On the manifold of tripotents in JB*-triples

László L. Stachó *

October 29, 2007

Abstract. Tripotents are natural generalizations of partial isometries in C*-algebras to the context of JB*-triples that is complex Banach spaces with symmetric unit ball. We give a survey on the main results papers [2,7,8,6] concerning the structure of the tripotents as a direct real-analytic submanifold in a JB*-triple. We also discuss some recent achievements.

Keywords: JB*-triple, tripotent, connection, Grassman manifold.
AMS 2000 Subject Classification: 17C27, 17C36, 17B60, 17B65, 17B66.

1. Preliminaries: symmetry, JB*-triples, tripotents

Topological algebraic structures concerning spatial symmetry have their obvious importance in mathematical physics and they have independent mathematical interest as well. The underlying space in this paper will be a so-called *JB*-triple*, a complex Banach space Z whose unit ball $B(Z) := \{x \in Z : \ \|x\| < 1\}$ is symmetric in the sense of holomorphy that is for every point $x \in B(Z)$ there is a biholomorphism $S_z : B(Z) \leftrightarrow B(Z)$ such that $S_z^2 = S_z \circ S_z = \mathrm{Id}_{B(Z)}$, $S_z(z) = z$ and $S_z'(z) = -\mathrm{Id}_Z$ for the Fréchet derivative of S_z. As result of a long development started with the Harish-Chandra realization of finite dimensional symmetric domains, in 1983 W. Kaup [9] established the following algebraic characterization. *The Banach spaces with symmetric unit ball are exactly those admitting a Jordan-Banach *-triple product* (JB*-triple product for short, hence the name JB*-triple). By a JB*-triple product we mean an operation $\{.,.,.\} : Z \times Z \times Z \to Z$ with three variables satisfying the axioms

(J1) $\quad \{x, y, z\}$ is symmetric bilinear in x, z and conjugate-linear in y,

(J2) $\quad \|\{x, x, x\}\| = \|x\|^3$,

and with the linear operators $D(a) : z \mapsto \{a, a, z\}$ we have

(J3) $\quad D(a)\{x, y, z\} = \{D(a)x, y, z\} - \{x, D(a)y, z\} + \{x, y, D(a)z\}$,

(J4) $\quad \big\| \exp\big(\zeta D(a)\big) \big\| \le 1 \quad$ whenever $\quad \mathrm{Re}\,\zeta \le 0$.

---

*Supported by the National Research Found OTKA Grant No. T 48753.
Address: SZTE Bolyai Institute, Aradi Vértanúk tere 1, 6725 Szeged, Hungary.
E-mail: stacho@math.u-szeged,hu.

As a typical example, each C*-algebra with its natural norm is a JB*-triple with the triple product $\{x,y,z\} := [xy^*z+zy^*x]/2$. It is remarkable that the JB*-triple product is unambiguously determined by the norm of the underlying space, furthermore any bounded symmetric domain is biholomorphically equivalent to the unit ball of some JB*-triple.

Henceforth Z will denote an arbitrarily fixed JB*-triple with norm $\|.\|$ and JB*-triple product $\{.,.,.\}$, respectively. We shall write

$$\mathrm{Der}(Z) := \big\{\delta\in\mathcal{L}(Z):\ \delta\{x,y,z\}=\{\delta x,y,z\}+\{x,\delta y,z\}+\{x,y,\delta z\}\big\}$$

for the set of all *derivations* of the triple product and

$$\mathrm{Her}(Z) := \big\{\alpha\in\mathcal{L}(Z):\ \|\exp(it\alpha)\| = 1 \quad (t\in\mathbb{R})\big\}$$

will stand for the set of all *hermitian* operators of the norm $\|.\|$. Axiom (J3) can be interpreted as the fact that that all the operators $iD(a)$ belong to $\mathrm{Der}(Z)$. In view of Sinclair's theorem on the norm of hermitian operators (for an elementary proof see [6, p. 245]), axiom (J4) is an equivalent formulation of the fact that the operators $D(a)$ are hermitian with non-negative spectra.[1]

The link between complex geometry and Jordan structure in Z is established by the fact that the family $\mathrm{aut}\,B(Z)$ of all *complete holomorphic vector fields* of the unit ball is spanned by derivations and polynomials of second degree of the triple product. In this paper, by a vector field on a domain $C\subset Z$ we simply mean a holomorphic mapping $C\to Z$ and, by definition, the vector field V is complete in C if its flow is defined on the whole phase set $D\times\mathbb{R}$. In particular $V\in\mathrm{aut}\,B(Z)$ if there is a necessarily real-analytic mapping $F_V : B(Z)\times\mathbb{R}\to B(Z)$ such that $F_V(p,0)=p$ and $\frac{d}{dt}F_V(p,t) = V\big(F_V(p,t)\big)$ for all $p\in B(Z)$ and $t\in\mathbb{R}$. Namely, in terms of the conjugate linear *quadratic representation* operators $Q(a): z\mapsto\{a,z,a\}$ we can write

$$\mathrm{aut}\,B(Z) = \big\{[z\mapsto a-Q(z)a+\delta z]:\ a\in Z,\ \delta\in\mathrm{Der}(Z)\big\}.$$

The main objectives of our work is the family

$$\mathrm{Tri}(Z) := \big\{e\in Z:\ \{e,e,e\}=e\neq 0\big\}$$

of the *tripotents* that is the idempotent elements of the triple product in Z. In case of Z being a C*-algebra $\mathrm{Tri}(Z)=\{e:\ ee^*e=e\neq 0\}$ is

[1] Proof. Let $\alpha\in\mathrm{Her}_+(Z) := \{\beta\in\mathrm{Her}(Z):\ \mathrm{Sp}(\beta)\geq 0\}$ and assume $\xi,\eta\in\mathbb{R}$ with $\xi\leq 0$. Then $\big\|\exp\big(\xi+i\eta)\alpha\big)\big\| = \big\|\exp(\xi\alpha)\big\|$ because the operator $\exp(i\eta\alpha)$ is unitary with respect to the norm $\|.\|$. Define $\mu_1 := \max\mathrm{Sp}(\alpha)$ and $\mu_2 := \min\mathrm{Sp}(\alpha)$ and consider the operator $\beta := \alpha - 2^{-1}(\mu_1+\mu_2)\mathrm{Id}$. We have $\beta\in\mathrm{Her}(Z)$ and $\alpha = \beta+2^{-1}(\mu_1+\mu_2)\mathrm{Id}$. By Sinclair's theorem, $\|\beta\| = \max\big\{|\max\mathrm{Sp}(\beta)|,|\min\mathrm{Sp}(\beta)|\big\} = 2^{-1}(\mu_1-\mu_2)$. Therefore $\exp\big(\xi+i\eta)\alpha\big)\big\| = \big\|\exp(\xi\alpha)\big\| = e^{2^{-1}\xi(\mu_1+\mu_2)}\big\|\exp(\xi\beta)\big\| \leq e^{2^{-1}\xi(\mu_1+\mu_2)}e^{|\xi|\|\beta\|} = e^{2^{-1}\xi(\mu_1+\mu_2)}e^{-2^{-1}\xi(\mu_1-\mu_2)} = e^{\xi\mu_2}\leq 1$. The converse is an easy consequence of the spectral mapping theorem. Let $\|\exp(\zeta\alpha)\|\leq 1$ for $\mathrm{Re}\,\zeta\leq 0$. Then, given any $\lambda\in\mathrm{Sp}(\alpha)$, we have $|e^{\zeta\lambda}|\leq\|\exp(\zeta\alpha)\|\leq 1$ that is $\mathrm{Re}\,\zeta\lambda\leq 0$ whenever $\mathrm{Re}\,\zeta\leq 0$ which is possible only if $\lambda\geq 0$.

the set of all partial isometries. It is a well-known consequence of axioms (J1),(J3) that the operators $D(e)$ and $Q(e)$ are semisimple and commute if $e \in \mathrm{Tri}(Z)$. Namely we have $D(e)\big(D(e) - 2^{-1}\mathrm{Id}\big)\big(D(e) - \mathrm{Id}\big) = 0$ and $Q(e)^3 = Q(e)$ and hence the *Peirce decomposition*[2]

$$Z = Z_0(e) \oplus Z_{1/2}(e) \oplus Z_1(e), \quad Z^0(e) = \oplus_{\lambda=0,1/2} Z_\lambda(e), \; Z_1(e) = \oplus_{\varepsilon=\pm 1} Z^\varepsilon(e)$$

with the eigenspaces

$$Z_\lambda(e) := \{z \in Z : \; D(e)z = \lambda z\}, \qquad Z^\varepsilon(e) := \{z \in Z : \; Q(e)z = \varepsilon z\}.$$

It is also a well-known consequence of axioms (J1),(J3) that $Q(e)$ acts as an involutive automorphism of the triple product on $Z_1(e)$: $Q(e)^2 x = x$ and $Q(e)\{x,y,z\} = \{Q(e)x, Q(e)y, Q(e)z\}$ for $x,y,z \in Z_1(e)$. This fact along with $iD(e) \in \mathrm{Der}(Z)$ entails the so-called *Peirce arithmetics*

$$\{Z_\xi(e), Z_\alpha(e), Z_\eta(e)\} \subset Z_{\xi-\alpha+\eta}(e), \quad \{Z^\varepsilon(e), Z^\varphi(e), Z^\psi(e)\} \subset Z^{\varepsilon\varphi\psi}(e),$$
$$\{Z_0(e), Z_1(e), Z\} = \{Z_1(e), Z_0(e), Z\} = \{0\}.$$

As a typical example, if Z is the C*-algebra of all complex $(m+n)$-square matrices then $e = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Tri}(Z)$ with the $m \times m$ identity matrix I and, in terms of the (m,n) matrix decomposition we have $Z^1(e) = \big\{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)\!: \; a$ real$\big\}$, $Z^{-1}(e) = iZ^1(e)$, $Z_{1/2}(e) = \big\{\left(\begin{smallmatrix} 0 & x \\ y & 0 \end{smallmatrix}\right)\!: \; x,y$ arbitrary$\big\}$, $Z_0(e) = \big\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right)\!: \; b$ arbitrary$\big\}$.

By the C*-axiom (J2), tripotents have norm one. In finite dimensions their geometric importance as distinguished boundary objects relies upon the fact [11, 12] that the *holomorphic boundary components* (faces in holomorphic sense)[3] of the unit ball have the form $e + B(Z_0(e))$, $e \in \mathrm{Tri}(Z)$ and the boundary $\partial B(Z)$ is their disjoint union. In infinite dimensions there may be no tripotents at all as e.g. in the case of the commutative C*-algebra $Z := \mathcal{C}_0(0,1)$ of the continuous functions $f\colon (-1,1) \to \mathbb{C}$ with $\lim_{|\omega|\to 1} f(\omega) = 0$. However, using the canonical embedding of Z into its bidual Z^{**}, we can regard $B(Z)$ as a weak*-dense norm-closed subset of $B(Z^{**})$. Actually Z^{**} is always a JB*-triple whose triple product admits plenty of tripotents and extends the triple product from Z in a

[2] We include a short simultaneous proof which cannot be found in the literature. Let $e \in \mathrm{Tri}(Z)$ and $\delta := D(e)$, $\mu := Q(e)$. Using only axioms (J1) and (J3), we have $\{x,e,e\} = \{x,e,\{e,e,e\}\} = 2\{\{x,e,e\},e,e\} - 2\{e,\{e,x,e\},e\}$. This means the relation $\delta = 2\delta^2 - \mu^2$ or which is the same (1) $\mu^2 = 2\delta(\delta - 2^{-1}\mathrm{Id})$. Similarly, from the three term expansion of $\{e,e,\{x,e,x\}\}$ we get (2) $\delta\mu = 2\mu - \mu\delta$. Expanding $\{e,x,e\} = \{e,x,\{e,e,e\}\}$ we also get (3) $\mu = 2\delta\mu - \mu\delta$. Equations (2),(3) imply immediately that (4) $\mu = \delta\mu = \mu\delta$. Hence $\mu^2(\delta - \mathrm{Id}) = 0$. In view of (1) this entails the first Peirce equation $2\delta(\delta - 2^{-1}\mathrm{Id})(\delta - \mathrm{Id}) = 0$. The second Peirce equation has the form $\mu^3 - \mu = 0$. This is immediate from (1) and (4). Indeed, $\mu^3 - \mu = \mu(\mu^2 - \mathrm{Id}) = \mu\delta(2\delta^2 - \delta - \mathrm{Id}) = \mu(2\delta^3 - \delta^2 - \delta) = 2\mu - \mu - \mu = 0$.

[3] The holomorphic boundary component of a point $p \in \partial B(Z) = \{z \in\!: \; \|z\| = 1\}$ is the union of all finite sequences $F_0, \ldots, F_n$ by holomorphic images of the unit disc $\mathbb{D} := \{\zeta \in \mathbb{C}\colon |\zeta| < 1\}$ such that $F_0, \ldots, F_n \subset \partial B(Z)$, $p \in F_0$ and $F_{j-1} \cap F_j \neq \emptyset$.

separately weak*-continuous manner [1]. Though the sets $e+B(Z_0^{**}(e))$, $e\in \mathrm{Tri}(Z^{**})$ do not cover $\partial B(Z)$ in general, we have [3]

$$\{\text{norm-exposed faces of } \overline{B}(Z^{**})\} = \{e+\overline{B}(Z^{**}):\ e\in \mathrm{Tri}(Z^{**})\}$$

where $\overline{B}$ denotes closed unit ball. For more on JB*-triples see [16,12,13].

2. $\mathrm{Tri}(Z)$ as a submanifold of Z

Recall that the *tangent cone* of a subset S in a real Banach space $(X,\|.\|)$ at the point $p\in S$ is the set $\mathrm{T}_p(S)$ of all vectors $v\in X$ such that $v=\lim_n \xi_n(p_n-p)$ for some sequences $p_1,p_2,\ldots\in S$ and $\xi_1,\xi_2,\ldots\in \mathbb{R}_+$. Notice that $\mathrm{T}_p(S)$ is always a closed cone in X. By definition, S is a direct analytic submanifold in X if for every point $p\in S$ there is a bianalytic mapping $\Phi_p : U_p\to V_p$ between some neighborhoods of the origin and the point p, respectively, along with a direct sum decomposition $X = X_p^{(0)}\oplus X_p^{(1)}$ into closed subspaces such that

$$V_p\cap S = \Phi_p\big(X^{(0)}\cap U_p\big).$$

It is a direct consequence of the inverse mapping theorem that S is a analytic submanifold in X if and only if, for any point $p\in S$, $\mathrm{T}_p(S)$ is a closed complemented subspace of X and there is an analytic mapping Ψ_p from some neighborhood U_p of the origin in X into X such that

$$\Psi_p'(p)=\mathrm{Id} \ \text{ and } \ \Psi_p(x)\in S \text{ if and only if } x\in \mathrm{T}_p(S)\cap U_p.$$

Actually, in the latter case one can find a family $\{U_p :\ p\in S\}$ of 0-neighborhoods in X such that the restricted mappings $\Psi_p|U_p$, $p\in S$ form an analytic atlas of S.

Given any tripotent $e\in \mathrm{Tri}(Z)$, in the sequel we shall write $P_\lambda(e)$ for the *Peirce projection* onto $Z_\lambda(e)$ along the complementary sum $\oplus_{\mu\neq\lambda}Z_\mu(e)$. We also introduce the spaces $Z_1^\sigma(e):=Z_1(e)\cap Z^\sigma(e)$ and

$$Z^{(-)}(e):=Z_{1/2}(e)\oplus Z_1^{-1}(e),\quad Z^{(+)}(e):=Z_0(e)\oplus Z_1^1(e)$$

and write $P_1^\sigma(e):=2^{-1}P_1(e)[\mathrm{Id}+\sigma Q(e)]$ respectively $P^{(\pm)}(e):=P_1^{\pm1}(e)+P_{1/2}(e)$ for the corresponding projections. Furthermore we shall keep fixed the notation $K(e,.)$ for the operator

$$K(e,z):=D\Big([2^{-1}P_1^1(e)+2P_{1/2}(e)]z,e\Big)-D\Big(e,[2^{-1}P_1^1(e)+2P_{1/2}(e)]z\Big).$$

Notice that $K(e,z)e=P^{(-)}(e)z$ for all $z\in Z$. Moreover, as being in the form $D(e,w)-D(w,e)$, we have $K(e,z)\in \mathrm{Der}(Z)$, $z\in Z$. Hence $\exp K(e,w)\in \mathrm{Aut}(Z)$ with the family of all (linear) automorphisms of the triple product $\{.,.,.\}$ (which coincides with the set of all surjective isometries $Z \to Z$). In particular $\exp K(e,w)\mathrm{Tri}(Z) = \mathrm{Tri}(Z)$ and $Z_\lambda\big(\exp K(e,w)e\big) = \big[\exp K(e,w)\big] Z_\lambda(e)$, $\lambda = 0, 1/2, 1$. Regarding the complex JB*-triple Z as a real Banach space and taking into account that $e + a + x$ cannot be a tripotent if $e \in \mathrm{Tri}(Z)$, $a \in Z_1^1(e)$, $x \in Z_0(e)$ and $\|a\|, \|x\| < 2^{-1}$we have the following observation.

2.1. Proposition [14]**.** $\mathrm{Tri}(Z)$ *is a real-analytic direct submanifold of* Z*. For any tripotent* e *we have* $\mathrm{T}_e(Z) = Z_{1/2} \oplus Z_1^{-1}(e) = Z_{1/2} \oplus iZ_1^1(e)$*. In terms of the operators* K*, the mappings* $\Psi_e : Z \to Z$*,* $e \in \mathrm{Tri}(Z)$ *are well-defined by*

$$\Psi_e(x + v + a + ib) := \big[\exp K(e, v + ib)\big](e + x + a),$$
$$x \in Z_0(e),\ v \in Z_{1/2}(e),\ a, b \in Z_1^1(e).$$

They are are real-analytic with the properties $\Psi_e' = \mathrm{Id}$ *(Fréchet derivative in real sense) and, for* $\|x\|, \|a\| < 2^{-1}$*, we have* $\Psi_e(x + v + a + ib) \in \mathrm{Tri}(Z) \Leftrightarrow x = a = 0$*.*

A fundamental consequence of this fact is the possibility that we can establish a canonical one-to-one correspondence $\mathcal{E}_e$ between the smooth curves in $Z_{1/2}(e) \oplus Z_1^{-1}(e)$ and those in $\mathrm{Tri}(Z)$ with starting point e as follows. Recall ([7] or [5]) that $\mathrm{Aut}(Z)$ is an algebraic Banach-Lie subgroup of $\mathcal{L}(Z)$ with $\mathrm{T}_{\mathrm{Id}}\big(\mathrm{Aut}(Z)\big) = \mathrm{Der}(Z)$. Hence each smooth function $F : \mathbb{R} \to \mathrm{Der}(Z)$ admits a (unique) *left multiplicative primite function* ${}^{\mathbf{L}}F : \mathbb{R} \to \mathrm{Aut}(Z)$ such that

$$\frac{d}{dt}{}^{\mathbf{L}}F(t) = \big[{}^{\mathbf{L}}F(t)\big]F(t), \quad {}^{\mathbf{L}}F(0) = \mathrm{Id}.$$

2.2. Theorem. *Given a smooth curve* $\gamma : \mathbb{R} \to Z^{(-)}$*, the curve*

$$\mathcal{E}_e(\gamma) := {}^{\mathbf{L}}K\big(e, \gamma(.)\big)e$$

ranges smoothly in $\mathrm{Tri}(Z)$*. Conversely, given any smooth curve* $\varepsilon : \mathbb{R} \to \mathrm{Tri}(Z)$*, there is a unique* $\gamma \in \mathcal{C}^\infty\big(\mathbb{R}, Z_{1/2}(e) \oplus Z_1^{-1}(e)\big)$ *with* $\mathcal{E}_{\varepsilon(0)}(\gamma) = \varepsilon$*.*

Proof. Since $t \mapsto K\big(e, \gamma(t)\big)$ ranges smoothly in $\mathrm{Der}(Z)$, its left multiplicative primitive function is well-defined and ranges smoothly in $\mathrm{Aut}(Z)$. Hence indeed $\mathcal{E}_e(\gamma)e \in \mathcal{C}^\infty\big(\mathbb{R}, \mathrm{Tri}(Z)\big)$. To prove the converse, we have to see that, given a smooth curve $\varepsilon : \mathbb{R} \to \mathrm{Tri}(Z)$ with starting point $e = \varepsilon(0)$, there is a unique smooth curve $g : \mathbb{R} \to \mathrm{Aut}(Z)$ such that $g(t)e = \varepsilon(t)$ and $\frac{d}{dt}g(t) = g(t)K\big(e, v(t)\big)$ for some smooth curve $v : \mathbb{R} \to Z^{(-)}(e)$. According to Proposition 2.1, the maps $Z^{(-)}(f) \ni w \to \exp K(f, e)f$, $f \in \mathrm{Tri}(Z)$ are real analytic local charts of $\mathrm{Tri}(Z)$. Hence it readily follows that $\varepsilon(t) = h(t)e$, $t \in \mathbb{R}$ with some smooth curve $h : \mathbb{R} \to \mathrm{Aut}(Z)$. Fixing such a curve h (and regarding g in the form $g = hk$), it suffices to see that there is a unique smooth curve $k : \mathbb{R} \to \mathrm{Aut}(Z)$ such that $k(t)e = e$ and $\frac{d}{dt}h(t)k(t) = h(t)k(t)K\big(e, w(t)\big)$, $t \in \mathbb{R}$ for some smooth curve $w : \mathbb{R} \to Z^{(-)}(e)$. By abbreviating $\frac{d}{dt}$ with $'$ as usually, this means the condition

$$k'(t) = k(t)K\big(e, w(t)\big) - \ell(t)k(t) \quad \text{where} \quad \ell(t) := h(t)^{-1}h'(t)$$

on $k(.)$ with suitale $w : \mathbb{R} \to Z^{(-)}(e)$. The requirement $k(t)e = e$ implies

$$\begin{aligned} 0 &= k(t)^{-1}(e)k'(t)e = K\big(e, w(t)\big)e - k(t)^{-1}\ell(z)k(t)e = \\ &= P^{(-)}(e)w(t) - k(t)^{-1}\ell(z)e = w(t) - k(t)^{-1}\ell(z)e. \end{aligned}$$

Thus necessarily $w(t) = k(t)^{-1}\ell(z)e = k(t)^{-1}\ell(z)k(t)e \in Z^{(-)}(e)$, $t \in \mathbb{R}$ if a required curve $k(.)$ exists. Since h ranges in $\mathrm{Aut}(Z)$, $\ell = h^{-1}h'$ ranges necessarily in the tangent of $\mathrm{Aut}(Z)$ that is $\ell(t) \in \mathrm{Der}(Z)$, $t \in \mathbb{R}$. As a consequence, also $\widetilde{k}^{-1}\ell(t)\widetilde{k} \in \mathrm{Der}(Z)$ and $\widetilde{k}^{-1}\ell(t)\widetilde{k}e \in \mathrm{T}_e\mathrm{Tri}(Z) = Z^{(-)}(e)$ whenever $\widetilde{k} \in \mathrm{Aut}(Z)$. Therefore the initial value problem $k'(t) = k(t)K\big(e, k(t)^{-1}\ell(z)e\big) - \ell(t)k(t)$, $k(0) = \mathrm{Id}$ is wellposed in $\mathcal{L}(Z)$, with a unique solution ranging in the isotropy subgroup of the point e in $\mathrm{Aut}(Z)$. Its boundedness ensures that its (maximal) domain is the whole $\mathbb{R}$. $\square$

The model of curves in $\mathrm{Tri}(Z)$ in the real vector space of curves in $Z^{(-)}(e)$ described by Theorem 2.2 is a powerful tool in the study of the natural differential geometry of $\mathrm{Tri}(Z)$. In 2000 Chu and Isidro [2] have found an interesting generalization of the classical Riemannian connection on surfaces to $\mathrm{Tri}(Z)$ by replacing the orthogonal projections to the tangent planes with the Pierce projections $P^{(-)}$. That is given two vector fields X, Y on $\mathrm{Tri}(Z)$ (functions $\mathrm{Tri}(Z) \to Z$ such that $X(e), Y(e) \in \mathrm{T}_e\mathrm{Tri}(Z) = Z^{(-)}(e)$, $e \in \mathrm{Tri}(Z)$) we define

$$\nabla_X Y := P^{(-)}Y'X$$

i.e. $\nabla_X Y(e) = P^{(-)}(e)Y'(e)X(e) = P^{(-)}(e)\frac{d}{dt}\big|_{t=0}Y\big(\exp K\big(e, tX(e)\big)\big)$, $e \in \mathrm{Tri}(Z)$. We shall refer to ∇ as the *algebraic connection* of $\mathrm{Tri}(Z)$. In [2] one have established partial results on the algebraic form of the geodesics of finite rank tripotents in some JB*-triples. In 2005 in [7, Lemma 1] we achieved the solution of the geodesic equation

$$P^{(-)}(\varepsilon(t))\varepsilon''(t) = 0$$

for ∇ with curves in the form $\varepsilon = \mathcal{E}_e(\omega)$ with the following arguments. Let $g(t) = {}^{\mathbf{L}}K\big(e, \omega(t)\big)$ and $\varepsilon(t) = g(t)e$. Then $\varepsilon' = gK(e,\omega)e = g\omega$ and $g'' = \varepsilon'\omega + g\omega' = g[K(e,\omega)\omega + \omega']$. Since $g(t) \in \mathrm{Aut}(Z)$ for any t, $P^{(-)}(\varepsilon(t)) = g(t)P^{(-)}(e)g^{(-)}(t)$ and hence

$$P^{(-)}(\varepsilon(t))\varepsilon''(t) = g(t)P^{(-)}(e)\big[K\big(e, \omega(t)\big)\omega(t) + \omega'(t)\big].$$

Given any vector $w = w_{1/2} + w_1 \in Z_{1/2}(e) \oplus Z_1^{-1}(e) = Z^{(-)}(e)$, from the Pierce rules it follows that $P_1^{-1}(e)K(e,w)w = 0$ and $P_{1/2}(e)K(e,w)w = 2^{-1}\{w_1, e, w_{1/2}\} + 2\{w_{1/2}, e, w_1\} - 2^{-1}\{e, w_1, w_{1/2}\} = 3\{w_1, e, w_{1/2}\}$. Thus for the components $\omega_\lambda(t) := P_\lambda(e)\omega(t)$, $\lambda = 2^{-1}, 1$ we get the linear differential equations $\omega_1\prime = 0$ and $\omega'_{1/2} = 3D(\omega_1, e)\omega_{1/2}$. Hence Theorem 2.2 yields the following result.

2.3. Theorem. *A curve ε in* $\mathrm{Tri}(Z)$ *is a ∇-geodesic if and only if*

$$\varepsilon(t) = {}^{\mathbf{L}}K\Big(e, w_1 + \exp[3t\,D(w_1, e)]w_{1/2}\Big)e$$

for some $e \in \mathrm{Tri}(Z)$, $w_1 \in Z_1^{-1}(e)$ and $w_{1/2} \in Z_{1/2}(e)$.

As an immediate consequence, we get the following minor correction to [2, Thm. 2.7]: for fixed $e \in \mathrm{Tri}(Z)$ and $w = w_{1/2} + w_1 \in Z_{1/2}(e) \oplus Z_1^{-1}(e)$, the curve $\varepsilon(t) := \exp K(e, tw)e$ is a ∇-geodesic if and only if $\{w_1, e, w_{1/2}\} = \{e, w_1, w_{1/2}\} = 0$. For a nontrivial example let $E := \left[\begin{smallmatrix}10\\01\end{smallmatrix}\right]$, $R := \left[\begin{smallmatrix}i0\\00\end{smallmatrix}\right]$, $A := \left[\begin{smallmatrix}00\\11\end{smallmatrix}\right]$ $B := \left[\begin{smallmatrix}01\\01\end{smallmatrix}\right]$ and let Z be the C*-algebra of all 4×4 matrices. Then with $e := \left[\begin{smallmatrix}E0\\00\end{smallmatrix}\right]$, $w_1 := \left[\begin{smallmatrix}R0\\00\end{smallmatrix}\right]$, $w_{1/2} := \left[\begin{smallmatrix}0A\\B0\end{smallmatrix}\right]$ we have $e \in \mathrm{Tri}(Z)$, $w_1 \in Z_1^{-1}(e)$, $w_{1/2} \in Z_{1/2}(e)$ and $\{w_1, e, w_{1/2}\} = \{w_{1/2}, e, w_1\} = 0$.

Another issue for an effective application of Theorem 2.2 can be the investigation of minimal and stationary curves with respect to the distance in $\mathrm{Tri}(Z)$ inherited from Z. By definition, a smooth curve $\varepsilon : [0,1] \to \mathrm{Tri}(Z)$ is a *minimal curve* if the length of any (smooth) curve in $\mathrm{Tri}(Z)$ joining the endpoints $\varepsilon(0)$ and $\varepsilon(1)$ is not less then that of the curve ε. We say that ε is a *stationary curve* if $\frac{d}{d\tau}\big|_{s=0}\mathrm{Length}(\varepsilon_\tau) = 0$ whenever $(\tau, t) \mapsto \varepsilon_\tau(t)$ is a smooth mapping $[0,1]^2 \to \mathrm{Tri}(Z)$ such that $\varepsilon_0(t) = \varepsilon(t)$ and $\varepsilon_\tau(a) = \varepsilon(a)$ for $\tau, t \in [0,1]$ and $a = 0, 1$. In contrast with the close relationship in classical surface geometry in Euclidean spaces between the stationary curves and the Riemannian connection, in our setting the situation seems to be more involved. For instance, in the case of the commutative C*-algebra $Z := \mathcal{C}[0,1]$, any curve $\varepsilon^\alpha(t) := \big[s \mapsto e^{i\alpha(s,t)}\big]$ is minimal joining the constant functions ε^0 and ε^1 whenever α is a smooth function $[0,1]^2 \to [0,1]$ such that each subfunction $\alpha(.,t)$ is maps increasingly the interval $[0,1]$ onto itself. Disregarding the few cases where $\mathrm{Tri}(Z)$ happens to be a Riemannian manifold, there seem to be no results in the literature on metric minimal and stationary curves of tripotents in general complex JB*-triples. Recently [15] we achieved the following reformulation of the length variational equation for tripotents by the aid of the techniqe with multiplicative primitive functions.

2.4. Proposition. *Let $\varepsilon : [0,1] \to \mathrm{Tri}(Z)$ be a smooth curve in the form $\varepsilon(t) = {}^{\mathbf{L}}K\big(e, \omega(t)\big)e$ where $e \in \mathrm{Tri}(Z)$ and $\omega : [0,1] \to Z^{(-)}(e)$*

is a smooth curve. Then $\mathrm{Length}(\varepsilon) = \int_0^1 \|\omega(t)\|\, dt$*. If the curve* ε *is stationary then we have*

$$\int_0^1 \Big[\xi(t)\delta\Big(\omega(t), K\big(e, \omega(t)\big)u\Big) + \xi'(t)\delta\Big(\omega(t), u\Big)\Big]\, dt = 0$$

for any vector $u \in Z^{(-)}(e)$ *and for any smooth function* $\xi : [0,1] \to \mathbb{R}$ *with* $xi(0) = \xi(1) = 0$ *where* $\delta(z,v) := \lim_{s\downarrow 0} s^{-1}\|z + sv\|$, $z, v \in Z$ *denotes the subgradient of the norm in* Z*.*

An immediate difficulty in the progress along these line is the fact that the bad smoothness properties of the norm in most JB*-triples do not allow to carry out a routine partial integration in the latter formula. Hence the following problem is still open. *In which JB*-triples are all* ∇*-geodesics curves stationary?*

3. The Grassmanian structure of the equivalence classes of tripotents

Since the tangent space $\mathrm{T}_e\mathrm{Tri}(Z) = Z_{1/2}(e) \oplus Z_1^{-1}(e)$ is no complex subspace in Z (in particular $ie \in Z_1^{-1}(e) = iZ_1^1(e)$ and $Z_1^{-1}(e) \cap Z_1^1(e) = \{0\}$), $\mathrm{Tri}(Z)$ is no complex submanifold of Z. Observe that if we "go in the wrong directions" in $\mathrm{Tri}(Z)$ in the sense that we consider curves in the form $\varepsilon(t) := g(t)$ with $g(t) := {}^{\mathbf{L}}K(e, \omega(t))$ and $\omega(t) \in Z_1^{-1}(e)$ then the operators $D\big(\varepsilon(t)\big)$ determining the Peirce subspaces do not change. Indeed, it is well-known that $D(e,w) = \sigma D(w,e)$ whenever $w \in Z_1^\sigma(e)$ whence $\frac{d}{dt}D(\varepsilon) = D(\varepsilon', \varepsilon\big) + D(\varepsilon, \varepsilon') = D(g\omega, ge) + D(ge, gw) = g[D(\omega, e) + D(e,\omega)]g^{-1} = 0$. The *equivalence of tripotents*

$$e \sim f \overset{\text{def}}{\iff} D(e) = D(f)$$

was introduced and studied already in 1985 by E. Neher [13]. Originally he formulated this relationship as $\{e,e,f\} = f$ and $\{f,f,e\} = e$ and called it "association" but established its equivalence with $D(e) = D(f)$ immediately. Since any automorphism of the triple product maps an equivalence class of $\sim$ onto another equivalence class and since the maps $\Psi_e : Z_{1/2} \oplus Z_1^{-1}(e) \ni w \mapsto \exp K(e,w)e$, $e \in \mathrm{Tri}(Z)$ are local charts on $\mathrm{Tri}(Z)$, it can be expected that the *quotient manifold*

$$\mathbb{M} := \mathrm{Tri}(Z)/\!\sim\, := \big\{e^\sim : e \in \mathrm{Tri}(Z)\big\} \quad \text{where} \quad e^\sim := \{f \in \mathrm{Tri}(Z) : f \sim e\}$$

equipped with the maps

$$\Psi_e^\sim : Z_{1/2}(e) \in w \mapsto \exp K(e,w)e^\sim = \big[\Psi_e(w)e\big]^\sim, \qquad e \in \mathrm{Tri}(Z)$$

becomes a real-analytic manifold. If so, $\mathbb{M}$ must be *symmetric* in the following sense (cf. [2]). The *Peirce reflections*

$$S(e) := 2P_{1/2}(e) - \mathrm{Id}, \qquad e \in \mathrm{Tri}(Z)$$

belong to $\mathrm{Aut}(Z)$ with $S_e|Z_0(e) \oplus Z_1(e) = \mathrm{Id}$ and $S_e|Z_{1/2}(e) = -\mathrm{Id}$. Easily seen, $S(e)$ commutes with the chart map Ψ_e that is $S(e)\Psi_e(w) = \Psi_e(-w)$ for all $w \in Z_{1/2}(e)$. Consequently, its quotient mapping $S^\sim(e)f^\sim := \big[S(e)f\big]^\sim$, $f \in \mathrm{Tri}(Z)$ is a welldefined symmetry of $\mathbb{M}$ that is $S^\sim(e)$ is holomorphic with $S^\sim(e)e^\sim = e^\sim$ and $\big[S^\sim(e)\big]'(e^\sim) = -\mathrm{Id}$. It is an *open problem* for the time being, in which cases does $\mathbb{M}$ become with this atlas a complex manifold (i.e. all the coordinate changing maps $\big[\Psi_e^\sim\big]^{-1} \circ \Psi_f^\sim$ are holomorphic). In 2001, Kaup [10] published a paper on the Grassmanian manifold

$$\mathbb{P} := \big\{ J_a : \ a \in Z, \ \exists V \subset Z \text{ subspace } \ J_a \oplus V = Z \big\}$$

of all *principal inner ideals* $J_a := \bigcap \big\{ J \subset Z : \ a \in J, \ \{J, Z, J\} = J \big\}$ which are complemented in Z. One of its main conclusions is that the maps

$$\Theta_e : Z_{1/2}(e) \ni u \mapsto \exp D(u, e) J_e, \quad e \in \mathrm{Tri}(Z)$$

form an atlas on $\mathbb{P}$ and $\mathbb{P}$ becomes a complex symmetric manifold with them. Notice that the equivalence $e \sim f$ of two tripotents can also be formulated in terms of their Peirce 1-subspaces as $Z_1(e) = Z_1(f)$ (as an easy consequence of $e \sim f \Leftrightarrow \{e, e, f\} = f \ \& \ \{f, f, e\} = e \Leftrightarrow D(e) = D(f)$). As it is also shown in [10], that actually we have

$$\mathbb{P} = \big\{ J_a : a \in \mathrm{Reg}(Z) \big\} = \big\{ J_{s(a)} : a \in \mathrm{Reg}(Z) \big\} = \big\{ Z_1(e) : e \in \mathrm{Tri}(Z) \big\}$$

where $\mathrm{Reg}(Z) := \big\{ a \in Z : \ \mathrm{Sp}\big(D(a)\big) > 0 \big\}$ denotes the set of all *von Neumann regular* elements in Z and

$$s(a) := \lim_{n \to \infty} \varphi_n\big(D(a)\big) a, \qquad a \in \mathrm{Reg}(Z)$$

is the *support tripotent* of $a \in \mathrm{Reg}(Z)$ welldefined with any sequence $\big(\varphi_n\big)$ of real polynomials such that $\varphi_n(x^2)x \to 1$ locally uniformly for $x > 0$. Thus, with the family $\mathbb{D} := \{iD(e) \colon \ e \in \mathrm{Tri}(Z)\}$ of triple derivations, the diagram of mappings

$$\begin{array}{ccccc} & & \mathrm{Tri}(Z) & & \\ & \swarrow & \downarrow & \searrow & \\ \mathbb{D} & \leftrightarrow & \mathbb{M} & \leftrightarrow & \mathbb{P} \end{array} \qquad\qquad \begin{array}{ccccc} & & e & & \\ & \swarrow & \downarrow & \searrow & \\ i\,D(e) & \leftrightarrow & e^\sim & \leftrightarrow & Z_1(e) \end{array}$$

is commutative. Thus the complex structure of $\mathbb{P}$ provided by the charts Θ_e on $\mathbb{P}$ can be translated to $\mathbb{D}$ and $\mathbb{M}$ by its means. Henceforth we shall be concerned with the problem how to describe holomorphy in $\mathbb{M}$ and $\mathbb{D}$ in intrinsic manners, not involving principal ideals explicitly. Such kind of an approach may have interest from the following view point: the algebraically less sophisticated maps $u \mapsto \exp D(u,e)$ in the construction of the charts of $\mathbb{P}$ apply to rather "big" objects such that we may have $J_e \cap J_f$ even if $e \not\sim f$ while $e^\sim \cap f^\sim = \emptyset$ and $D(e) \neq D(f)$ simply in the latter case. As a first natural question we can raise is how $\mathbb{D}$ does behave topologically in its covering real-linear operator space $\mathrm{Der}(Z)$. We gave the following answer in terms of decompositions with the projections

$$\pi_{k\ell}(e) : \mathcal{L}(Z) \ni L \mapsto P_{k/2}(e) L P_{\ell/2}(e), \quad \Pi_m(e) := \sum_{|k-\ell|=m} \pi_{k\ell}(e).$$

3.1. Theorem [8]**.** *Π_0 and Π_1 map $\mathrm{Der}(Z)$ into itself and we have*

$$\mathrm{Der}(Z) = \Delta_1(e) \oplus \Delta_0(e) \quad \textit{where} \quad \Delta_m(e) := \Pi_m(e)\mathrm{Der}(Z).$$

$\mathbb{D}$ is a real-analytic direct submanifold of $\mathrm{Der}(Z)$. For any $e \in \mathrm{Tri}(Z)$, the map $u \mapsto K(e,u)$ is a bijection $Z_{1/2}(e) \leftrightarrow \Delta_1(e)$ and $\mathrm{T}_{iD(e)}\mathbb{D} = \Delta_1(e)$. The families

$$\big\{ \big[Z_{1/2}(e) \ni u \mapsto iD\big(\exp K(e,u)e\big)\big] : \; e \in \mathrm{Tri}(Z) \big\},$$
$$\big\{ \big[Z_{1/2}(e) \ni u \mapsto \big(\exp K(e,u)e\big)^\sim \big] : \; e \in \mathrm{Tri}(Z) \big\}$$

are real-analytic atlases for $\mathbb{D}$ and $\mathbb{M}$, respectively.

The main topological properties of the natural map $\mathbb{D} \leftrightarrow \mathbb{M}$ can be established by a fine estimate as follows.

3.2. Proposition [8]. *If $e, f \in \mathrm{Tri}(Z)$ and we have $\|D(e) - D(f)\| < \frac{1}{66}$ then there exists $f' \in f^\sim$ such that $\|e - f'\| \le 16\|D(e) - D(f)\|$.*

As a consequence, by writing $d(z, A) := \inf_{a \in A} \|z - a\|$, $z \in Z$, $A \subset Z$ for the point-set distance in Z, the quotient topology of the equivalence classes in $\mathbb{M}$ inherited from the norm topology of $\mathrm{Tri}(Z)$ coincides with the topology by the bias $d_0(e^\sim, f^\sim) := \inf_{e' \in e^\sim} d(e, f^\sim)$. It coincides also with the topology by the Hausdorff metric $d_H(e^\sim, f^\sim) := \max\big\{ \sup_{e' \in e^\sim} d(e', f^\sim), \sup_{f' \in f^\sim} d(f', e^\sim) \big\}$. Moreover the mapping $e^\sim \mapsto iD(e)$ is bilipschitzian $\mathbb{M} \leftrightarrow \mathbb{D}$ with respect to d_H.

Next we proceed to the question if the real-analytic structures given in Theorem 3.1 are compatible with those inherited from Kaup's complex manifold structure on $\mathbb{P}$. There is a natural candidate for a canonical technique to translate the coordinate map $\Theta_e(u) := \exp(u,e)Z_1(e) = J_{\exp D(u)e}$ into $\mathbb{M}$ and $\mathbb{D}$. Namely we can project the range of $\exp D(.,e)$ into $\mathrm{Tri}(Z)$ by using support tripotents resulting in the mappings

$$\widetilde{\Theta}_e(u) := \big[s\big(\exp D(u,e)e\big)\big]^\sim, \quad \widehat{\Theta}_e(u) := iD\big(s\big(\exp D(u,e)e\big)\big)$$

into from $Z_{1/2}(e)$ into $\mathbb{M}$ and $\mathbb{D}$, respectively. Are they real-analytic with respect to the atlases of $\mathbb{M}$ and $\mathbb{D}$ given in Theorem 3.1?

3.3. Theorem [8]. *If Z is a JC*-triple, $e \in \mathrm{Tri}(Z)$ and $u \in Z_{1/2}(e)$ then the function $s_t := s\big([\exp tD(u,e)]e\big)$ is the solution of the initial value problem*

$$\frac{d}{dt}s_t = P_{1/2}(s_t)\{u,e,s_t\}, \qquad s_0 = e. \tag{3.4}$$

By a JC*-triple we mean a JB*-triple which is isomorphic to a subtriple of some C*-algebra $\mathcal{L}(H)$ with a suitable Hilbert space H. It is a well-known consequence of the Gelfand-Neumark theorem of JB*-triples due to Friedman and Russo [4] that to establish Theorem 3.3 for general JB*-triples, it suffices to prove its statement additionally only in the special case $Z = \mathcal{H}_3(\mathbf{O})$ of the 27-dimensional exceptional JB*-triple. It seems that any JB*-subtriple of $\mathcal{H}_3(\mathbf{O})$ generated by a tripotent and an element from its Peirce (1/2)-subspace must be a JC*-triple. The solution of (3.4) passes in such a subtriple necessarily and the theorem is valid in general. However, we have no complete proof for the moment.

To bypass this difficulty, in [8] we construct holomorphic atlases on $\mathbb{M}$ by means of the solutions of (3.4), leading to some results of independent interest. To this aim, first we have to understand the connection between the tangent vector fields of $\mathrm{Tri}(Z)$ and those of $\mathbb{M}$. Consider a flow $\big[\phi_t : t \in \mathbb{R}\big]$ of mappings $\phi_t : \mathrm{Tri}(Z) \to \mathrm{Tri}(Z)$ which preserve the equivalence classes of $\sim$ (i.e. $e \sim f \Rightarrow \phi_t(e) \sim \phi_t(f)$) such that $\phi_0 = \mathrm{Id}$ and each curve $t \mapsto \phi_t(e)$ is smooth. Then the vector field $e \mapsto X(e) := \frac{d}{dt}\big|_{t=0}\phi_t(e) \in Z^{(-)}(e)$ has the property

$$D\big(X(e),e\big) = D\big(X(f),f\big), \quad D\big(e,X(e)\big) = D\big(f,X(f)\big) \quad \text{whenever } e \sim f.$$

We shall call such tangent vector fields *equivariant*. Different flows $\big[\phi_t\big]$, $\big[\phi_t\big]$ may give rise to the same mappigs of equivalence classes in the sense that $\phi_t(e^\sim) = \psi_t(e^\sim)$ for all $t \in \mathbb{R}$, $e \in \mathrm{Tri}(Z)$. Then, for the generator vector fields $X := \frac{d}{dt}\big|_{t=0}\phi_t$ and $X := \frac{d}{dt}\big|_{t=0}\phi_t$, we have

$$D\big(X(e),e\big) = D\big(Y(e),e\big), \quad D\big(e,X(e)\big) = D\big(e,Y(e)\big), \quad e \in \mathrm{Tri}(Z).$$

We call this property the *equivalence* of the fields X, Y and write $X \approx Y$ for it.

3.5. Proposition [8]. (1) *Given an equivariant field X, its projection*

$$P_{1/2}X : e \mapsto P_{1/2}(e)X(e)$$

is the unique equivariant field Y with $X \approx Y$ and $Y(e) \in Z_{1/2}(e)$, $e \in \mathrm{Tri}(Z)$.

(2) *A bounded locally Lipschitzian tangent vector field X on $\mathrm{Tri}(Z)$ is equivariant if and only if $\exp tX$ preserves the equivalence classes of $\sim$ for all $t \in \mathbb{R}$.*

(3) *The family of all smooth equivariant vector fields in $\mathrm{Tri}(Z)$ is a Lie algebra with the operation $\big[X,Y\big]_* := \nabla_X Y - \nabla_Y X$ and we have $\big[X,Y\big]_* \approx \big[\widehat{X},\widehat{Y}\big]_*$ whenever $X \approx \widehat{X}$ and $Y \approx \widehat{Y}$.*

Define the auxiliary manifolds

$$\mathbf{S}_\lambda := \big\{(e,x):\ e\in \mathrm{Tri}(Z),\ x\in Z_\lambda(e)\big\},\quad \lambda=1,1/2,0.$$

Heuristically, $\mathbf{S}_1$ can serve as a "disjointification" of the Grassmanian $\mathbb{P}$. Its main features for the study of the solutions of (3.4) can be summarized as follows.

3.6. Proposition [8]**.** *$\mathbf{S}_1$ is a real-analytic direct submanifold of $Z\times Z$ with*

$$\mathrm{T}_{(e,x)}\mathbf{S}_1 = Z_1^{(-)}(e)\times Z_1(e)\supset \Big\{\Big(P_{1/2}(e)D(a,b)e,\ D(a,b)x\Big):\ a,b\in Z\Big\}.$$

Given two smooth vector fields $C:\mathrm{Tri}(Z)\to Z$, $D:Z\to Z$ being complete in $\mathrm{Tri}(Z)$ *and Z, respectively, the statements* (1),(2),(3) *below are equivalent.*

(1) $[\exp tD]x\in Z_1\big([\exp tC]e\big)$ *for all* $(e,x)\in\mathbf{S}_1$ *and* $t\in\mathbb{R}$,
(2) $D(x)=\{\{C(e),e,x\}+\{e,C(e),x\}+\{e,e,D(x)\}$ *for all* $(e,x)\in\mathbf{S}_1$,
(3) $[\exp(tD)]Z_1(e)=Z_1\big([\exp tC]e\big)$ *for all* $(e,x)\in\mathbf{S}_1$ *and* $t\in\mathbb{R}$.

For any couple $(e,u)\in\mathbf{S}_{1/2}$, let us introduce the tangent vector field

$$C_u^{(e)}(f):=P_{1/2}(f)D(u,e)f,\qquad f\in\mathrm{Tri}(Z).$$

On the basis of Propositions 3.5 and 3.6 we can complete the argument.

3.7. Theorem [8]**.** *For each $e\in\mathrm{Tri}(Z)$ there exists a neighborhood W of 0 in $Z_{1/2}(e)$ and a real-analytic map $T_e\colon W\to\mathrm{Tri}(Z)$ such that*

$$T_e(0)=e\ ,\quad \exp D\,(u,e)\,e\in Z_1\big(T_e(u)\big),\qquad u\in W.$$

Proof. Fix any $u\in Z_{1/2}(e)$ and set $x^u:=\exp D\,(u,e)\,e$. Notice that the vector field $E_u(f):=P_{1/2}(f)D\,(u,f)\,f$, $f\in\mathrm{Tri}(Z)$ is a tangent to $\mathrm{Tri}(Z)$ and its exponential is a welldefined mapping $\mathrm{Tri}(Z)\to\mathrm{Tri}(Z)$. Let

$$T_e(u):=(\exp E_u)e,\qquad u\in Z_{1/2}(e).$$

Then the curve $t\mapsto e_t:=T_e(tu)$, $t\in\mathbb{R}$, is the solution of the initial value problem $e_0=e$, $\frac{d}{dt}e_t=P_{1/2}(e_t)D\,(u,e_t)$. Consider the mapping

$$F(f,y):=\big(P_{1/2}(f)D\,(u,e)\,f,\ D\,(u,e)\,y\big),\qquad (f,y)\in\mathbf{S}_1.$$

From Proposition 3.6 we see that F is a tangent vector field to $\mathbf{S}_1$ and its exponential is a well-defined mapping $\mathbf{S}_1\to\mathbf{S}_1$. In particular, there is a curve $t\mapsto(f_t,y_t)\in\mathbf{S}_1$, $t\in\mathbb{R}$, such that $(f_0,y_0)=(e,e)$ and $\frac{d}{dt}(f_t,y_t)=F(f_t,y_t)$. Then we have $\frac{d}{dt}y_t=D(u,e)y_t$, $y_0=e$ and $\frac{d}{dt}f_t=P_{1/2}(f_t)D(u,e)f_t$, $f_0=e$. By the uniqueness of solutions of initial value problems, $y_t=(\exp tD\,(u,e))e$ and $f_t=e_t$ for all $t\in\mathbb{R}$. Since $(f_t,y_t)\in$

$\mathbf{S}_1$, we have $y_t \in Z_1(f_t)$ for all $t \in \mathbb{R}$. In particular $\exp D\,(u,e)\,e = y_1 \in Z_1(f_1) = Z_1\big(T_e(u)\big)$ which completes the proof. $\square$

On the basis of these results we can describe the holomorphic atlases corresponding to Kaup's coordinatization for $\mathbb{P}$ both on $\mathbb{D}$ and $\mathbb{M}$ as follows. Consider the vector fields

$$C_u^{(e)} : \mathrm{Tri}(Z) \ni f \mapsto P_{1/2}(f)D(u,e)f, \quad (e,u) \in \mathbf{S}_{1/2}.$$

According to the results of Section 2, they are real-analytic (with respect to the coordinates $Z^{(-)}(e) \ni w \mapsto \exp K(e,w)$). and tangent to $\mathrm{Tri}(Z)$. Hence the curves $t \mapsto [\exp tC_u^{(e)}]e$ are well-defined on the whole $\mathbb{R}$ and range in $\mathrm{Tri}(Z)$. By definition they are solutions of (3.4). Also the maps

$$Y_e Z_{1/2}(e) \ni u \mapsto [\exp C_u^{(e)}]e, \qquad e \in \mathrm{Tri}(Z)$$

are all well-defined. real-analytic and range in $\mathrm{Tri}(Z)$. Using Propositions 3.5, 3.6 and Theorem 3.7 we conclude the following.

3.8. Theorem [8]**.** *The vector fields $C_u^{(e)}$ are equivariant and complete in $\mathrm{Tri}(Z)$. For any tripotent e, there exists a neighborhood W_e of the origin in $Z_{1/2}(e)$ such that the restricted map $Y_e|W_e$ is real-bianalytic with*

$$Y_e(0) = e, \quad [\exp D(u,e)]J_e = J_{Y_e(u)} = Z_1\big(Y_e(u)\big), \qquad u \in W_e\ .$$

By setting $\widehat{Y}^{(e)}(u) := iD\big(Y_e(u)\big)$, $\widetilde{Y}^{(e)}(u) := Y_e(u)^\sim$, $\overline{Y}^{(e)}(u) := J_{Y_e(u)}$, the families

$$\big\{\widehat{Y}^{(e)} :\ e \in M\big\},\ \big\{\widetilde{Y}^{(e)} :\ e \in M\big\},\ \big\{\overline{Y}^{(e)} :\ e \in M\big\}$$

are holomorphic atlases for $\mathbb{D}, \mathbb{M}$ and $\mathbb{P}$ with commuting diagram

$$\begin{array}{ccc} & Z_{1/2}(e) & \\ \swarrow & \downarrow & \searrow \\ \mathbb{D} \leftrightarrow & \mathbb{M} & \leftrightarrow \mathbb{P} \end{array} \qquad \begin{array}{ccc} & u & \\ \swarrow & \downarrow & \searrow \\ \widehat{Y}^{(e)}(u) \leftrightarrow & \widetilde{Y}^{(e)}(u) & \leftrightarrow \overline{Y}^{(e)}(u) \end{array}\ .$$

Since the points of $\mathbb{M}$ are actually pairwise disjoint subsets in Z, it is natural to ask how can we describe the holomorphy of a function $\mathbb{M} \to \mathbb{C}$ (and hence holomorphy to general Banach spaces) in terms of holomorphy in Z.

3.9. Theorem [8]**.** *Let $\mathbf{U}$ be an open subset of $\mathbb{M}$ and let $U := \bigcup_{e^\sim \in \mathbf{U}} e^\sim$ denote its trace in Z. A function $\Phi : \mathbf{U} \to \mathbb{C}$ is holomorphic if and only if for any point $e \in U$, there exists an open neighborhood V of e in Z along with a holomorphic function $\phi : V \to \mathbb{C}$ such that $\phi(f) = \Phi(f^\sim)$ whenever $f \in U \cap V$.*

References

[1] T. Barton - R. Timoney, *Weak*-continuity of Jordan triple products and its applications,* Math. Scand. **59** (1986) 177-191.

[2] C. H. Chu - J.-M. Isidro, *Manifolds of tripotents in JB*-triples,* Math. Z. **233**, (2000) 741-754.

[3] C.M. Edwards - G.T. Rüttimann, *On the facial structure of the unit balls in a JBW*-triple and its predual,* J. London Math. Soc. (2) **38** (1988) 317-332.

[4] Y. Friedmann - B. Russo, *The Gelfand-Naimark theorem for JB*-triples,* Duke Math. J. **53** (1986) 139-148.

[5] L.A. Harris - W. Kaup, *Linear algebraic groups in infinite dimensions,* Illinois J. Math. **21/3** (1977) 666-674.

[6] J.-M. Isidro - L.L. Stachó, Holomorphic automorphism groups in Banach spaces, North Holland Mat. Studies 105, North Holland, New York - Amsterdam - Oxford, 1985.

[7] J.-M. Isidro - L.L. Stachó, *On the manifold of tripotents in JB*-triples,* J. Math. Anal. Appl. **304/1** (2005) 147-157.

[8] J.-M. Isidro - L.L. Stachó, *On the manifold of complemented principal inner ideals in JB*-triples,* Quart. J. Math (Oxford) **57** (2006) 505-525.

[9] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces,* Math. Z. **138** (1983) 503-509.

[10] W. Kaup, *On Grassmannians associated with JB*-triples,* Math. Z. **236** (2001) 567-584.

[11] W. Kaup - J. Sauter, *Boundary structure of bounded symmetric domains,* Manuscripta Math. **101** (2000) 351-360.

[12] O. Loos, Bounded symmetric domains and Jordan pairs. Math. Lectures, University of California at Irvine, 1977.

[13] E. Neher, Grids in Jordan triple systems, Lecture Notes in Math. Vol. 1280, Springer-Verlag, Berlin - Heidelberg - New York, 1987.

[14] J. Sauter, Randstrukturen beschränkter symmetrischer Gebiete, Ph.D. Dissertation, Eberhard-Karls-Universität, Tübingen, 1995.

[15] L.L. Stachó, *On metric geodesics of tripotents in JB*-triples,* preprint (2007), www.math.u-szeged.hu/∼stacho.

[16] H. Upmeier, Symmetric Banach manifolds and Jordan C^*-algebras. North-Holland Mathematics Studies 104. Notas de Matemática (Mathematical Notes 96). North-Holland Publishing Co., Amsterdam, 1985.