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A counterexample concerning C_0-semigroups of holomorphic Carathéodory isometries

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Abstract: We give an example for a C_0-semigroup of non-linear 0-preserving holomorphic Carathéodory isometries of the unit ball.

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1. Introduction

It is a well-known consequence of Cartan's classical Uniqueness Theorem [5] that given a bounded circular domain D in the N-dimensional complex space $\mathbb{C}^N$ any holomorphic mapping $F : D \to D$ with $F(0) = 0$ and preserving the Carathéodory (or Kobayashi) distance associated with D is necessarily linear and surjective. In contrast, in 1994 E. Vesentini [10](p. 508),[11](Sec. 3) found various examples, even with holomorphic families, showing that the infinite dimensional version of this fact is no longer valid in general Banach space setting. Actually Cartan's result ensured the linearity of the one-parameter semigroups of holomorphic automorphisms fixing the origin and hence gave rise to a Lie theoretic approach by means of the infinitesimal generators to the precise algebraic description of the group of holomorphic automorphisms of a finite dimensional bounded homogeneous circular domain. However, Vesentini's techniques seem unsuitable in constructing a C_0-semigroup $[F^t : t \geq 0]$ of non-linear Carathéodory isometries $F^t \in \mathrm{Hol}(\mathbf{D}, \mathbf{D})$ on a bounded circular domain $\mathbf{D}$ contained in some complex Banach space $\mathbf{E}$. Our aim in this short note is a C_0-semigroup construction (Lemma 2) done with slight modifications of familiar methods used in the theory of C_0-semigroups of linear operators [3] resp. delay equations [1] in the fading memory space $C_0(\mathbb{R}_+, \mathbf{E})$. Our examples involve bounded convex circular domains $\mathbf{D}$ but relies upon some auxiliary remarks with independent interest on holomorphic invariant distances associated to domains for the type $\mathcal{D} = \{x \in \mathcal{X} : \mathrm{range}(f) \subset \mathbf{D}\}$ in a function space $\mathcal{X} = C_0(\Omega, \mathbf{E})$ with some bounded convex domain $\mathbf{D}$ containing $0 \in \mathbf{E}$. Actually, our arguments require no deep knowledge of symmetric spaces and invariant distances.

As for the background of motivation: The approach by von Neumann to classical Quantum Mechanics proposed modeling the evolution of wave functions with one-parameter C_0-groups of unitary operators in complex Hilbert spaces. Toward the beginning of the 1970-s, exigences occure to extend the related framework beyond the setting of linear operators and regard not necessarily reversible evolution. To this aim naural candidates are one-parameter C_0-semigroups of holomorphic self-mappings preserving some automorphism invariant distance on a bounded Banach space domain. Physical symmetry properties can be played by the circularity or more generally by the holomorphic symmetry of the underlying domain. According to Kaup's celebrated Riemann Mapping Theorem [7], up to holomorphic equivalence, bounded symmetric domains are circular and convex.

At first sight Theorem 2 seems a negative result. However, the construction may reveal interesting geometric properties and links to delay equations for further investistigation.

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2. Preliminaries

To establish terminology: by a *one-parameter C_0-semigroup* on a topological space X we mean an indexed family $\big[F^t : t \in \mathbb{R}_+\big]$ of mappings $F^t : X \to X$ with the semigroup properties $F^0 = \mathrm{Id}_X = \big[X \ni x \mapsto x\big]$, $F^t \circ F^s(x) = F^t\big(F^s(x)\big) = F^{t+s}(x)$ $(s,t \in \mathbb{R}_+)$ and the continuity of all orbits $t \mapsto F^t(x)$ for any $x \in X$. Given two metric spaces (X_j, d_j) $(j = 1,2)$ a mapping $f : X_1 \to X_2$ is a $d_1 \to d_2$ *contraction* if $d_2\big(f(x), f(y)\big) \le d_1(x,y)$ $(x,y \in X_1)$.

A subset D in a complex topological vector space E is said to be *circular* if it is connected, contains the origin of E and $D = e^{it}D = \{e^{it}x : x \in D\}$ $(t \in \mathbb{R})$.

Throughout this work let $\mathbf{E}$ denote an arbitrarily fixed complex Banach space with norm $\|\cdot\|$ and open unit ball $B_1(\mathbf{E})$. As standard notation, we write $\mathbb{C}$ for the complex plane regarded as a 1-dimensional space normed with the absolute value and unit disc $\Delta = B_1(\mathbb{C}) = \{\zeta : |\zeta| < 1\}$ equipped with the *Poincaré metric* $d_\Delta(\alpha,\beta) = \tanh^{-1}\big|(\beta-\alpha)/(1-\overline{\alpha})\big|$ $\big(\alpha,\beta \in \Delta\big)$. Given any domain (connected open set) $\mathbf{D} \subset \mathbf{E}$,

$$d_{\mathbf{D}}(p,q) = \sup\Big\{d_\Delta\big(f(p), f(q)\big) : \ f \in \mathrm{Hol}(\mathbf{D},\Delta)\big)\Big\} \quad (p,q \in \mathbf{D})$$

is the associated *Carathéodory distance* where $\mathrm{Hol}(\mathbf{D}_1,\mathbf{D}_2)$ stands for the family of all *holomorphic* maps between two Banach space domains $\mathbf{D}_1 \subset \mathbf{E}_1$ resp. $\mathbf{D}_2 \subset \mathbf{E}_2$. In the cases of our interests, a function $f : \mathbf{D}_2 \to \mathbf{E}_2$ with bounded range is holomorphic if and only if for any point $p \in \mathbf{D}$ and any unit vector $v \in \mathbf{E}$, it admits a uniformly convergent directional Taylor expansion $\zeta \mapsto f(p+\zeta v) = \sum_{n=0}^{\infty} \zeta^n a_n$ $\Big(a_n \in \mathbf{E}_1,\ \sum_{n=0}^{\infty} \|a_n\|\rho^n < \infty\Big)$ whenever the closed ball $p + \rho\overline{B_1(\mathbf{E})}$ is contained in $\mathbf{D}$. A fundamental feature of Carathéodory metrics [5] is that all holomorphic maps $\mathbf{D}_2 \to \mathbf{D}_2$ are $d_{\mathbf{D}_1} \to d_{\mathbf{D}_2}$ contractions, furthermore if the domain $\mathbf{D} \subset \mathbf{E}$ is bounded then $\big(\mathbf{D}, d_{\mathbf{D}}\big)$ is a complete metric space giving rise to the same topology as the distance by the norm on $\mathbf{D}$.

For a locally compact Hausdorff space Ω, $C_0(\Omega,\mathbf{E})$ will denote the Banach space of all continuous functions $f : \Omega \to \mathbf{E}$ vanishing at infinity (i.e. $f^{-1}\{p \in \mathbf{E} : \|p\| \ge \varepsilon\}$ is a compact subset for any $\varepsilon > 0$) equipped with the norm $\|f\| = \max_{\omega\in\Omega} \|f\|$. In particular $C_0(\mathbb{R}_+,\mathbf{E})$ consists of functions with limit 0 at infinity. It is immediate that, given any domain $\mathbf{D}_0$ in some Banach space $\mathbf{E}_0$, a mapping $f : \mathbf{D}_0 \to C_0(\Omega,\mathbf{E})$ with bounded range is holomorphic if and only if all pointwise evaluations $\delta_\omega f : \mathbf{D}_0 \ni z \mapsto f(z)(\omega)$ $(\omega \in \Omega)$ are holomorphic.

Given a bounded convex domain $\mathbf{D} \subset \mathbf{E}$ with $0 \in \mathbf{D}$, we also introduce the figure $C_0(\Omega,\mathbf{D}) = \{f \in C_0(\Omega,\mathbf{E}) : \mathrm{range}(f) \subset \mathbf{D}\}$ which is easily seen a bounded convex domain in $C_0(\Omega,\mathbf{E})$. In course of the verification of Carathéodory isometry properties of holomorphic self-maps of domains $\mathcal{D}$ of the type $C_0(\Omega,\mathbf{D})$, we shall use the following plausible but highly non-trivial relation.

Lemma 1. *For the Carathéodory distance of the domain $\mathcal{D} = C_0(\Omega,\mathbf{D})$ with $0 \in \mathbf{D} \subset \mathbf{E}$ we have*

$$d_{\mathcal{D}}(x,y) = \max_{\omega\in\Omega} d_{\mathbf{D}}\big(x(\omega), y(\omega)\big) \qquad (x,y \in \mathcal{D}) \tag{1}$$

provided the underlying topological space Ω has countable base and the target space $\mathbf{E}$ is separable.

Remark 1. The special case of (1) with $\mathcal{D} = C_0(\mathbb{R}_+,\Delta)$ appears in [5] with a proof relying upon Möbius transformations. Similar arguments can be applied in the case when $\mathbf{D}$ is a (necessarily convex) *holomorphically symmetric* bounded circular domain even without countability restrictions using Kaup's JB*-triple calculus [6–8].

In its full generality, Lemma 1 can be deduced from a far-reaching theorem [2] due to Dineen-Timoney and Vigué (extending Lempert's result [9] on the coincidence of the Carathéodory- and Kobayashi pseudometrics in finite dimensions) for convex domains in separable locally convex spaces. Since we do not know a reference, we give a detailed proof in Section 4.

3. Results

Throughout this section $\mathbf{D}$ denotes an arbitrarily fixed bounded convex domain in $\mathbf{E}$ containing the origin. For short we write

$$\mathcal{X} = C_0(\mathbb{R}_+, \mathbf{E}) \quad \text{and} \quad \mathcal{D} = C_0(\mathbb{R}_+, \mathbf{D}) = \{x \in \mathcal{X} : \text{range}(x) \subset \mathbf{D}\}.$$

Lemma 2. *Let $[\varphi^t : t \in \mathbb{R}_+]$ be a C_0-semigroup of (norm)-contractions $\mathbf{D} \to \mathbf{D}$. Then the maps $\Phi^t : \mathcal{D} \to \mathcal{X}$ $(t \in \mathbb{R}_+)$ defined by*

$$\Phi^t(x) : \mathbb{R}_+ \ni \tau \mapsto \Big[\varphi^{t-\tau}\big(x(0)\big) \textit{ if } 0 \le \tau \le t, \quad x(\tau - t) \textit{ if } \tau \ge t\Big]$$

form a C_0-semigroup of isometries $\mathcal{D} \to \mathcal{D}$.

Proof. Consider any function $x \in \mathcal{D}$. Since, by definition, the function $\tau \mapsto \varphi^t\big(x(0)\big)$ is continuous and ranges in $\mathbf{D}$, we have $\Phi^t(x) \in \mathcal{D}$. Given another function $y \in \mathcal{D}$,

$$\begin{aligned} \big\|\Phi^t(x) - \Phi^t(y)\big\| &= \max\Big\{\max_{0\le\tau\le t}\big\|\varphi^{t-\tau}\big(x(0)\big) - \varphi^{t-\tau}\big(y(0)\big)\big\|, \max_{\sigma\ge t}\big\|x(\sigma-t) - y(\sigma-t)\big\|\Big\} \\ &\le \big\|x(0) - y(0)\big\|, \max_{\sigma\ge t}\big\|x(\sigma-t) - y(\sigma-t)\big\|\Big\} \\ &= \max_{\tau\ge 0}\big\|x(\tau) - y(\tau)\big)\big\| = \|x - y\|. \end{aligned}$$

Since trivially

$$\big\|\Phi^t(x) - \Phi^t(y)\big\| \ge \max_{\sigma\ge t}\big\|x(\sigma-t) - y(\sigma-t)\big\|\Big\} = \max_{\tau\ge 0}\big\|x(\tau) - y(\tau)\big\|\Big\} = \|x - y\|,$$

we conclude that each map Φ^t is a $\mathcal{D}$-isometry.

Next we check the semigroup property of $[\Phi^t : t \in \mathbb{R}_+]$. Let $s, t \ge 0$. Then we have

$$\begin{aligned} \Phi^s \circ \Phi^t(x) &: \tau \mapsto \Big[\varphi^{s-\tau}\big(\Phi^t(x)(0)\big) \text{ if } \tau \le s, \quad \varphi^t(x)(\tau - s) \text{ if } \tau \ge s\Big], \\ \Phi^{s+t}(x) &: \tau \mapsto \Big[\varphi^{(s+t)-\tau}\big(x(0)\big) \text{ if } \tau \le s+t, \quad x\big(\tau - (s+t)\big) \text{ if } \tau \ge s+t\Big]. \end{aligned}$$

Thus if $0 \le \tau \le s$ then

$$\begin{aligned} \Phi^s \circ \Phi^t(x)(\tau) &= \varphi^{s-\tau}\Big(\Phi^t(x)(0)\Big) = \varphi^{s-\tau}\Big(\varphi^t\big(x(0)\big)\Big) \\ &= \varphi^{s-\tau} \circ \varphi^t\big(x(0)\big) = \varphi^{(s+t)-\tau}\big(x(0)\big) = \Phi^{s+t}(x)(\tau). \end{aligned}$$

If $s \le \tau \le s + t$ then

$$\begin{aligned} \Phi^s \circ \Phi^t(x)(\tau) &= \Phi^t(x)(\tau - s) \overset{\tau-s\le t}{=\!=} \varphi^{t-(\tau-s)}\big(x(0)\big) \\ &= \varphi^{(s+t)-\tau}\big(x(0)\big) = \Phi^{s+t}(x)(\tau). \end{aligned}$$

If $s + t \le \tau$ then

$$\Phi^s \circ \Phi^t(x)(\tau) = \Phi^t(x)(\tau - s) \overset{\tau-s\ge t}{=\!=} x\big((\tau - s) - t\big) = \Phi^{s+t}(x)(\tau).$$

We complete the proof by checking strong continuity, that is that $\|\Phi^t(x) - \Phi^s(x)\| \to 0$ whenever $s \to t$ in $\mathbb{R}_+$. Recall that the moduli of continuity

$$M(z, \delta) := \max_{|t_1 - t_2| \le \delta}\|z(t_1) - z(t_2)\|, \qquad m(e, \delta) := \max_{|t_1 - t_2| \le \delta}\|\varphi^{t_1}(e) - \varphi^{t_2}(e)\|$$

associated to any function $z \in \mathcal{X}$ resp. any vector $e \in \mathbf{E}$ are well-defined and converge to 0 as $\delta \searrow 0$. Let $0 \le t_1 \le t_2$. Since we have

$$\Phi^{t_1}(x)(\tau) - \Phi^{t_2}(x)(\tau) = \begin{cases} \varphi^{t_1-\tau}(x(0)) - \varphi^{t_2-\tau}(x(0)) & \text{if } \tau \le t_1, \\ \varphi^{t_2-\tau}(x(0)) - x(\tau - t_1) & \text{if } t_1 \le \tau \le t_2, \\ x(\tau - t_2) - x(\tau - t_1) & \text{if } t_2 \le \tau, \end{cases}$$

it follows

$$\|\Phi^{t_1}(x) - \Phi^{t_2}(x)\| \le \begin{cases} m\big(x(0), t_2 - t_1\big) & \text{if } \tau \le t_1, \\ \|\varphi^{t_2-\tau}(x(0)) - x(0)\| + \|x(\tau - t_1) - x(0)\| \le & \\ \le m\big(x(0), t_2 - t_1\big) + M(x, t_2 - t_1) & \text{if } t_1 \le \tau \le t_2, \\ M(x, t_2 - t_1) & \text{if } t_2 \le \tau. \end{cases}$$

Hence we see the uniform continuity of the function $t \mapsto \Phi^t(x)$ with modulus of continuity $\delta \mapsto m\big(x(0), \delta\big) + M(x, \delta)$. □

Remark 2. The conclusion of Lemma 2 holds even if $\mathbf{E}$ is only assumed to be a *real* Banach space.

Proposition 1. *Under the hypothesis of Lemma 1, if the maps φ^t above are additionally holomorphic and leave the origin of $\mathbf{E}$ fixed, furthermore the underlying Banach space $\mathbf{E}$ is separable or $\mathbf{D}$ is a circular holomorphically symmetric domain then each term Φ^t is a holomorphic 0-preserving $d_{\mathcal{D}} \to d_{\mathcal{D}}$-isometry.*

Proof. Since the domain $\mathbf{D}$ is bounded, the holomorphy of the maps Φ^t with holomorhic terms φ^t is an immediate consequence of the fact that all the pointwise evaluations $\delta_\omega \Psi : \mathcal{D} \ni x \mapsto \Psi(x)(\omega)$ $(\omega \in \Omega)$ are holomorphic. Indeed we have $\delta_\tau \Phi^t = \big[x \mapsto x(\tau - t)\big]$ or $\delta_\tau \Phi^t = \big[x \mapsto \varphi^{\tau - t}\big(x(0)\big)\big]$ with holomorhic maps by assumption.

Since the maps $\varphi \in \mathrm{Hol}(\mathbf{D}, \mathbf{D})$ are $d_{\mathcal{D}} \to d_{\mathcal{D}}$ contractions, by the aid of Lemma 1 we can see that each term Φ^t is a $d_{\mathcal{D}}$-isometry as follows. Given any pair of functions $x, y \in \mathcal{D}$ we have $d_{\mathbf{D}}\big(\varphi^t\big(x(0)\big), \varphi^t\big(y(0)\big) \le d_{\mathbf{D}}\big(x(0), y(0)\big)$ $(t \ge 0)$. Hence

$$\begin{aligned} d_{\mathcal{D}}\big(\Phi^t(x), \Phi^t(y)\big) &= \max_{\tau \ge 0} d_{\mathbf{D}}\big(\delta_\tau \Phi^t(x)(\tau), \delta_\tau \Phi^t(y)(\tau)\big) = \\ &= \max\Big\{d_{\mathbf{D}}\big(\varphi^{[t-\tau]_+}\big(x(0)\big), \varphi^{[t-\tau]_+}\big(y(0)\big)\big), d_{\mathbf{D}}\big(x([\tau - t]_+), y([\tau - t]_+)\big) : t \ge 0\Big\} = \\ &= d_{\mathbf{D}}\big(x(\tau - t), y(\tau - t)\big), \max\Big\{d_{\mathbf{D}}\big(x(0), y(0)\big), d_{\mathbf{D}}\big(x(\tau), y(\tau)\big) : \tau \ge 0\Big\} = \\ &= \max_{\tau \ge 0} d_{\mathbf{D}}\big(x(\tau), y(\tau)\big) = d_{\mathcal{D}}(x, y) \end{aligned}$$

which completes the proof. □

Remark 3. It is well-known from [4] that, given a continuously differentiable function $f \in \mathcal{X}$, we have

$$\frac{d^+}{dt}\big\|f(t)\big\| := \limsup_{h \searrow 0} \big[\|f(t+h)\| - \|f(t)\|\big]/h = \sup_{L \in \mathcal{S}(f(t))} \mathrm{Re}\big\langle L, f'(t)\big\rangle$$

in terms of the family of supporting bounded linear functionals

$$\mathcal{S}(y) := \big\{L \in \mathbf{E}^* : \|L\| = 1,\ \langle L, y\rangle = \|y\|\big\} \qquad (y \in \mathbf{E}).$$

In particular f is non-increasing whenever $\mathrm{Re}\big\langle L, f'(t)\big\rangle \le 0$ for any $t \in \mathbb{R}_+$ and for any functional $L \in \mathcal{S}(f(t))$.

Lemma 3. *Let $V : U \to \mathbf{E}$ be a Lipschitzian continuously differentiable map (regarded as a vector field) on some open neighborhood U of the closed unit ball $\overline{B_1(\mathbf{E})}$ with $V(0) = 0$ and let $\mu \geq \mathrm{Lip}\big(V|\overline{B(\mathbf{E})}\big) = \sup_{f_1,f_2 \in \overline{B(\mathbf{E})}} \|f_1 - f_2\|^{-1}\|V(f_1) - V(f_2)\|$. Then the maximal flow of the vector field $W : B_1(\mathbf{E}) \ni e \mapsto V(e) - \mu e$ is a well-defined uniformly continuous one-parameter semigroup $[\varphi^t : t \in \mathbb{R}_+]$ consisting of contractive (non-expansive) self maps of $B_1(\mathbf{E})$.*

Proof. By definition, any flow of W is a family $[\varphi^t : t \in I]$ of self maps $\varphi^t : B_1(\mathbf{E}) \to B_1(\mathbf{E})$ where I is some (relatively) open subinterval of $\mathbb{R}_+$ and, for any point $e \in B_1(\mathbf{E})$, the function $I \ni t \mapsto \varphi^t(e)$ is the solution of the initial value problem

$$\frac{d}{dt}z(t) = W\big(z(t)\big), \quad z(0) = e. \tag{2}$$

By writing I_e for the maximal solution of (2), it is well-known that $\sup I_e > 0$ in any case, furthermore we have $\lim_{t\to\sup I_e} \|z(t)\| = 1$ whenever $\sup I_e < \infty$.

Let $e_1, e_2 \in B_1(\mathbf{E})$ and consider the function $f(t) := \varphi^t(e_1) - \varphi^t(e_2)$ defined on the interval $I_{e_1} \cap I_{e_2}$. Observe that, given any functional $L \in \mathcal{S}\big(\varphi^t(e_1) - \varphi^t(e_2)\big)$, we have

$$\begin{aligned}
&\mathrm{Re}\big\langle L, f'(t)\big\rangle = \mathrm{Re}\big\langle L, W\big(\varphi^t(e_1)\big) - W\big(\varphi^t(e_2)\big)\big\rangle = \\
&= \mathrm{Re}\big\langle L, V\big(\varphi^t(e_1)\big) - V\big(\varphi^t(e_2)\big)\big\rangle - \mu\mathrm{Re}\big\langle L, \varphi^t(e_1) - \varphi^t(e_2)\big\rangle = \\
&= \mathrm{Re}\big\langle L, V\big(\varphi^t(e_1)\big) - V\big(\varphi^t(e_2)\big)\big\rangle - \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| \leq \\
&\leq \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| - \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| = 0.
\end{aligned}$$

Hence we conclude that the function $t \mapsto f(t)$ is decreasing, in particular we have the contraction property $\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| \leq \big\|\varphi^0(e_1) - \varphi^0(e_2)\big\| = \big\|e_1 - e_2\big\|$ for $t \in I_{e_1} \cap I_{e_2}$. By assumption $W(0) = V(0) = 0$ implying $\varphi^t(0) \equiv 0$ with $I_0 = [0, \infty) = \mathbb{R}_+$. Hence we see also that $\big\|\varphi^t(e)\big\| = \big\|\varphi^t(e) - \varphi^t(0)\big\| \leq \big\|e - 0\big\| = \big\|e\big\| < 1$ for all $e \in B_1(\mathbf{E})$ and $t \in I_e$. This is possible only if $\sup I_e = \infty$. Therefore the maximal flow of W is defined for all (time) parameters $t \in \mathbb{R}_+$ and consists of $B_1(\mathbf{E})$-contractions φ^t.

It is well-known that flows parametrized on $\mathbb{R}_+$ are strongly continuous semigroups automatically. The uniform continuity of in our case is a consequence of the fact that $\big\|\varphi^{t_2}(e) - \varphi^{t_1}(e)\big\| \leq \int_{t_1}^{t_2} \big\|\frac{d}{dt}\varphi^t(e)\big\|dt = \int_{t_1}^{t_2} \big\|W\big(\varphi^t(e)\big)\big\|dt \leq \int_{t_1}^{t_2} 4\mu\, dt \quad (0 \leq t_1 \leq t_2)$, which shows that $\omega(e, \delta) \leq 4\mu\delta \quad (e \in B_1(\mathbf{E}),\ \delta \in \mathbb{R}_+)$. □

Example 1. Let $\mathbf{E} := \mathbb{C}$ with $B_1(\mathbf{E}) = \Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and let $V(z) \equiv z^2$. Since $|z_1^2 - z_2^2| = |z_1 - z_2| \cdot |z_1 + z_2| \leq 2|z_1 - z_2|$, we can apply Lemma 3 with $W(z) := z^2 - 2z$. For the flow $[\varphi^t : t \in \mathbb{R}_+]$ of W we obtain the holomorphic maps

$$\varphi^t(z) = \frac{2z}{(1 - e^{2t})z + 2e^{2t}} \qquad (z \in \Delta,\ t \geq 0).$$

Indeed, the solution of the initial value problem

$$\frac{d}{dt}x(t) = x(t)^2 - 2x(t), \quad x(0) = z \tag{3}$$

is $x(t) = 2z/\big[(1 - e^{2t})z + 2e^{2t}\big]$ as one can check by direct computation. As for heuristics, we get a real valued solution with real calculus for (3) with initial values $-1 < z < 1$, and the obtained formula extends holomorphically to Δ.

Theorem 2. *Given a complex Banach space $\mathbf{E}$ with symmetric or separable unit ball, there is a C_0-semigroup of non-linear holomorphic 0-preserving norm and Carathéodory isometries of the open unit ball of the function space $\mathcal{X} := C_0(\mathbb{R}_+, \mathbf{E})$.*

Proof. We can apply the construction of Proposition 1 with a semigroup $[\varphi^t : t \in \mathbb{R}_+]$ obtained with the construction of Lemma 3 with any **E**-polynomial polynomial vector field $V : \mathbf{E} \to \mathbf{E}$. □

Example 2. Let $\mathbf{E} := \mathbb{C}$ and $\mathbf{X} := C_0(\mathbb{R}_+, \mathbb{C})$. Then the maps

$$\Phi^t(x) : \mathbb{R}_+ \ni \tau \mapsto \left[\frac{2x(0)}{(1-e^{2(t-\tau)})x(0) + 2e^{2(t-\tau)}} \text{ if } \tau \leq t, \;\; x(\tau - t) \text{ if } \tau \geq t \right]$$

form a C_0-semigroup of non-linear holomorphic 0-preserving norm and Carathéodory isometries of the unit ball $B_1(\mathbf{X})$.

Question 1. Is any holomorphic norm-isometry of the unit ball of a complex Banach space automatically a Carathéodory isometry as well?

4. Appendix: proof of Lemma 1

Notice that our assumptions imply the separability of the space $\mathcal{X} = C_0(\Omega, \mathbf{E})$. Thus we can apply the main result in [2] to $\mathcal{D}$ with the conclusion that

$$\begin{aligned} d_{\mathcal{D}}(x,y) &= \max\left\{ d_\Delta\big(f(x), f(y)\big) : f \in \mathrm{Hol}(\mathcal{D}, \Delta) \right\} = \\ &= \inf\left\{ d_\Delta(\zeta,\eta) : \exists f \in \mathrm{Hol}(\Delta,\mathcal{D}) \text{ with } f(\zeta) = x, f(\eta) = y \right\} = \\ &= \inf\left\{ \tanh^{-1}(\eta) : \eta > 0 \text{ and } \exists f \in \mathrm{Hol}(\Delta,\mathcal{D}) \text{ with } f(0) = x, f(\eta) = y \right\} \end{aligned}$$

for any pair $x, y \in \mathcal{D}$. In the case of the space $\mathcal{X}$ consisting of functions $\Omega \to \mathbf{E}$, the evaluations $\delta_\omega : x \mapsto (\omega)$ are linear mappings with $\delta_\omega(\mathcal{D}) \subset \mathbf{D}$. Since all holomorphic functions $\mathcal{D} \to \mathbf{D}$ are $d_{\mathcal{D}} \to d_{\mathbf{D}}$ contractions, hence we conclude that

$$d_{\mathcal{D}}(x,y) \geq \sup_{\omega \in \Omega} d_{\mathbf{D}}\big(x(\omega), y(\omega)\big) \qquad (x, y \in \mathcal{D}).$$

It is well-known [5] that the Carathéodory pseudodistance is a continuous metric on any bounded Banach space domain, being locally equivalent to the natural distance defined by the underlying norm. Therefore we can replace the term *sup* with *max* in the above formula and to complete the proof it suffices to see that the following approximate version of the inf-expression of $d_{\mathcal{D}}(x,y)$.

Let $\varepsilon > 0$ *and* $\eta > \tanh\big(d_{\mathcal{D}}(x,y)\big)$. *Then given any pair of functions* $x, y \in \mathcal{D}$, *there exists a mapping* $\Delta \ni \zeta \mapsto z_\zeta \in \mathcal{E}$ *such that for any location* $\omega \in \Omega$, *we have*

$$\|z_0(\omega) - x(\omega)\|, \|z_\eta(\omega) - y(\omega)\| < \varepsilon, \quad \big[\zeta \mapsto z_\zeta(\omega)\big] \in \mathrm{Hol}(\Delta, \mathbf{D}).$$

Construction of a suitable function $\zeta \mapsto z_\zeta$: Let $\Omega^* = \Omega \cup \{\infty\}$ be the one point compactification of Ω. For each location $\omega \in \Omega^*$, we can find a neighborhood $\Gamma_\omega \subset \Omega^*$ such that

$$d_{\mathbf{D}}\big(x(\gamma), x(\omega)\big), d_{\mathbf{D}}\big(y(\gamma), y\|(\omega)\big), \|x(\gamma) - x(\omega)\|, \|y(\gamma) - y(\omega)\| < \epsilon \quad (\gamma \in \Gamma_\omega).$$

Due to the compactness of Ω^*, there exists a finite partition of unity subordinated to the covering $\{\Gamma_\omega : \omega \in \Omega^*\}$. That is we can choose a finite subset $\{\omega_n\}_{n=0}^N \subset \Omega^*$ along with a family $\{w_n\}_{n=0}^N$ of continuous functions $\Omega^* \to \mathbb{R}_+$ such that

$$\sum_{n=0}^N w_n(\omega) = 1 \quad (\omega \in \Omega^*), \qquad \mathrm{supp}(w_n) \subset \Gamma_{\omega_n}.$$

Consider the points $p_n = x(\omega_n)$, $q_n = y(\omega_n)$. Notice that

$$d_{\mathbf{D}}(p_n, q_n) \leq \max_{\omega \in \Omega} d_{\mathbf{D}}(p_n, q_n) = d_{\mathcal{D}}(x,y) < \eta \quad (n = 0, \ldots, N).$$

Since $d_{\mathbf{D}}(p,q) = \inf\big\{d_\Delta(0,\eta' : \eta, \in (0,1),\ \ \exists f \in \mathrm{Hol}(\Delta,\mathbf{D}) f(0) = p,\ f(\eta') = q\big\}$, we can find functions $f_0,\ldots,f_N$ such that

$$f_n \in \mathrm{Hol}(\Delta,\mathbf{D}), \quad f_n(0) = p_n,\ \ f_n\big(\tanh(\eta+\varepsilon)\big) = q_n.$$

In terms of $f_0,\ldots,f_N$ we can finish the construction by setting

$$z_\zeta(\omega) := \sum_{n=0}^{N} w_n(\omega) f_n(\zeta) \qquad \big(\zeta \in \Delta,\ \omega \in \Omega\big).$$

For any fixed location $\omega \in \Omega$, the function $\zeta \mapsto z_\zeta(\omega)$ is holomorphic as being a linear combination of the holomorphic functions f_n. For any fixed scalar $\zeta \in \Delta$, the function $\omega \mapsto z_\zeta(\omega)$ belongs to $\mathcal{D}$ as being a convex combination of the continuous functions $\big[\Omega \ni \omega \mapsto f_n(\omega)\big)$ vanishing at ∞. Finally, since $f_n(0) = p_n = x(\omega_n)$ and $f_n(\eta+\varepsilon) = q_n = y(\omega_n)$, for any location $\omega \in \Omega$ we have the following estimates:

$$\begin{aligned} \big\|z_0(\omega) - x(\omega)\big\| &= \Big\|\sum_n w_n(\omega)\big[f_n(0) - x(\omega)\big]\Big\| = \Big\|\sum_n w_n(\omega)\big[x(\omega_n) - x(\omega)\big]\Big\| \le \\ &\le \sum_{n:w_n(\omega)>0} w_n(\omega)\big\|x(\omega_n) - x(\omega)\big\| < \sum_n w_n(\varepsilon) = \varepsilon\,; \\ \big\|z_{\eta+\varepsilon}(\omega) - y(\omega)\big\| &= \Big\|\sum_n w_n(\omega)\big[f_n(\zeta+\varepsilon) - y(\omega)\big]\Big\| = \Big\|\sum_n w_n(\omega)\big[y(\omega_n) - y(\omega)\big]\Big\| \le \\ &\le \sum_{n:w_n(\omega)>0} w_n(\omega)\big\|y(\omega_n) - y(\omega)\big\| < \sum_n w_n(\varepsilon) = \varepsilon. \end{aligned}$$

which completes the proof.

References

1. Bátkai, András; Piazzera, Susanna; *Semigroups for Delay Equations*, Research Notes in Mathematics 10; A K Peters, Wellesley, Massachusetts (2005).
2. Dineen, Seán; Timoney, Richard M.; Vigué, Jean-Pierre. Pseudodistances sur les domaines d'un espace localement convexe, *Annali della Scuola Normale Superiore di Pisa* **1985**, *12/4*, 515–529.
3. Engel, Klaus-Jochen; Nagel, Rainer: *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics. 194; Springer, Berlin, 2000.
4. Federer, Herbert; *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften Band 153, Springer-Verlag New York, Inc., New York, 1969.
5. Franzoni, Tullio; Vesentini, Edoardo; *Holomorphic Maps and Invariant Distances*, North Holland Math. Studies Vol. 40; Elsevier North Holland, New York, 1980.
6. Isidro, José-Maria; *Jordan Triple Systems in Complex and Functional Analysis*, AMS Mat. Surveys and Monographs 243, Providence RI, 2019.
7. Kaup, Wilhelm. A Riemann Mapping Theorem for Bounded Symmetric Domains in Complex Banach Spaces, *Mathematische Zeitschrift* **1983**, *183*, 503–530.
8. Kaup, Wilhelm; Sauter, Joachim. Boundary structure of bounded symmetric domains, *Manuscripta Math.* **2000**, *101*, 351–360.
9. Lempert, László; La métric de Kobayashi et la représentation des domaines sur la boule, *Bull. Soc. Math. France* **1981**, *109*, 427–474.
10. Vesentini, Edoardo; Semigroups of holomorphic isometries, In *Complex potential theory*; Eds. Gauthier, Paul M.; Sabidussi, G. Eds.; NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. Vol. 439: Kluwer Acad. Publ., Dordrecht, 1994, 475–548.
11. Vesentini, Edoardo; On the Banach-Stone Theorem, *Advances in Mathematics* **1985**, *112*, 135–146.