 2.1. According to Corollary 3.12, for some tuple $\mathfrak{a} := \big(\mathbf{H}, e, [V_0^t : t\in\mathbb{R}_+], b_0, \lambda, \mu\big)$ we have $\Phi^t = \Phi_{\mathfrak{a}}^t = \mathfrak{F}(\mathcal{U}^t)$ $(t \in \mathbb{R}_+)$ with

$$\mathrm{gen}[\mathcal{U}^t : t \in \mathbb{R}_+] = \begin{bmatrix} iR & b \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} i(S_0 + \mu) & -b_0 & b_0 \\ b_0^* & 2i\mu & \lambda - i\mu \\ b_0^* & \lambda + i\mu & 0 \end{bmatrix}. \tag{5.3}$$

in terms of $\big[\mathbf{H}_0 \oplus (\mathbb{C}e) \oplus \mathbb{C}\big]$-matrices. Let $[\widehat{V}_0^t : t \in \mathbb{R}]$ be the dilation C_0G of $[V_0^t : t \in \mathbb{R}_+]$ consisting of unitary operators of a covering Hilbert space $\widehat{\mathbf{H}}_0$ of $\mathbf{H}_0$ with the skew self-adjoint extension $i\widehat{S}_0 = \mathrm{gen}[\widehat{V}_0^t : t \in \mathbb{R}]$ of iS_0 guaranteed by Corollary 5.2 Also by Remark 3.12 the tuple $\widehat{\mathfrak{a}} := \big(\widehat{\mathbf{H}}, e, [\widehat{V}_0^t : t\in\mathbb{R}_+], b_0, \lambda, \mu\big)$ where $\widehat{\mathbf{H}} := \widehat{\mathbf{H}}_0 \oplus (\mathbb{C}e)$ gives rise to a C_0G $\Phi_{\widehat{\mathfrak{a}}}^t : t \in \mathbb{R}]$ such that $\Phi_{\widehat{a}}^t = \mathfrak{F}(\widehat{\mathcal{U}}^t)$ $(t \in \mathbb{R})$ whose infinitesimal generator can be written in the form of the right hand side of (5.3) when the entry S_0 is replaced with $\widehat{S}_0$. Hence, by Theorem 2.1, the transformations Φ_a^t can be written in the form (2.2) with $\widehat{S}_0$ in place of S_0 and $\widehat{V}_0^t$ in place of V_0^t. Since $\widehat{V}_0^t\big|\mathbf{H}_0 = V_0^t$ $(t \in \mathbb{R}_+)$, it readily follows $\Phi_{\widehat{\mathfrak{a}}}^t\big|\mathbf{H} = \Phi_{\mathfrak{a}}^t$ $(t \in \mathbb{R}_+)$ which completes the proof of Corollary 2.10.

To prove Corollary 2.11, consider any C_0S $[\Phi_a^t : t \in \mathbb{R}_+]$ with its dilation group $[\widehat{\Phi}_{\mathfrak{a}}^t : t \in \mathbb{R}_+]$ as above. By construction, the dilation C_0G $[\widehat{V}_0^t : t \in \mathbb{R}]$ consists of $\widehat{\mathbf{H}}_0$-unitary operators. Thus, in view of Stone's classical theorem, we can apply the functional calculus [8] with its skew self-adjoint generator $i\widehat{S}_0$ when evaluating the transformations $\Phi_{\mathfrak{a}}^t$ by means of (3.12). Actually, for any $t \in \mathbb{R}$ we have

$$\int_0^t e^{-\lambda\tau}\widehat{V}_0^\tau d\tau = g_{1,t}(\widehat{S}_0), \quad \int_0^t e^{-2\lambda\tau}\int_0^\tau e^{\lambda\sigma}\widehat{V}_0^\sigma \, d\sigma \, d\tau = g_{2,t}(\widehat{S}_0)$$

with the functions $\mathbf{s}\mapsto\int_0^t e^{-\lambda\tau}e^{i\tau\mathbf{s}}d\tau$ resp. $\mathbf{s}\mapsto\int_0^t e^{-2\lambda\tau}\int_0^\tau e^{\lambda\sigma}e^{i\sigma\mathbf{s}}d\sigma\, d\tau$ which are real analytic $\mathbb{R} \to \mathbb{C}$. Straightforward calculation establishes their algebraic form and the Taylor series appearing in Corollary 2.11.

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References

[1] J. Deddens, On extending semigroups of contractions, Trans. Amer. Math. Soc. 145 (1969) 233–239.

[2] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Vol. 194 of Texts in Mathematics, Springer-Verlag, New York, 2000.

[3] T. Franzoni, E. Vesentini, Holomorphic Maps and Invariant Distances, Vol. 40 of North Holland Math. Studies, North Holland Publ. Co., Amsterdam – New York – Oxford, 1980.

[4] V. Katshkevich, S. Reich, S. D., One-parameter semigroups of fractional-linear transformations, Operator Theory Advances and Applications 123 (2001) 401–411.

[5] W. Kaup, Über die klassifikation der symmetrischen hermiteschen mannigfaltigkeiten unendlicher dimension i., Math. Ann. 357 (1981) 463–481.

[6] W. Kaup, A riemann mapping theorem for bounded symmetric domains in complex banach spaces, Math. Z. 83 (1983) 503–529.

[7] L. Stachó, A fixed point approach to unitary c_0-groups in hilbert space, Rev. Roum. Math. Pures Appl.Submitted 2015.

[8] B. Sz.-Nagy, C. Foias, Harmonic Analysis of Operators on Hilbert Space, Akadémiai Kiadó/North-Holland Publ. Co., Amsterdam – New York – Oxford, 1970.

[9] E. Vesentini, Semigrioups of holomorphic isometries, Advances in Mathemetics 65 (1987) 272–306.

[10] E. Vesentini, Semigrioups of holomorphic isometries, in: P. Gauthier (Ed.), Complex Potential Theory, Kluwer Academic Pulbl., Netherlands, 1994, pp. 475–548.