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On Hermitian interpolation of first order data with locally generated C1-splines over triangular meshes

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Abstract. Given a system of triangles in the plane $\mathbb{R}^2$ along with given data of function and gradient values at the vertices, we describe the general pattern of local linear methods involving only four smooth standard shape functions which results in a spline function fitting the given value and gradient data value with $\mathcal{C}^1$-coupling along the edges of the triangles. We characterize their invariance properties with relevance for the construction of interpolation surfaces over triangularizations of scanned 3D data. The numerically simplest procedures among them leaving invariant all polynomials of 2-variables with degree 0 resp 1 involve only polynomials of 5-th resp. 6-th degree, but the characterizations give rise to a huge variety of procedures with non-polynomial shape functions.

1. Introduction

Recently [7] we published a $\mathcal{C}^1$-spline interpolation method over 2-dimensional triangular meshes with polynomials of 5-th degree with low operational costs: using first order data (value and gradient) at the mesh vertices for input. The result relies on the following basic tool: given a non-degenerate triangle with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^2$ along with three values $f_1, f_2, f_2 \in \mathbb{R}$ respectively three linear functionals $A_1, A_2, A_3 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ and three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^2$ such that $\mathbf{u}_k \not\parallel \mathbf{p}_i - \mathbf{p}_j$, we can construct a polynomial $F : \mathbb{R}^2 \to \mathbb{R}$ of 5-th degree of the

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form $F = F_0 - H$ where

$$F_0(\mathbf{x}) = \sum_{i=1}^{3} \Big[\Phi\big(\lambda_i(\mathbf{x})\big) f_i + \Theta\big(\lambda_i(\mathbf{x})\big) A_i(\mathbf{x}-\mathbf{p}_i)\Big],$$
$$H(\mathbf{x}) = \lambda_1(\mathbf{x})^2\lambda_2(\mathbf{x})^2\lambda_3^2(\mathbf{x})^2 \sum_{k=1}^{3} \lambda_k(\mathbf{x})^{-1} \frac{M_k\mathbf{u}_k}{G_k\mathbf{u}_k} \tag{1.1}$$

in terms of the *barycentric weights* $\lambda_1, \lambda_2, \lambda_3 : \mathbb{R}^2 \to \mathbb{R}$ of the triangle $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ associated with the vertices and the *shape functions*

$$\Phi(t) = t^3(10 - 15t + 6t^2), \qquad \Theta(t) = t^3(4 - 3t) \tag{1.2}$$

for an Hermite interpolant F_0 on $\mathbf{T}$ with the data $f_i \in \mathbb{R}, A_i \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ $(i = 1, 2, 3)$. The correction term H is defined by means of the linear functionals $\mathbb{R}^2 \to \mathbb{R}$

$$G_k\mathbf{u} = \lambda_k'\mathbf{u} = \frac{d}{dt}\Big|_{t=0} \lambda_k(\mathbf{x} + t\mathbf{u}) = \lambda_k(\mathbf{x} + \mathbf{u}) - \lambda_k(\mathbf{x}),$$
$$M_k\mathbf{u} = \sum_{\{i,j\}=\{1,2,3\}\setminus\{k\}} [G_i\mathbf{u}]\big(30f_i + 12A_i(\mathbf{p}_j - \mathbf{p}_i)\big).$$

It is worth to notice that the function F_0 in (1.1) restricted to the edge $[\mathbf{p}_i, \mathbf{p}_k]$ depends only on the terms f_i, f_j, A_i, A_j and the impact of the correction by adding H results in the *reduced side derivatives* (**RSD**)

$$F'(\mathbf{x}_t)\mathbf{u_k} = \Theta(t)A_i\mathbf{u}_k + \Theta(1-t)A_j\mathbf{u}_k \quad \text{for} \quad \mathbf{x}_t = t\mathbf{p}_i + (1-t)\mathbf{p}_j.$$

As a consequence, if we choose any family $\big\{\mathbf{u_E} : \mathbf{E} \text{ being a mesh edge}\big\}$ of vectors such that $\mathbf{u_E} \not\parallel \mathbf{E}$, by applying the construction (1.1) with the associated data over every triangle of the mesh, we obtain a $\mathcal{C}^1$-smooth (continuously differentiable) function on the union of the mesh triangles.

The shape functions Φ, Θ and also the RSD method described above appeared in [7] without heuristic introduction. Actually they arose from our earlier study [8] with somewhat restrictive postulates on the possible polynomial $\mathcal{C}^1$-interpolations over triangular meshes based on computer algebraic analysis of technics developed in [2],[5],[6],[11].

Our goal here is to characterize all RSD methods with shape functions $\Psi_0, \Psi_1 \in \mathcal{C}^1\big([0,1]\big)$ instead of (1.2). This requires a completely different approach as that in [8] relying heavily upon polynomial identities. From our main result

Theorem 3.5 it turns out that there is a very general pattern behind (1.1) with

$$\begin{aligned} F_0(\mathbf{x}) &= \sum_{k=1}^{3} \Big[\Psi_0\big(\lambda_k\big) f_k + \Psi_1\big(\lambda_k\big) A_k(\mathbf{x}-\mathbf{p}_k)\Big], \\ H(\mathbf{x}) &= \sum_{\{\ell,m,n\}=\{1,2,3\}} \frac{G_\ell \mathbf{u}_n}{G_n \mathbf{u}_n} \Big\{ f_\ell \chi_0(\lambda_\ell,\lambda_m,\lambda_n) + A_\ell(\mathbf{p}_m - \mathbf{p}_\ell)\chi_1(\lambda_\ell,\lambda_m,\lambda_n)\Big\}. \end{aligned} \tag{1.3}$$

If Ψ_0, Ψ_1 fulfill the minimal necessary condition (established in (2.1) later) for giving rise to a Hermite interpolation of the type F_0 in (1.3) then we can find plenty of suitable functions $\chi_0, \chi_1 \in \mathcal{C}_0^1(\mathbb{R}^3_+)$, namely those satisfying only the not too restrictive conditions (3.6),(3.7), in order for H being an RSD correction for F_0. This degree of freedom enables various canonical constructions for the modifiers χ_0, χ_1 in terms of the shape functions Ψ_0, Ψ_1 to ensure specific properties of the interpolation operator $\mathfrak{F}\colon \big\{(\mathbf{p}_k, f_k, A_k) : k=1,2,3\big\} \mapsto \big[F_0 + H \text{ by } (1.3)\big]$.

We complete the paper with the investigation of two essential properties of the interpolant $\mathfrak{F}$ having obvious importance in applications used to construct smooth surfaces in the form of the graph of a function $\mathbb{R}^2 \to \mathbb{R}$ passing through a finite family of points in $\mathbb{R}^3$ (obtained as vertices of a triangularization from scanned data):

(i) The *range shift property* $\mathfrak{F}\big\{(\mathbf{p}_k, 1, 0) : k = 1,2,3\big\} \equiv 1$, that is all constant functions remain invariant when interpolated with $\mathfrak{F}$. In this case the graph of $\mathfrak{F}\big\{(\mathbf{p}_k, f(\mathbf{p}_k) + c, f'(\mathbf{p}_k)) : k = 1,2,3\big\}$ is just a shifted form of the graph of $\mathfrak{F}\big\{(\mathbf{p}_k, f(\mathbf{p}_k), f'(\mathbf{p}_k)) : k = 1,2,3\big\}$ for any function $f \in \mathcal{C}^1(\mathbb{R}^2)$.

(ii) *Affinity invariance*: $\mathfrak{F}\big\{(\mathbf{p}_k, A\mathbf{x}+b, A) : k = 1,2,3\big\} \equiv A\mathbf{x}+b$. That is all affine (constant+linear) functions remain invariant when interpolated with $\mathfrak{F}$. In particular if the graph of a function $\mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \to \mathbb{R}$ is a 3D plane triangle then the function obtained with interpolation by $\mathfrak{F}$ has the same graph.

In Section 4 we give a parametric classification of the procedures with range shift property. Concerning affinity invariance, we have no complete results yet: in Section 5 we provide several necessary and sufficient algebraic conditions. As a by no means obvious fact, it turns out that the procedure (1.1) with the polynomials (1.2) has range shift property but fails to be affinity invariant. We also provide an affinity invariant procedure with polynomials of 6-th degree: namely that of the form (1.3) with $\Psi_0 = \Psi_1 = \Phi$ and $\chi_0(t_1,t_2,t_3) = 30t_1^2t_2^2t_3$, $\chi_1(t_1,t_2,t_3) = 30t_1^2t_2^3t_3$.

Hermite interpolation involving triangular meshes became an attractive topics nowadays, motivated by the enormous power of recent computer architectures. Most highly intersting new works like [3],[4],[10] focus on methods with spectacular accuacy but involving large data for the values at generic points. In contrast, we primarily investigate the algebraic aspects of constructions resulting in interpolants over each mesh triangle being independent of the data outside its vertices. Such constructions are suited in fitting a $\mathcal{C}^1$-surface to scanned data of points and normal vectors of 2D surfaces in 3D not being homeomorphic to a plane (e.g. sphere or torus). Though "minimalist" approaches compromise the accuracy of approximation, by modifying some ideas of [3, Section 2], one can prove that the interpolation accuracy by an affinity invariant operator can be majorized with the maximum norm of the second derivative of the underlying functions.

2. Preliminaries

For standard terminology, $\mathbb{R}^N$ resp. $\mathbb{R}^N_+$ stand for the set of all real resp. non-negative N-tuples $[\xi_1, \dots, \xi_N]$. Actually, we shall only be interested in the cases of dimensions $N = 1, 2, 3$.

Given a subset Ω of $\mathbb{R}^N$, we write $\mathcal{C}(\Omega)$ for the family of all continuous functions with domain Ω. If the interior Ω° of Ω is dense in Ω, we define $\mathcal{C}^1(\Omega)$ as the set consisting of all functions $f \in \mathcal{C}(\Omega)$ with continuous partial derivatives $D_1 f, \dots, D_N f$ on Ω° *admitting a continuous extension to* Ω with the value denoted also by $D_k f(\mathbf{x})$ at the points $\mathbf{x} \in \Omega \setminus \Omega^\circ$. As a folklore consequence of Whitney's extension theorem [9], if Ω is closed in $\mathbb{R}^N$ then every function $f \in \mathcal{C}^1(\Omega)$ can be regarded as the restriction of a continuously differentiable function defined on $\mathbb{R}^N$. For any function $f \in \mathcal{C}^1(\Omega)$, we write f' for its *Fréchet derivative* defined at any point $\mathbf{x} \in \Omega$ as the linear functional

$$f'(\mathbf{x})\mathbf{u} = \sum_{k=1}^{N} D_k f(\mathbf{x}) \upsilon_k \qquad \big(\mathbf{u} = [\upsilon_1, \dots, \upsilon_N]\big).$$

In particular $f'(\mathbf{x})\mathbf{u} = \lim_{t \to 0} t^{-1}\big[f(\mathbf{x}+t\mathbf{u}) - f\mathbf{u}\big]$ is the familiar directional derivative at the points $\mathbf{x} \in \Omega^\circ$. If Ω has dense interior, for $k = 0, 1$ we define $\mathcal{C}_0^k(\Omega) = \big\{f \in \mathcal{C}^k(\Omega) : f(\mathbf{x}) = 0 \ \big(\mathbf{x} \in \Omega \setminus \Omega^o\big)\big\}$ (so that e.g. $xy/(x^2+y^2) \in \mathcal{C}_0^1(\mathbb{R}^2_+ \setminus \{(0,0)\})$.

Given any subset $\mathbf{P} \subset \mathbb{R}^2$, we write $\mathrm{Conv}(\mathbf{P})$ for its convex hull. If $\mathbf{P} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and the triangle $\mathbf{T} = \mathrm{Conv}(\mathbf{P})$ is non-degenerate, the *barycentric weights* (weight functions associated to the vertices) [1] are the functions

$$\lambda_i = \lambda^{\mathbf{T}}_{\mathbf{p}_i} : \mathbf{x} \mapsto \frac{\det\big[\mathbf{x} - \mathbf{p}_j, \mathbf{x} - \mathbf{p}_k\big]}{\det\big[\mathbf{p}_i - \mathbf{p}_j, \mathbf{p}_i - \mathbf{p}_k\big]} \quad \text{with } (i, j, k) \in S_3$$

where

$$S_3 = \big\{(1,2,3), (2,1,3), (2,3,1), (3,2,1), (3,1,2), (1,3,2)\big\}$$

denotes the set of all permutations of the indices $1,2,3$. For the sake of brevity, in the sequel we omit the background parameters in most formulas (like writing $\lambda_i = \lambda^{\mathbf{T}}_{\mathbf{p}_i}$ above) without danger of confusion. It is well-known that the mapping $\mathbf{x} \mapsto [\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})]$ is a homeomorphism between $\mathbf{T}$ and the *3D-unit simplex*

$$\Delta_3 := \big\{[t_1,t_2,t_3] \in \mathbb{R}^3_+ : t_1+t_2+t_3 = 1\big\},$$

moreover $[\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})]$ is the unique triple of non-negative numbers such that $\mathbf{x} = \sum_k \lambda_k(\mathbf{x})\mathbf{p}_k$ and $\sum_k \lambda_k(\mathbf{x}) = 1$. Furthermore $\lambda_1, \lambda_2, \lambda_3 : \mathbb{R}^2 \to \mathbb{R}$ are affine (linear+constant) functions with the necessarily constant Fréchet derivatives

$$G_i\mathbf{u} := \lambda_i'(\mathbf{x})\mathbf{u} = \frac{d}{dt}\Big|_{t=0} \lambda_i(\mathbf{x}+t\mathbf{u}) \quad \text{independently of } \mathbf{x}.$$

Given any pair $\mathbf{\Psi} = [\Psi_0, \Psi_1]$ with $\Psi_0, \Psi_1 \in \mathcal{C}^1\big([0,1]\big)$ such that

$$\Psi_0(0) = \Psi_0'(0) = \Psi_1(0) = \Psi_1'(0) = \Psi_0'(1) = 0, \quad \Psi_0(1) = \Psi_1(1) = 1 \tag{2.1}$$

we introduce the *basic triangular interpolation of first order with the shape functions* Ψ_0, Ψ_1 as the operator

$$\mathfrak{F}_0^{\mathbf{\Psi}} : \big\{(\mathbf{p}_k, f_k, A_k)\big\}_{k=1}^3 \mapsto \sum_{k=1}^3 \Big\{\Psi_0\big(\lambda_k\big)f_k + \Psi_1\big(\lambda_k\big)A_k\big(\mathbf{x}-\mathbf{p}_k\big)\Big\} \tag{2.2}$$

defined for all *first order function germs* $\big\{(\mathbf{p}_k, f_k, A_k) : k = 1,2,3\big\}$ with $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1\} \subset \mathbb{R}^2$ being a non-degenerate triangle, $f_k \in \mathbb{R}$ and $A_k \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ in terms of the weights $\lambda_k = \lambda_k^{\mathbf{T}}$. By definition, the domain of the function $F_0 = \mathfrak{F}_0^{\mathbf{\Psi}}\big\{(\mathbf{p}_k, f_k, A_k)\big\}_{k=1}^3$ is only the triangle $\mathbf{T}$ and $F_0 \in \mathcal{C}^1(\mathbf{T})$. By straightforward calculation, for its Fréchet derivative we have

$$F_0' = \sum_{k=1}^3 \Big\{\big[\Psi_0'\big(\lambda_k\big)f_k + \big[\Psi_1'\big(\lambda_k\big)A_k\big(\mathbf{x}-\mathbf{p}_k\big)\big]G_k + \Psi_1\big(\lambda_k\big)A_k\Big\}. \tag{2.3}$$

Remark. 2.4. (a) Conditions (2.1) do not restrict the value of $\Psi_1'(1)$.

(b) In view of (2.2) and (2.3), it is not hard to see that (2.1) is sufficient and necessary for F_0 to satisfy the relations

$$F_0\big(\mathbf{p}_i\big) = f_i, \quad F_0'\big(\mathbf{p}_i\big) = A_i \qquad (i = 1,2,3) \tag{2.5}$$

under any choice of the first order function germs $\big\{\mathbf{p}_k, f_k, A_k)\big\}_{k=1}^3$.

(c) We can express the terms $A_k(\mathbf{x}-\mathbf{p}_k)$ appearing in (1.3), (2.2) resp. (2.3) in the form of linear combination of the weights due to the identity $\mathbf{x} = \sum_{i=1}^3 \lambda_i(\mathbf{x})\mathbf{p}_i$ as

$$A_k(\mathbf{x}-\mathbf{p}_k) = \sum_{i=1}^3 A_k(\mathbf{p}_i - \mathbf{p}_k)\lambda_i.$$

Definition. 2.6. Henceforth we say that $\mathbf{\Psi} = [\Psi_0, \Psi_1]$ is a *pair of admissible shape functions* if $\Psi_0, \Psi_1 \in \mathcal{C}^1([0,1])$ and (2.1) holds. We also say for short that $\big\{(\mathbf{p}_k, f_k, A_k)\big\}_{k=1}^3$ is an *admissible function germ* if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^2$ form a non-degenete triangle, $f_k \in \mathbb{R}$ and $A_k \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$.

As an immediate consequence of (2.3), since

$$\begin{aligned}
&\lambda_i\big(\mathbf{p}_j\big) = \delta_{ij}, \quad G_i\big(\mathbf{p}_j - \mathbf{p}_k\big) = \delta_{ij} - \delta_{ik},\\
&\lambda_i(\mathbf{x}_t^k) = t, \quad \lambda_j(\mathbf{x}_t^k) = 1-t, \quad \lambda_k(\mathbf{x}_t^k) = 0,\\
&\mathbf{x}_t^k - \mathbf{p}_i = (1-t)(\mathbf{p}_j - \mathbf{p}_i), \quad \mathbf{x}_t^k - \mathbf{p}_j = t(\mathbf{p}_i - \mathbf{p}_j),
\end{aligned} \tag{2.7}$$

we obtain the following observation.

Lemma. 2.8. *Given an admissible pair $\mathbf{\Psi} = [\Psi_0, \Psi_1]$ of shape functions along with an admissible function germ $\mathfrak{g} = \big\{(\mathbf{p}_k, f_k, A_k)\big\}_{k=1}^3$, if $i, j, k \in \{1,2,3\}$ are three different indices then, at the generic point*

$$\mathbf{x}_t^k := t\mathbf{p}_i + (1-t)\mathbf{p}_j \qquad (0 \le t \le 1) \tag{2.9}$$

of the edge $[\mathbf{p}_i, \mathbf{p}_j]$ in the triangle $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, for the function $F_0 = \mathfrak{F}_0^{\mathbf{\Psi}}\mathfrak{g}$ we have

$$\begin{aligned}
F_0\big(\mathbf{x}_t^k\big) &= \Psi_0(t) f_i + \Psi_1(t)(1-t) A_i(\mathbf{p}_j - \mathbf{p}_i) + \\
&\qquad + \Psi_0(1-t) f_j + \Psi_1(1-t) t A_j(\mathbf{p}_i - \mathbf{p}_j);\\
F_0'\big(\mathbf{x}_t^k\big) &= \big[\Psi_0'(t) f_i + \Psi_1'(t)(1-t) A_i(\mathbf{p}_j - \mathbf{p}_i)\big] G_i + \Psi_1(t) A_i + \\
&\quad + \big[\Psi_0'(1-t) f_j + \Psi_1'(1-t) t A_j(\mathbf{p}_i - \mathbf{p}_j)\big] G_j + \Psi_1(1-t) A_j.
\end{aligned} \tag{2.10}$$

3. Generic algorithm of reduced side derivatives (RSD)

In this section we are looking for $\mathcal{C}^1$-spline interpolation procedures analogous to those described in [7] but with more general shape functions $\Psi_0, \Psi_1 \in \mathcal{C}^1[0,1]$ instead of Φ, Θ there.

Henceforth let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^2$ be the vertices of some (arbitrarily fixed) non-degenerate triangle $\mathbf{T}$ with respective weight functions $\lambda_m : \mathbb{R}^2 \to \mathbb{R}$ and derivative weights $G_m := \lambda_m' \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ $(m=1,2,3)$ and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^2$ be given vectors such that $\mathbf{u}_k \not\parallel \mathbf{p}_i - \mathbf{p}_j$ whenever $(i,j,k) \in S_3$.

Furthermore $\mathbf{\Psi} = [\Psi_0, \Psi_1]$ resp. $\mathfrak{g} = \big\{(\mathbf{p}_k, f_k, A_k)\big\}_{k=1}^3$ denote a fixed admissible pair of shape functions and a function germ. As earlier, we write

$$F_0 = \mathfrak{F}_0^{\mathbf{\Psi}}\mathfrak{g}.$$

Our starting point is the following immediate consequence of (2.10):

Remark. 3.1. Let $H \in \mathcal{C}_0^1(\mathbf{T})$. Then the function $F := F_0 - H$ coincides with F_0 along the edges of $\mathbf{T}$ and hence, at the vertices, it has also the properties $F(\mathbf{p}_i) = f_i$, $F(\mathbf{p}_i) = A_i$ $(i = 1, 2, 3)$ analogous to (2.5).

Definition. 3.2. We say that the function

$$F = F_0 - H, \quad H \in \mathcal{C}_0^1(\mathbf{T}) \tag{3.3}$$

is an *RSD modification* of F_0 (with respect to the directions $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ along the edges $[\mathbf{p}_2, \mathbf{p}_3], [\mathbf{p}_3, \mathbf{p}_1], [\mathbf{p}_1, \mathbf{p}_2]$ respectively) if for any index pemutation $(i, j, k) \in S_3$,

$$F'(\mathbf{x})\mathbf{u}_k = \Psi_1(\lambda_i(\mathbf{x}))A_i\mathbf{u}_k + \Psi_1(\lambda_j(\mathbf{x}))A_j\mathbf{u}_k \text{ whenever } \mathbf{x} \in [\mathbf{p}_i, \mathbf{p}_j].$$

Conveniently, in this case we refer to H as an *RSD modifier of* F_0.

According to Lemma 2.8, we have the following.

Corollary. 3.4. *A function $H \in \mathcal{C}_0^1(\mathbf{T})$ is an RSD modifier of F_0 with respect to the directions $\mathbf{u}_k$ along the edges $[\mathbf{p}_i, \mathbf{p}_j]$ if and only if*

$$\begin{aligned} H'(\mathbf{x}_t^k)\mathbf{u}_k = &\left[\Psi_0'(t)f_i + \Psi_1'(t)(1-t)A_i(\mathbf{p}_j - \mathbf{p}_i)\right][G_i\mathbf{u}_k] + \\ &+ \left[\Psi_0'(1-t)f_j + \Psi_1'(1-t)tA_j(\mathbf{p}_i - \mathbf{p}_j)\right][G_j\mathbf{u}_k] \end{aligned}$$

whenever $\mathbf{x}_t^k = t\mathbf{p}_i + (1-t)\mathbf{p_j}$ with $(i, j, k) \in S_3$ and $0 \le t \le 1$.

Theorem. 3.5. *Let $\Omega \subset \mathbb{R}^3$ be a set whose interior contains $\Delta_3 \cap (0,1)^3$. Assume $\chi_0, \chi_1 \in \mathcal{C}^1(\Omega)$ are functions vanishing along the edges $\Delta_{3,k} = \{(t_1, t_2, t_3) \in \Delta_3 : t_k = 0\}$ $(k = 1, 2, 3)$ of Δ_3 such that*

$$D_3\chi_0(t, 1-t, 0) = \Psi_0'(t), \quad D_3\chi_1(t, 1-t, 0) = \Psi_1'(t) \cdot (1-t) \tag{3.6}$$

with the following marginal conditions on the derivatives

$$\chi_r'(\mathbf{t}) = 0 \ \ (\mathbf{t} \in \Delta_{3,1} \cup \Delta_{3,2}); \quad D_m\chi_r(\mathbf{t}) = 0 \ \ (\mathbf{t} \in \Delta_{3,3}, \ m = 1, 2). \tag{3.7}$$

Then the function H in (1.3) is an RSD modifier for F_0 under any choice of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with $\mathbf{u}_i \not\parallel \mathbf{p}_j - \mathbf{p}_k$ $((i, j, k) \in S_3)$.

Proof. Consider any permutation $(i, j, k) \in S_3$. According to Corollary 3.4, it suffices to see that, at the points $\mathbf{x}_t = t\mathbf{p}_i + (1-t)\mathbf{p}_j$ we have

$$\begin{aligned} &H(\mathbf{x}_t) = 0, \\ &H'(\mathbf{x}_t)\mathbf{u}_k = \left[\Psi_0'(t)f_i + \Psi_1'(t)(1-t)A_i(\mathbf{p}_j - \mathbf{p}_i)\right]G_i\mathbf{u}_k + \\ &\qquad + \left[\Psi_0'(1-t)f_j + \Psi_1'(1-t)tA_j(\mathbf{p}_i - \mathbf{p}_j)\right]G_j\mathbf{u}_k. \end{aligned} \tag{3.8}$$

Notice that, in terms of the triples

$$\mathbf{t}_{\ell,m,n} = \big(\lambda_\ell(\mathbf{x}_t), \lambda_m(\mathbf{x}_t), \lambda_n(\mathbf{x}_t)\big) \quad \big((\ell,m,n) \in S_3\big),$$

and by setting $(\ell,m,n)_1 = \ell$, $(\ell,m,n)_2 = m$, $(\ell,m,n)_3 = n$, we can write

$$H(\mathbf{x}_t) = \sum_{(\ell,m,n)\in S_3} \frac{G_\ell \mathbf{u}_n}{G_n \mathbf{u}_n} \Big[f_\ell \chi_0(\mathbf{t}_{\ell,m,n}) + A_\ell(\mathbf{p}_m - \mathbf{p}_\ell)\chi_1(\mathbf{t}_{\ell,m,n}) \Big],$$

$$H'(\mathbf{x}_t)\mathbf{u}_k = \sum_{(\ell,m,n)\in S_3} \frac{G_\ell \mathbf{u}_n}{G_n \mathbf{u}_n} \Big[f_\ell \Big[\sum_{q=1}^{3} D_q \chi_0(\mathbf{t}_{\ell,m,n}) G_{(\ell,m,n)_q} \mathbf{u_k} \Big] +$$

$$+ A_\ell(\mathbf{p}_m - \mathbf{p}_\ell) \Big[\sum_{q=1}^{3} D_q \chi_1(\mathbf{t}_{\ell mn}) G_{(\ell,m,n)_q} \mathbf{u_k} \Big] \Big\}.$$

Observe that we have $\mathbf{t}_{ijk} = (t, 1-t, 0) \in \Delta_{3,3}$, $\mathbf{t}_{jik} = (1-t, t, 0) \in \Delta_{3,3}$, $\mathbf{t}_{ikj} = (t, 0, 1-t) \in \Delta_{3,2}$, $\mathbf{t}_{kij} = (0, t, 1-t) \in \Delta_{3,1}$, $\mathbf{t}_{jki} = (1-t, 0, t) \in \Delta_{3,2}$, $\mathbf{t}_{kji} = (0, 1-t, t) \in \Delta_{3,1}$. Hence the relation $H(\mathbf{x}_t) = 0$ is immediate. According to (3,7), all the terms $D_q\chi_r(\mathbf{t}_{\ell,m,n})$ vanish except for the cases with $\lambda_n(\mathbf{x}_t) = 0$ and $q = 3$ that is for $(q,r,\ell,m,n) \in \big\{(3,r,i,j,k), (3,r,j,i,k) : r = 0,1\big\}$. In view of (3.6) we have $D_3\chi_0(\mathbf{t}_{ijk}) = \Psi_0'(t)$, $D_3\chi_0(\mathbf{t}_{jik}) = \Psi_0'(1-t)$, $D_3\chi_1(\mathbf{t}_{ijk}) = \Psi_1'(t)(1-t)$, $D_3\chi_1(\mathbf{t}_{jik}) = \Psi_1'(1-t)t$ which completes the proof of (3.8) and hence the theorem. □

Definition. 3.9. Let $[\Psi_0, \Psi_1]$ be an admissible pair of shape functions and let $\chi_0, \chi_1 \in \mathcal{C}^1(\Omega)$ on a set $\Omega \subset \mathbb{R}^3$ containing Δ_3. We say that $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ is an *RSD tuple* if the expression H in (1.3) is an RSD modifier for F_0 under any choice $\mathbf{u}_i \not\parallel \mathbf{p}_j - \mathbf{p_k}$ $((i,j,k) \in \mathrm{S}_3)$. In particular we say that $\mathbf{\Pi}$ is an *RSD$_*$ tuple* if χ_0, χ_1 are functions satisfying the requirements of Theorem 3.5. Clearly $[\Psi_0, \Psi_1, \chi_{0|\Delta_3}, \chi_{1|\Delta_3}]$ is and RSD tuple if $\mathbf{\Pi}$ is an RSD$_*$ tuple.

Lemma. 3.10. *Given any function $h \in \mathcal{C}_0(\mathbb{R}^2_+)$, there exists $\chi \in \mathcal{C}^1_0(\mathbb{R}^3_+)$ such that for all $t_1, t_2, t_3 \geq 0$ we have*

$$0 = D_m\chi(0, t_2, t_3) = D_m(t_1, 0, t_3) = D_m(t_1, t_2, 0) \quad (m = 1, 2),$$
$$h(t_1, t_2) = D_3\chi(t_2, t_2, 0), \quad 0 = D_3\chi(0, t_2, t_3) = D_3\chi(t_1, 0, t_3).$$

Proof. For $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_+$, define the smoothing

$$\widehat{h}(\mathbf{t}) = \frac{1}{t_3^2} \int_{s_1=t_1}^{t_1+t_3} \int_{s_2=t_2}^{t_2+t_3} h(s_1, s_2)\, ds_2 ds_1 \quad (t_3 > 0), \qquad \widehat{h}(\mathbf{t}) = h(t_1, t_2) \quad (t_3 = 0)$$

of $h(t_1,t_2,0)$ and let us fix a function $\phi \in \mathcal{C}^1\big([0,1]\big)$ (e.g. $\phi(t)=3t^2-2t^3$) with bounded derivative and being such that $\phi(0)=\phi'(0)=\phi'(1)=0$ and $\phi(1)=1$. We show that the function

$$\chi(t_1,t_2,t_3)=\phi\Big(\frac{t_1}{t_1+t_3}\Big)\phi\Big(\frac{t_2}{t_2+t_3}\Big)t_3\widehat{h}(t_1,t_2,t_3) \qquad (t_1,t_2\geq 0,\ t_3>0)$$

extended with $\chi(t_1,t_2,0)=0$ suits the requirements of the lemma.

It is folklore that, by the Newton-Leibniz theorem, the function $\widehat{h}$ is continuous, moreover its restriction to $\mathbb{R}^2_+\times\mathbb{R}_{++}\big(=\{(t_1,t_2,t_3): t_1,t_2\geq 0,\ t_3>0\}\big)$ is $\mathcal{C}^1$ smooth. In particular, for any point $\mathbf{t}=(t_1,t_2,t_3)$ with $t_3>0$ and $t_1=0$ or $t_2=0$ we have $\chi(\mathbf{t})=0$ and $\chi'(\mathbf{t})=0$. Furthermore, for any point $\mathbf{t}\in\mathbb{R}_+\times\mathbb{R}_{++}$, with the indices $m=1,2$ resp. 3 we can write

$$\begin{aligned}
D_m\chi(\mathbf{t}) &= \phi\Big(\frac{t_{3-m}}{t_{3-m}+t_3}\Big)\Big[\phi'\Big(\frac{t_m}{t_m+t_3}\Big)\frac{t_3^2}{(t_m+t_3)^2}\widehat{h}(\mathbf{t})+\phi\Big(\frac{t_m}{t_m+t_3}\Big)\frac{\partial[t_3\widehat{h}]}{\partial t_1}\Big],\\
D_3\chi(\mathbf{t}) &= -\phi'\Big(\frac{t_1}{t_1+t_3}\Big)\phi\Big(\frac{t_2}{t_2+t_3}\Big)\frac{t_1t_3}{(t_1+t_3)^2}\widehat{h}(\mathbf{t})\ -\\
&-\phi\Big(\frac{t_1}{t_1+t_3}\Big)\phi'\Big(\frac{t_2}{t_2+t_3}\Big)\frac{t_2t_3}{(t_2+t_3)^2}\widehat{h}(\mathbf{t})+\phi\Big(\frac{t_1}{t_1+t_3}\Big)\phi\Big(\frac{t_2}{t_2+t_3}\Big)\frac{\partial[t_3\widehat{h}]}{\partial t_3}.
\end{aligned}$$

Therefore it only remains to prove that given any point $\mathbf{t}^*=(t_1^*,t_2^*,0)\in\mathbb{R}^3_+$ we have

$$\begin{aligned}
&D_m\chi(\mathbf{t})\to 0\ (m=1,2), \quad D_3\chi(\mathbf{t})\to h(t_1^*,t_2^*)\\
&\text{whenever} \quad \mathbb{R}^2_+\times\mathbb{R}_{++}\ni\mathbf{t}\to\mathbf{t}^*.
\end{aligned} \tag{3.11}$$

Actually, these relation follow from the mean value expressions

$$\begin{aligned}
\frac{\partial[t_3\widehat{h}]}{\partial t_m} &= \frac{1}{t_3}\int_{s_2=t_{3-m}}^{t_{3-m}+t_3}\big[h(t_m+t_3,s_2)-h(t_m,s_2)\big]\,ds_2 =\\
&= h\big(t_m+t_3,r_{3-m}(\mathbf{t})\big)-h\big(t_m,q_{3-m}(\mathbf{t})\big),\\
\frac{\partial[t_3\widehat{h}]}{\partial t_3} &= -\frac{1}{t_3^2}\int_{s_1=t_1}^{t_1+t_3}\int_{s_2=t_2}^{t_2+t_3}h(s_1,s_2)\,ds_2\,ds_1\ +\\
&\quad+\frac{1}{t_3}\int_{s_2=t_2}^{t_2+t_3}h(t_1+t_3,s_2)\,ds_2\ +\frac{1}{t_3}\int_{s_1=t_1}^{t_1+t_3}h(s_1,t_2+t_3)\,ds_1 =\\
&= -h\big(p_1(\mathbf{t}),p_2(\mathbf{t})\big)+h\big(t_1+t_3,r_2(\mathbf{t})\big)+h\big(r_1(\mathbf{t}),t_2+t_3\big)
\end{aligned}$$

with suitable

$$p_1(\mathbf{t}),q_1(\mathbf{t}),r_1(\mathbf{t})\in[t_1,t_1+t_3], \quad p_2(\mathbf{t}),q_2(\mathbf{t}),r_2(\mathbf{t})\in[t_2,t_2+t_3].$$

Indeed, let $\mathbb{R}_+^2\times\mathbb{R}_{++}\ni\mathbf{t}\to\mathbf{t}^*=(0,t_2^*,t_3^*)$. Then $\mathbf{p}(\mathbf{t})\to(t_1^*,t_2^*)$, $\mathbf{q}_m(\mathbf{t}),\mathbf{r}_m(\mathbf{t})\to t_m^*$ $(m=1,2)$, whence $\partial[t_3\widehat{h}]/\partial t_m\to 0$ $(m=1,2)$ and $\partial[t_3\widehat{h}]/\partial t_3\to h(t_1^*,t_2^*)$. Since the function $(s,t)\mapsto s/(s{+}t)$ is analytic on the open half plane $\{(s,t)\in\mathbb{R}^2 : s{+}t>0\}$ containing the rays $R_1=\mathbb{R}_{++}\times\{0\}$ resp. $R_2=\{0\}\times\mathbb{R}_{++}$, (3.11) is immediate in the cases $\mathbf{t}^*\in R_m$ $(m=1,2)$. If $\mathbb{R}_+^2\times\mathbb{R}_{++}\ni\mathbf{t}\to(0,0,0)$, we can deduce (3.11) from the facts that, for $\ell=1,2,3$, the functions $t_\ell t_3/(t_\ell+t_3)^2, t_3^2/(t_\ell+t_3)^2$ resp. $\Phi\big(t_\ell/(t_\ell{+}t_3)\big),\Phi'\big(t_\ell/t_\ell{+}t_3)\big)$ are bounded, furthermore $\partial[t_3\widehat{h}]/\partial t_\ell\to h(0,0)=0$. $\square$

Corollary. 3.12. *By Theorem* 3.5 *and Lemma* 3.10*, for any admissible pair* $[\Psi_0,\Psi_1]$*, there exist* $\chi_0,\chi_1\in\mathcal{C}_0^1\big(\mathbb{R}_+^3\setminus\{(0,0,0\}\big)$ *such that* $[\Psi_0,\Psi_1,\chi_0,\chi_1]$ *is an RSD*$_*$*-tuple.*

Remark. 3.13. It is an immediate corollary of Theorem 3.5 that given any RSD$_*$ tuple $[\Phi_0,\Psi_1,\chi_0,\chi_1]$, $[\Psi_0,\Psi_0,\chi_0,\chi_0(t_1,t_2,t_3)t_2]$ is also an RSD$_*$ tuple.

Unfortunately, our convolution type generic construction provided by the proof of Lemma 3.10 is far from being optimal in most cases from numerical view points. In contrast, in view of Theorem 3.5, in many cases we may apply the following satisfactory construction:

Proposition. 3.14. *If* $[\Psi_0,\Psi_1]$ *is an admissible pair of shape functions and* $h_0,h_1\in\mathcal{C}_0^1(\mathbb{R}_+^2)$ *are functions such that* $h_0(t,1-t)=\Psi_0'(t)$ *and* $h_1(t,1-t)=\Psi_1'(t)(1-t)$ $(0\le t\le 1)$ *then* $[\Psi_0,\Psi_1,\chi_0,\chi_1]$ *is an RSD*$_*$ *tuple with*

$$\chi_r(t_1,t_2,t_3)=h_r(t_1,t_2)t_3 \qquad (r=0,1).$$

Example. 3.15. Cases with a factorization

$$\Psi_0'(t)=w_{01}(t)w_{02}(1-t),\quad \Psi_1'(t)(1-t)=w_{11}(t)w_{12}(1-t)$$

for suitable functions $w_{rk}\in\mathcal{C}_0^1(\mathbb{R}_+)$ such that $w_{rk}(0)=w_{rk}'(0)=0$.

(a) The procedure (1.1) with the shape functions (1.2) corresponds to the case $[\Psi_0,\Psi_1]=[\Phi,\Theta]$, $\chi_0(t_1,t_2,t_3)=30t_1^2t_2^2t_3$, $\chi_1(t_1,t_2,t_3)=12t_1^2t_2^2t_3$ with $w_{01}(t)=30t^2$, $w_{11}(t)=12t^2$, $w_{02}(t)=w_{12}(t)=t^2$ since $\Phi'(t)=30t^2(1-t)^2$ and $\Theta'(t)=12t^2(1-t)$.

(b) $\big[\Phi,\Phi,30t_1^2t_2^2t_3,30t_1^2t_2^3t_3\big]$ is an RSD$_*$ tuple by Proposition 3.14, corresponding to the case $w_{01}(t)=w_{11}(t)=30t^2$, $w_{02}(t)=t^2$, $w_{12}(t)=t^3$.

(c) Let $\Psi_H(t)=3t^2-2t^3$ be the 1-dimensional basic Hermite polynomial. Then $\big[\Psi_H,\Psi_H,\chi_H,\chi_H(t_1,t_2,t_3)t_3\big]$ is an RSD$_*$ tuple with the choice

$$\chi_H(t_1,t_2,t_3)=\frac{3t_3}{2}\prod_{k=1}^{2}\left[\frac{3t_k^2}{(t_k+t_3)^2}-\frac{2t_k^3}{(t_k+t_3)^3}\right](2t_k+t_3)$$

obtained by taking $h(t_1,t_2) = 6t_1t_2$ and $\phi = \Psi_H$ for smoothing weight function. Notice that, by the classification in [8], there is no RSD tuple of the form $[\Psi_H, \Psi_H, *, *]$ since $\deg(\Psi_H) < 5$.

(d) As for an RSD$_*$ tuple with non-polynomial shape functions, the perhaps simplest example is $\big[\Psi_S, \Psi_S, \chi_S, \chi_S(t_1,t_2,t_3)t_2\big]$ with $\Psi_S(t) = \sin^2(\pi t/2)$ and

$$\chi_S(t_1,t_2,t_3) = \frac{16}{\pi t_3}\sin^2\Big(\frac{\pi}{4}t_3\Big)\prod_{k=1}^{2}\sin^2\Big(\frac{\pi}{2}\frac{t_k}{t_k+t_3}\Big)\sin\Big(\frac{\pi}{4}(2t_k+t_3)\Big)$$

obtained with the construction in the proof of Lemma 3.5 using $h(t_1,t_2) = \pi\sin\big(\pi t_1/2\big)\sin\big(\pi t_2/2\big)$ and $\Phi(t) = \Psi_S(t)$, respectively.

(e) Since $16(\pi t_3)^{-1}\sin^2(\pi t_3/4) = \pi t_3 + o(t_3^2) = 2\sin(\pi t_3/2) + o(t_3^2)$, the function

$$\widetilde{\chi}_S(t_1,t_2,t_3) = 2\sin\Big(\frac{\pi}{2}t_3\Big)\prod_{k=1}^{2}\sin^2\Big(\frac{\pi}{2}\frac{t_k}{t_k+t_3}\Big)\sin\Big(\frac{\pi}{4}(2t_k+t_3)\Big)$$

also satisfies the relations (3.6) and (3.7). That is $\big[\Psi_S, \Psi_S, \widetilde{\chi}_S, t_2\widetilde{\chi}_S\big]$ is an RSD$_*$ triple with algebraically simpler expressions for the shape functions in (d).

(f) By restricting χ_S to the simplex Δ_3, due to the relations $\sum_i t_i = 1$ $\big((t_1,t_2,t_3)\in\Delta_3\big)$, we can write $\prod_{k=1}^{2}\sin\big((\pi/4)(2t_k+t_3)\big) = \cos\big((\pi/2)(t_1-t_2)\big)/2$. Thus the tuple $\mathbf{\Pi}_{S\Delta} = \big[\Psi_S, \Psi_S, \chi_{S\Delta}, t_2\chi_{S\Delta}\big]$ defined in terms of the expression

$$\chi_{S\Delta}(t_1,t_2,t_3) = \sin\Big(\frac{\pi}{2}t_3\Big)\cos\Big(\frac{\pi}{2}(t_1-t_2)\Big)\prod_{k=1}^{2}\sin^2\Big(\frac{\pi}{2}\frac{t_k}{1-t_{3-k}}\Big)$$

is an RSD$_*$ tuple: despite its different algebraic form, it fulfills the marginal conditions (3.6), (3.7). By replacing the variables t_i with the weight functions λ_i, we obtain the same interpolating functions as with the formulas in (e).

(g) Given any RSD$_*$ tuple $\mathbf{\Pi} = \big[\Psi_0, \Psi_1, \chi_0, \chi_1\big]$ along with a function $\vartheta \in \mathcal{C}^1(\mathbb{R})$ such that $\vartheta(1) = 0 \neq \vartheta'(1)$, its perturbation $\mathbf{\Pi}_\vartheta = \big[\Psi_0, \Psi_1, \chi_0 + \vartheta(s), \chi_1\big]$ where $s = t_1 + t_2 + t_3$ is an RSD tuple but no RSD$_*$ tuple.

Remark. 3.16. It is an easy consequence of Whitney's extension theorem [9] that for any function $\psi \in \mathcal{C}([0,1])$ with $\psi(0) = \psi'(0) = \psi'(1) = 0$ there exists a function $h \in \mathcal{C}_0^1(\mathbb{R}_+^2)$ such that $\psi'(t) = h(t, 1-t)$. Hence every admissible pair $[\Psi_0, \Psi_1]$ with the property $\Psi_0'(1) = 0$ admits functions $h_0, h_1 \in \mathcal{C}(\mathbb{R}_+^2)$ such that $[\Psi_0, \Psi_1, h_0(t_1,t_2)t_3, h_1(t_1,t_2)t_3]$ is an RSD$_*$ tuple.

We complete this section with a brief description of the local approximation operator corresponding to Theorem 3.5 and its use in constructing $\mathcal{C}^1$-splines over a 2D triangular mesh analogously as done in [7, Algorithm].

Definition. 3.17. Let $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ be an RSD tuple and write $\mathbf{\Psi} = [\Psi_0, \Psi_1]$ resp. $\boldsymbol{X} = [\chi_0, \chi_1]$. We introduce the *RSD modification of* $\mathfrak{F}^{\mathbf{\Psi}}$ *by means of the complementary shape functions* $\boldsymbol{X}$ as the operator

$$\mathfrak{F}^{\mathbf{\Pi}}\colon \big\{(\mathbf{p}_k,f_k,A_k,\mathbf{u}_k)\big\}_{k=1}^3 \mapsto \mathfrak{F}^{\mathbf{\Psi}}\big\{(\mathbf{p}_k,f_k,A_k)\big\}_{k=1}^3 - \mathfrak{H}^{\boldsymbol{X}}\big\{(\mathbf{p}_k,f_k,A_k,\mathbf{u}_k)\big\}_{k=1}^3$$

defined for all tuples $\big\{(\mathbf{p}_k,f_k,A_k,\mathbf{u}_k)\big\}_{k=1}^3$ where $\big\{(\mathbf{p}_k,f_k,A_k)\big\}_{k=1}^3$ is a non-degenerate function germ and $\mathbf{u}_k \not\parallel \mathbf{p}_i - \mathbf{p}_j$ $\big((i,j,k) \in S_3\big)$ and

$$\mathfrak{H}^{\boldsymbol{X}}\big\{(\mathbf{p}_k,f_k,A_k,\mathbf{u}_k)\big\}_{k=1}^3 = \Big[\ H \text{ in } (1.3)\ \Big].$$

Recall [7] that, given a set $\mathbf{V} = \big\{\mathbf{v}_1, \ldots, \mathbf{v}_R\big\}$ of points in $\mathbb{R}^2$, by a *triangular mesh* over $\mathbf{V}$ we mean a family $\big\{\mathbf{T}_1, \ldots, \mathbf{T}_M\big\} \subset \mathbb{R}^2$ of non-degenerate triangles of the form

$$\mathbf{T}_m = \mathrm{Conv}\big\{\mathbf{v}_{i(m,1)}, \mathbf{v}_{i(m,2)}, \mathbf{v}_{i(m,3)}\big\}, \quad i(m,1) < i(m,2) < i(m,3)$$

with pairwise disjoint interiors whose intersections are either empty or a common vertice or a common edge, furthermore $\mathbf{V} \subset \bigcup_{m=1}^M \mathbf{T}_m$. A crucial ingredient of the construction, we enumerate the edges of the mesh triangles $\mathbf{T}_m$ in the form $\mathbf{E}_1, \ldots, \mathbf{E}_S$ and associate a vector $\mathbf{u}_s \not\parallel \mathbf{E}_s$ with each edge $\mathbf{E}_s$, furthermore let $s(m,k)$ denote the index of the *opposite* edge to the vertex $\mathbf{v}_{m(i,k)}$ in the triangle $\mathbf{T}_m$. By replacing Φ, Θ with Ψ_0, Ψ_1, respectively the functions $30\lambda_i^2\lambda_j^2\lambda_k$, $12\lambda_i^2\lambda_j^2\lambda_k$ with χ_0, χ_1 in a straightforward manner in [7, Proof of Thm. 2], we can conclude the following: Given any germ $\big\{(\mathbf{v}_i, f_i, A_i)\big\}_{i=1}^R$ of first order function data over $\mathbf{V}$, the union of the functions

$$F_m = \mathfrak{F}^{\mathbf{\Pi}}\Big\{\big(\mathbf{v}_{i(m,k)}, f_{i(m,k)}, A_{i(m,k)}, \mathbf{u}_{s(m,k)}\big)\Big\}_{k=1}^3 \in \mathcal{C}^1(\mathbf{T}_m) \quad (m=1,\ldots,M)$$

is continuously differentiable on the mesh domain $\mathbf{D} = \bigcup_{m=1}^M \mathbf{T}_m$.

4. Range shift property

One of the most frequent applications of splines over triangular meshes is reconstructing approximately a smooth surface with a function graph passing through a family of points. Such procedures correspond to a model of the following pattern: We are given a smooth surface $\mathbf{S} \subset \mathbb{R}^3$ which can be represented in the form $\mathbf{S} = \mathrm{graph}(f) = \big\{\big[\mathbf{p}, f(\mathbf{p})\big] : \mathbf{p} \in \mathbb{R}^2\big\}$ with some (a priori unknown) function $f \in \mathcal{C}^1(\mathbb{R}^2)$. Accurate data (obtained with scanner equipments) are available for a finite family of points $\mathbf{v}_1, \ldots, \mathbf{v}_N \in \mathbb{R}^2$ concerning the *first order data* $f^{[1]}(\mathbf{p}) = \big[\mathbf{p}, f(\mathbf{p}), f'(\mathbf{p})\big]$ and we approximate a piece of $\mathbf{S}$ with the graph of a $\mathcal{C}^1$-spline function $F \in \mathcal{C}^1(\mathbf{D})$ on the domain $\mathbf{D}$ of a triangular mesh with vertices $\mathbf{v}_n$. The procedures used in constructing F are usually investigated from the view points of estimating various biases between graph(F) and $\mathbf{S}$, respectively from exhibiting features of algebraic-geometric character. In this note we only discuss two kinds of the last category in the context of RSD methods: if the spline graps are invariant with respect to tranlations of first order data and the property that plane surfaces remain invariant.

Definition. 4.1. Throughout this section $[\mathbf{P}, \mathbf{U}]$ denotes an admissible pair of triples $\mathbf{P} = \big\{\mathbf{p}_i\big\}_{i=1}^3, \mathbf{U} = \big\{\mathbf{u}_i\big\}_{i=1}^3$ with $\mathbf{p}_i, \mathbf{u}_i \in \mathbb{R}^2$. We write $\mathbf{T} = \mathrm{Conv}(\mathbf{P})$ and $\lambda_1, \lambda_2, \lambda_3$ resp. G_1, G_2, G_3 for the weight functions resp. their derivatives associated with the vertices of the triangle $\mathbf{T}$. Given any function $f \in \mathcal{C}^1(\mathbb{R}^2)$, we shall use the shorthand notations

$$f^{[1]}_{\mathbf{P},\mathbf{U}} = \big\{\big[\mathbf{p}_i, f(\mathbf{p}_i), f'(\mathbf{p}_i), \mathbf{u}_i\big]\big\}_{i=1}^3, \quad \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}} f = \mathfrak{F}^{\mathbf{\Pi}} f^{[1]}_{\mathbf{P},\mathbf{U}}.$$

An RSD tuple $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ has the *range shift property* if the procedure $\mathfrak{F}^{\mathbf{\Pi}}$ leaves the constant functions invariant, that is if for all admissible pairs $\mathbf{P}, \mathbf{U}$ and the constant unit function $\mathbf{1} : \mathbb{R}^2 \ni \mathbf{x} \mapsto 1$ we have

$$\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}} \mathbf{1} = \mathbf{1}_{|\mathbf{T}}$$

Remark. 4.2. Since $\mathbf{1}(\mathbf{p}_i) = 1$, $\mathbf{1}'(\mathbf{p_i})\mathbf{u}_i = 0$, we can write

$$\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}} \mathbf{1} = \sum_{i=1}^3 \Psi_0(\lambda_i) - \sum_{(\ell,m,n)\in \mathrm{S}_3} \frac{G_\ell \mathbf{u}_n}{G_n \mathbf{u}_n} \chi_0(\lambda_\ell, \lambda_m, \lambda_n) =$$
$$= \sum_{n=1}^3 \left\{ \Psi_0(\lambda_n) - \left[\frac{G_{\ell_n} \mathbf{u}_n}{G_n \mathbf{u}_n} \chi_0(\lambda_{\ell_n}, \lambda_{m_n}, \lambda_n) + \frac{G_m \mathbf{u}_n}{G_n \mathbf{u}_n} \chi_0(\lambda_{m_n}, \lambda_{\ell_n}, \lambda_n) \right] \right\}$$

with $(\ell_1, m_1) = (2,3)$, $(\ell_2, m_2) = (1,3)$, $(\ell_3, m_3) = (1,2)$.

Lemma. 4.3. *For any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ there exist $\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1 \in \mathbb{R}^2$ such that*

$$\left[\frac{G_i \mathbf{u}_j}{G_j \mathbf{u}_j}\right]_{i,j=1}^3 = \begin{bmatrix} 1 & -1-\alpha_2 & \alpha_3 \\ \alpha_1 & 1 & -1-\alpha_3 \\ -1-\alpha_1 & \alpha_2 & 1 \end{bmatrix}.$$

Proof. A suitable choice is $\mathbf{u}_1 = \mathbf{p}_1 - \frac{1}{2}[\mathbf{p}_2 + \mathbf{p}_3] + (\frac{1}{2} + \alpha_1)[\mathbf{p}_2 - \mathbf{p}_3]$, $\mathbf{u}_2 = \mathbf{p}_2 - \frac{1}{2}[\mathbf{p}_3 + \mathbf{p}_1] + (\frac{1}{2} + \alpha_2)[\mathbf{p}_3 - \mathbf{p}_1]$, $\mathbf{u}_3 = \mathbf{p}_3 - \frac{1}{2}[\mathbf{p}_1 + \mathbf{p}_2] + (\frac{1}{2} + \alpha_3)[\mathbf{p}_1 - \mathbf{p}_2]$. □

Proposition. 4.4. *An RSD tuple* $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ *has range shift property if and only if*

$$\begin{aligned} &\Psi_0(t) + \Psi_0(1-t) = 1, \quad \Psi_0'(t) = \Psi_0'(1-t) && (0 \le t \le 1), \\ &\chi_0(t_1, t_2, t_3) = \chi_0(t_2, t_1, t_3) && \big((t_1, t_2, t_3) \in \Delta_3\big) \end{aligned} \tag{4.5}$$

and $$\mathbf{1} = \sum_{i=1}^{3} \Psi_0(\lambda_i) + \sum_{(\ell,m,n)\in \mathrm{S}_3^+} \chi_0(\lambda_\ell, \lambda_m, \lambda_n).$$

Proof. Consider the points $\mathbf{x}_t = t\mathbf{p}_1 + (1-t)\mathbf{p}_2$ $(0 \le t \le 1)$. Since $\mathbf{1}' \equiv 0$, $\mathbf{1}(\mathbf{x}_t) = 1$, $\lambda_1(\mathbf{x}_t) = t$, $\lambda_2(\mathbf{x}_t) = 1-t$, and $\lambda_3(\mathbf{x}_t) = 0$, by assumption

$$1 = \mathbf{1}(\mathbf{x}_t) = \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\mathbf{1}\,(\mathbf{x}_t) = \sum_{i=1}^{3} \Psi_0\big(\lambda_i(\mathbf{x}_t)\big) = \Psi_0(t) + \Psi_0(1-t) + \Psi_0(0)$$

where $\Psi_0(0) = 0$ due to (2.1). Thus in view of Remark 4.2 and Lemma 4.3, the expression of $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\mathbf{1}$ has the form

$$\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\mathbf{1} = \sum_{n=1}^{3} \big\{\Psi_0(\lambda_n) - \big[\alpha_n \chi_0(\lambda_{\ell_n}, \lambda_{m_n}, \lambda_n) + (-1-\alpha_n)\chi_0(\lambda_{m_n}, \lambda_{\ell_n}, \lambda_n)\big]\big\}$$

where the coefficients $\alpha_1, \alpha_2, \alpha_3$ may assume any real value. This is possible only if $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\mathbf{1} = \sum_{i=1}^{3} \Psi_0(\lambda_i) + \chi_0(\lambda_\ell, \lambda_m, \lambda_n) + \chi_0(\lambda_\ell, \lambda_m, \lambda_n) + \chi_0(\lambda_\ell, \lambda_m, \lambda_n)$ and

$$\chi_0(\lambda_{\ell_n}, \lambda_{m_n}, \lambda_n) = \chi_0(\lambda_{m_n}, \lambda_{\ell_n}, \lambda_n) \qquad (n = 1, 2, 3).$$

In particular, for $n = 3$ and given a generic point $\mathbf{x} = \sum_{k=1}^{3} t_k \mathbf{p}_k$ $\big($with $\sum_k t_k = 1,\ t_k \ge 0\big)$ of the triangle $\mathbf{T}$ we get $\chi_0(t_1, t_2, t_3) = \chi_0\big(\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})\big) = \chi_0\big(\lambda_2(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_3(\mathbf{x})\big) = \chi_0(t_2, t_1, t_3)$ which completes the proof. □

Lemma. 4.6. *Given any smooth function* ϕ *defined on a domain* $\Omega \subset \mathbb{R}^3$ *containing* Δ_3 *and given any point* $\mathbf{x} \in \mathbf{T}$, *for the function* $f = \phi(\lambda_1, \lambda_2, \lambda_3)$ *we have* $f'(\mathbf{x}) = 0$ *if and only if*

$$D_1\phi(\mathbf{t}) = D_2\phi(\mathbf{t}) = D_3\phi(\mathbf{t}) \quad \text{where} \quad \mathbf{t} = \big(\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})\big).$$

PROOF. For any vector $\mathbf{u}\in\mathbb{R}^2$ we have $f'(\mathbf{x})\mathbf{u}=\sum_{k=1}^3 D_k\phi(\mathbf{t})G_k\mathbf{u}$. Hence the statement is immediate from the relation $\big\{(G_1\mathbf{u},G_2\mathbf{u},G_3\mathbf{u}) : \mathbf{v}\in\mathbb{R}^2\big\} = \big\{(\alpha_1,\alpha_2,\alpha_3) : \sum_k \alpha_k = 0\big\}$. $\big($Actually, given any triple $(i,j,k)\in S_3$ of indices, by taking $\mathbf{u}=\mathbf{p}_i-\mathbf{p}_j$ we have $G_i\mathbf{u}=1, G_j\mathbf{u}=-1, G_k\mathbf{u}=0$ implying $D_i\phi(\mathbf{t}) = D_j\phi(\mathbf{t})$ if $f'(\mathbf{x})\mathbf{u}=0\big)$. □

Corollary. 4.7. *An RSD$_*$ tuple* $\mathbf{\Pi}=[\Psi_0,\Psi_1,\chi_0,\chi_1]$ *has range shift property if and only if* (4.5) *holds and*

$$\frac{\partial\Sigma}{\partial t_1}(t_1,t_2,t_3)=\frac{\partial\Sigma}{\partial t_2}(t_1,t_2,t_3)=\frac{\partial\Sigma}{\partial t_3}(t_1,t_2,t_3)\quad\big((t_1,t_2,t_3)\in\Delta_3\big), \tag{4.8}$$

$$\Sigma(t_1,t_2,t_3)=\sum_{i=1}^3\Psi_0(t_i)+\sum_{(\ell,m,n)\in S_3^+}\chi_0(t_\ell,t_m,t_n),\quad S_3^+=\{(1,2,3),(2,3,1),(3,1,2)\}.$$

PROOF. We established that $\mathbf{\Pi}$ has range shift property if and only if

$$\Sigma\big(\lambda_1(\mathbf{x}),\lambda_2(\mathbf{x}),\lambda_3(\mathbf{x})\big)=1 \qquad \big(\mathbf{x}\in\mathrm{Conv}(\mathbf{P})\big). \tag{4.9}$$

Since $\mathbf{\Pi}$ is an RSD tuple, in any case we have $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}(\mathbf{p}_i)=1$ $(i=1,2,3)$. Hence the relation (4.9) holds if and only if the Fréchet derivative of the function $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\mathbf{1}$ vanishes, that is if

$$\big[\Sigma(\lambda_1,\lambda_2,\lambda_3)\big]'(\mathbf{x})=0 \qquad \big(\mathbf{x}\in\mathrm{Conv}(\mathbf{P})\big). \tag{4.9$'$}$$

An application of Lemma 4.6 with $\phi=\Sigma$ completes the proof. □

Example. 4.10. The RSD procedures corresponding to tuples of the form $\big[t^3(10-15t+6t^2),*,30t_1^2t_2^2t_3,*\big]$ as in Examples 3.15 have range shift property. This highly non-trivial fact can easily be established by verifying (4.8) as follows. In such cases, we have

$$\Sigma(t_1,t_2,t_3)=\sum_{i=1}^3 t_i^3(10-15t_i+6t_i^2)+30\big(t_1^2t_2^2t_3+t_2^2t_3^2t_1+t_3^2t_1^2t_2\big)$$

and it suffices to show that the polynomial $\frac{\partial\Sigma}{\partial t_1}$ is symmetric in t_1,t_2,t_3 when replacing $t_1=1-t_2-t_3$. Since $\big[t^3(10-15t+6t^2)\big]'=30t^2(1-t)^2$, if $(t_1,t_2,t_3)\in\Delta_3$ then $\big[t_1^3(10-15t_1+6t_1^2)\big]'=30t_1^2(t_2+t_3)^2$ and

$$\begin{aligned}\frac{1}{30}\frac{\partial\Sigma}{\partial t_1}&=t_1^2(t_2+t_3)^2+2t_1t_2^2t_3+t_2^2t_3^2+2t_3^2t_1t_2=\\&=t_1^2t_2^2+t_1^2t_3^2+t_2^2t_3^2+2t_1t_2t_3(t_1+t_2+t_3)=t_1^2t_2^2+t_1^2t_3^2+t_2^2t_3^2+2t_1t_2t_3.\end{aligned}$$

Definition. 4.11. We say that a function $\Psi \in \mathcal{C}^1([0,1])$ is *d-symmetric* if

$$\Psi'(t) = \Psi'(1-t) \quad (0 \le t \le 1).$$

Corollary. 4.12. *If* $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ *is an RSD tuple with d-symmetric shape function* Ψ_0 *then* $\mathbf{\Pi}^{[s]} = [\Psi_0, \Psi_1, \chi_0^{[s]}, \chi_1]$ *with*

$$\chi_0^{[s]}(t_1,t_2,t_3) = \frac{1}{2}\chi_0(t_1,t_2,t_3) + \frac{1}{2}\chi_0(t_2,t_1,t_3) \tag{4.13}$$

is also an RSD tuple such that

$$\mathfrak{F}_{\mathbf{P},\mathbf{U}}^{\mathbf{\Pi}^{[s]}}\mathbf{1} = \sum_{(\ell,m,n)\in S_3} \Big\{\frac{1}{2}\Psi_0(\lambda_\ell) + \chi^{[s]}(\lambda_\ell,\lambda_m,\lambda_n)\Big\}. \tag{4.14}$$

Proof. The marginal conditions (3.7) for $\chi_0^{[s]}$ are immediate. As a consequence of the d-symmetry of Ψ_0, we have

$$\begin{aligned} D_3\chi_0^{[s]}(t,1-t,0) &= \frac{1}{2}D_3\chi_0(t,1-t,0) + \frac{1}{2}D_3\chi_0(1-t,t,0) = \\ &= \frac{1}{2}\Psi_0'(t) + \frac{1}{2}\Psi_0'(1-t) = \Psi_0'(t) \end{aligned}$$

establishing that $\mathbf{\Pi}^{[s]}$ is an RSD tuple. Due to the symmetry of $\chi_0^{[s]}$ in the variables t_1, t_2 in Remark 4.2 applied to $\chi_0^{[s]}$, we can write

$$\mathfrak{F}_{\mathbf{P},\mathbf{U}}^{\mathbf{\Pi}}\mathbf{1} = \sum_{n=1}^{3} \left\{\Psi_0(\lambda_n) - \left[\left[\frac{G_{\ell_n}\mathbf{u}_n}{G_n\mathbf{u}_n} + \frac{G_{\ell_m}\mathbf{u}_n}{G_n\mathbf{u}_n}\right]\chi_0(\lambda_{\ell_n},\lambda_{m_n},\lambda_n)\right]\right\}.$$

Taking Lemma 4.3 into account, we obtain (4.14). □

Theorem. 4.15. *Let* $[\Psi_0, \Psi_1]$ *be an admissible pair of shape functions, with* Ψ_0 *being d-symmetric. Then we can find* $\chi_0^* \in \mathcal{C}_0(\mathbb{R}_+^3)$ *such that* $\mathbf{\Pi}^* = [\Psi_0, \Psi_1, \chi_0^*, \chi_1^*]$ *is an* RSD_* *tuple with range shift property where* χ_0^* *is symmetric in its first two variables.*

Proof. According to Theorem 3.5 there exists an RSD_* tuple of the form $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ with $\chi_0, \chi_1 \in \mathcal{C}_0^1(\mathbb{R}_0^3)$. By Corollary (4.12), $\mathbf{\Pi}^{[s]} = [\Psi_0, \Psi_1, \chi_0^{[s]}, \chi_1]$ is also an RSD_* tuple where the complementary shape function $\chi_0^{[s]}$ is symmetric in its first two variables and such that (4.14) holds. Then, independently of the choice of the vector triple $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, we can write the difference function of $\mathbf{1}$ and $\mathfrak{F}^{\mathbf{\Pi}^{[s]}}_{\mathbf{P},\mathbf{U}}\mathbf{1}$ in the form

$$\begin{aligned} &d = \mathbf{1} - \mathfrak{F}^{\mathbf{\Pi}^{[s]}}_{\mathbf{P},\mathbf{U}}\mathbf{1} = \delta(\lambda_1, \lambda_2, \lambda_3) \quad \text{with} \\ &\delta(\mathbf{t}) = d\big(\textstyle\sum_{k=1}^3 t_k \mathbf{p}_k\big) = 1 - \displaystyle\sum_{i=1}^3 \Psi_(t_i) - \sum_{(\ell,m,n)\in \mathrm{S}_+} \chi_0(t_\ell, t_m, t_n). \end{aligned} \tag{4.16}$$

Observe that

$$d(\mathbf{x}) = 0, \quad d'(\mathbf{x}) = 0 \qquad (\mathbf{x} \in \partial \mathbf{T}). \tag{4.17}$$

Indeed, at the generic point (2.9) of the edge opposite to the vertex $\mathbf{p}_k$, by (2.2) we have $d(\mathbf{x}^k_t) = 1 - [\Psi_0(t) + \Psi_0(1-t)] = 0$. In particular the functions $\mathbf{1}$ and $\mathfrak{F}^{\mathbf{\Pi}^{[s]}}_{\mathbf{P},\mathbf{U}}$ coincide along the edge $[\mathbf{p}_i, \mathbf{p}_j]$, thus $d'(\mathbf{x}^k_t)(\mathbf{p}_j - \mathbf{p}_i) = 0$ with $d(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial \mathbf{T}$. On the other hand, since trivially $\mathbf{1}'(\mathbf{p}) = 0$ everywhere, the RSD construction ensures that $d'(\mathbf{x}^k_t)\mathbf{u}_k = \mathbf{1}'(\mathbf{x}^k_t)\mathbf{u}_k - \big[\Psi_0(t)1'(\mathbf{p}_k))\mathbf{u}_k + \Psi_0(1-t)1'(\mathbf{p}_k))\mathbf{u}_k\big] = 0$. Thus $\delta'(\mathbf{x}^k_t)$ vanishes in two linearly independent directions implying that $d'(\mathbf{x}^k_t)\mathbf{u} = 0$ $(\mathbf{u} \in \mathbb{R}^2)$ i.e. $d'(\mathbf{x}^k_t) = 0$.

Notice that $\mathbf{x} \in \partial \mathbf{T}$ if and only if $\big(\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})\big) \in \bigcup_{k=1}^3 \Delta_{3,k}$. Hence, due to (4.17), we can apply Lemma 4.6 with $f = d$ and $\phi = \delta$ to conclude that

$$\delta(\mathbf{t}) = 0, \quad D_1\delta(\mathbf{t}) = D_2\delta(\mathbf{t}) = D_3\delta(\mathbf{t}) \quad \text{whenever} \quad \mathbf{t} \in \bigcup_{k=1}^3 \Delta_{3,k}. \tag{4.17$'$}$$

On the domain $\Omega := \mathbb{R}^3_+ \setminus \{0\}$, define

$$\begin{aligned} &\chi_0^* = \chi_0^{[s]} + \frac{1}{3}\widehat{\delta} \quad \text{where} \\ &\widehat{\delta}(\mathbf{t}) = \delta\Big(\frac{t_1}{t_1+t_2+t_3}, \frac{t_2}{t_1+t_2+t_3}, \frac{t_3}{t_1+t_2+t_3}\Big). \end{aligned}$$

By writing $s(\mathbf{t}) = t_1 + t_2 + t_3$ for short, for $k = 1, 2, 3$ we have

$$\begin{aligned} D_k\widehat{\delta}(\mathbf{t}) &= \frac{\partial}{\partial t_k}\delta\Big(\frac{t_1}{s(\mathbf{t})}, \frac{t_2}{s(\mathbf{t})}, \frac{t_3}{s(\mathbf{t})}\Big) = \\ &= D_1\delta\Big(\frac{\mathbf{t}}{s(\mathbf{t})}\Big)\frac{\partial}{\partial t_k}\frac{t_1}{s(\mathbf{t})} + D_2\delta\Big(\frac{\mathbf{t}}{s(\mathbf{t})}\Big)\frac{\partial}{\partial t_k}\frac{t_2}{s(\mathbf{t})} + D_3\delta\Big(\frac{\mathbf{t}}{s(\mathbf{t})}\Big)\frac{\partial}{\partial t_k}\frac{t_3}{s(\mathbf{t})}. \end{aligned}$$

In particular, if $0 \neq \mathbf{t} \in \mathbb{R}^3_+$ then $s(\mathbf{t})^{-1}\mathbf{t} \in \Delta_3$, furthermore $s(\mathbf{t})^{-1}\mathbf{t} \in \bigcup_{k=1}^3 \Delta_{3,k}$ whenever $0 \neq \mathbf{t} \in \partial\mathbb{R}^3_+$. Thus, as a consequence of (4.17), we get

$$D_k\widehat{\delta}(\mathbf{t}) = \sum_{\ell=1}^3 D_1\delta\Big(\frac{\mathbf{t}}{s(\mathbf{t})}\Big)\frac{\partial}{\partial t_k}\frac{t_\ell}{s(\mathbf{t})} = D_1\delta\Big(\frac{\mathbf{t}}{s(\mathbf{t})}\Big)\frac{\partial}{\partial t_k}\frac{s(\mathbf{t})}{s(\mathbf{t})} = 0 \quad \big(0 \neq \mathbf{t} \in \partial\mathbb{R}^3_+\big)$$

It follows $D_k\chi_0^*(\mathbf{t}) = D_k\chi_0(\mathbf{t})$ for $0 \neq \mathbf{t} \in \partial\mathbb{R}_0^3 = \big\{(t_1,t_2,t_3) \in \mathbb{R}_+^3 : t_1t_2t_3 = 0\big\}$ and $k = 1,2,3$ whence, in view of Theorem 3.5 we conclude that $\mathbf{\Pi}^*$ with $\chi_1^* = \chi_1$ is indeed an RSD$_*$ tuple. □

Remark. 4.18. Since $s(\mathbf{t}) = 1$ for $\mathbf{t} \in \Delta_3$ and since $\sum_{k=1}^3 \lambda_k \equiv 1$ in $\mathbf{T}$, we simply have $\chi_0^*\big(\lambda_1(\mathbf{x}),\lambda_2(\mathbf{x}),\lambda_3(\mathbf{x})\big) = \big[\chi_0^{[s]} - \frac{1}{3}\delta\big]\big(\lambda_1(\mathbf{x}),\lambda_2(\mathbf{x}),\lambda_3(\mathbf{x})\big)$ for $\mathbf{x}\in\mathbf{T}$.

Example. 4.19. From the function $\chi_{S\Delta}$ in Example 3.15(f), the construction of the proof of Theorem 4.15 results in

$$\chi_{S\Delta}^* = \frac{1}{3}\left[1 - \sum_{k=1}^3 \sin^2\Big(\frac{\pi}{2}t_k\Big)\right] +$$
$$+ \sum_{(\ell,m,n)\in S_3^+} \frac{3\delta_{\ell,3}-1}{3} \sin\Big(\frac{\pi}{2}t_\ell\Big) \cos\Big(\frac{\pi}{2}(t_m - t_n)\Big) \prod_{j=m,n} \sin^2\Big(\frac{\pi}{2}\frac{t_{6-\ell-j}}{1-t_j}\Big).$$

Thus $\big[\Psi_S, \Psi_S, \chi_{S\Delta}^*, t_2\chi_{S\Delta}^*\big]$ is an RSD tuple with range shift property.

5. Affinity invariance

Definition. 5.1. An RSD tuple $\mathbf{\Pi} = [\Psi_0,\Psi_1,\chi_0,\chi_1]$ (and the related procedure $\mathfrak{F}^{\mathbf{\Pi}}$) is *affinity invariant* if

$$\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}} f = f_{|\mathrm{Conv}(\mathbf{P})}$$

for all admissible pairs $[\mathbf{P},\mathbf{U}]$ (i.e. $\mathrm{Conv}(\mathbf{P})$ is a non-degenerate triangle, $\mathbf{p}_\ell - \mathbf{p}_m \not\parallel \mathbf{p}_n$ $\big((\ell,m,n) \in S_3)\big)$ and affine (linear+constant) functions $f{:}\,\mathbb{R}^2 \to \mathbb{R}$.

Remark. 5.2. (a) Affinity invariance implies range shift property.

(b) Given any non-degenerate triangle $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\} \subset \mathbb{R}^2$, any affine function $\mathbb{R}^2 \to \mathbb{R}$ is a linear combination of the weights $\lambda_1,\lambda_2,\lambda_3$. Hence to verify the affinity invariance of $\mathbf{\Pi}$, it suffices to prove the relation $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\lambda_1(\mathbf{x}) = \lambda_1(\mathbf{x})$ $\big(\mathbf{x} \in \mathrm{Conv}(\mathbf{P})\big)$ for all admissible pairs $[\mathbf{P},\mathbf{U}]$.

(c) In view of Remark 2.4(c) and the identities $G_1(\mathbf{x} - \mathbf{p}_k) = \lambda_1(\mathbf{x}) - \delta_{1k}$ resp. $G_i(\mathbf{p}_j - \mathbf{p}_k) = \lambda_i(\mathbf{p}_j) - \lambda_i(\mathbf{p}_k) = \delta_{ij} - \delta_{ik}$, we can write

$$\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\lambda_1 = \sum_{k=1}^3 \Big[\lambda_1(\mathbf{p}_k)\Psi_0(\lambda_k) + G_1(\mathbf{x} - \mathbf{p}_k)\Psi_1(\lambda_k)\Big] -$$
$$- \sum_{(\ell,m,n)\in S_3} \frac{G_\ell\mathbf{u}_n}{G_n\mathbf{u}_n}\Big[\lambda_1(\mathbf{p}_\ell)\chi_0(\lambda_\ell,\lambda_m,\lambda_n) + G_1(\mathbf{p}_m - \mathbf{p}_\ell)\chi_1(\lambda_\ell,\lambda_m,\lambda_n)\Big] =$$

$$= \Psi_0(\lambda_1) + \Psi_1(\lambda_1)(\lambda_1 - 1) + \Psi_1(\lambda_2)\lambda_1 + \Psi(\lambda_3)\lambda_1 -$$
$$- \frac{G_1\mathbf{u}_3}{G_3\mathbf{u}_3}\Big\{\chi_0(\lambda_1,\lambda_2,\lambda_3) - \chi_1(\lambda_1,\lambda_2,\lambda_3)\Big\} - \frac{G_2\mathbf{u}_3}{G_3\mathbf{u}_3}\chi_1(\lambda_2,\lambda_1,\lambda_3) -$$
$$- \frac{G_1\mathbf{u}_2}{G_2\mathbf{u}_2}\Big\{\chi_0(\lambda_1,\lambda_3,\lambda_2) - \chi_1(\lambda_1,\lambda_3,\lambda_2)\Big\} - \frac{G_3\mathbf{u}_2}{G_2\mathbf{u}_2}\chi_1(\lambda_3,\lambda_1,\lambda_2) - 0 - 0.$$

with the ordering (1,2,3), (2,1,3), (1,3,2), (3,1,2), (2,3,1), (3,2,1) for S_3.

Lemma. 5.3. *If the RSD tuple* $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ *is affinity invariant then necessarily*

$$\begin{aligned} &\Psi_1(t) = \Psi_0(t) \ \text{ and } \ \Psi_0(t) + \Psi_0(1-t) = 1 \quad (0 \le t \le 1), \\ &\chi_0(t_1,t_2,t_3) = \chi_1(t_1,t_2,t_3) + \chi_1(t_2,t_1,t_3) \qquad \big((t_1,t_2,t_3) \in \Delta_3\big). \end{aligned} \tag{5.4}$$

Proof. Since $\mathbf{\Pi}$ is an RSD tuple, by Definition 3.2, the directional derivatives $F'(\mathbf{x}_t^3)\mathbf{u}_3$ along the side $[\mathbf{p}_2, \mathbf{p}_3]$ with generic point $\mathbf{x}_t^1 = t\mathbf{p}_2 + (1-t)\mathbf{p}_3$ of the function $F = \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\lambda_1$ are the linear combinations

$$\begin{aligned} F'(\mathbf{x}_t^1)\mathbf{u}_1 &= \Psi_1(t)F'(\mathbf{p}_2)\mathbf{u}_1 + \Psi_1(1-t)F'(\mathbf{p}_3)\mathbf{u}_1 = \\ &= \Psi_1(t)G_1\mathbf{u}_1 + \Psi_1(1-t)G_1\mathbf{u}_1 = \big[\Psi_1(t) + \Psi_1(1-t)\big]G_1\mathbf{u}_1. \end{aligned}$$

Due to the admissibility of $[\mathbf{P},\mathbf{U}]$, $G_1\mathbf{u}_1 \neq 0$. In case of the affinity invariance of $\mathbf{\Pi}$, $F = \lambda_1$ with $F'(\mathbf{x}_t^1)\mathbf{u}_1 = G_1\mathbf{u}_1$ implying $\Psi_1(t) + \Psi_1(1-t) = 1$. Also if $F = \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\lambda_1 = \lambda_1$ then, as a consequence of the relations (2.7), at the points $\mathbf{x}_t^3 = t\mathbf{p}_1 + (1-t)\mathbf{p}_2$ we have

$$\begin{aligned} t = \lambda_1(\mathbf{x}_t^3) = F(\mathbf{x}_1^3) &= \Psi_0(t) - \Psi_1(t)(1-t) + \Psi_1(1-t)t, \\ &= \big[\Psi_0(t) - \Psi_1(t)\big] + t\big[\Psi_1(t) + \Psi_1(1-t)\big] = \big[\Psi_0(t) - \Psi_1(t)\big] + t \end{aligned}$$

whence the first statement in (5.4) is immediate.

The proof of the second statement relies upon the fact that the formula for $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}$ in Remark 5.2(c) must be independent of the choice of the vectors in $\mathbf{U}$. Using the argument of the proof of Proposition 4.4, this means that the expression

$$\Psi_0(\lambda_1) + \Psi_1(\lambda_1)(\lambda_1 - 1) + \Psi_1(\lambda_2)\lambda_1 + \Psi(\lambda_3)\lambda_1 -$$
$$-\alpha_3\Big\{\chi_0(\lambda_1,\lambda_2,\lambda_3) - \chi_1(\lambda_1,\lambda_2,\lambda_3)\Big\} - (-1-\alpha_3)\chi_1(\lambda_2,\lambda_1,\lambda_3) -$$
$$-\alpha_2\Big\{\chi_0(\lambda_1,\lambda_3,\lambda_2) - \chi_1(\lambda_1,\lambda_3,\lambda_2)\Big\} - (-1-\alpha_2)\chi_1(\lambda_3,\lambda_1,\lambda_2)$$

must be independent of the scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. In particular it follows $\chi_0(\lambda_1,\lambda_2,\lambda_3) - \chi_1(\lambda_1,\lambda_2,\lambda_3) - \chi_1(\lambda_3,\lambda_1,\lambda_3) = 0$ which completes the proof. □

Remark. 5.5. (a) According to (5.4), affinity invariant tuples are of the form

$$\mathbf{\Pi} = \big[\Psi, \Psi, \chi_0, \chi_1)\big], \qquad \chi_0\big|\Delta_3 = 2\chi_1^{[s]}\big|\Delta_3 \tag{5.6}$$

with a d-symmetric shape function Ψ and the symmetrization (4.13) of χ_1.

(b) From the final argument in the proof of Lemma 5.3 it readily follows the below converse statement.

Corollary. 5.7. *Assume* $\mathbf{\Pi} = [\Psi, \Psi, \chi_0, \chi_1]$ *is an RSD-tuple with* (5.6). *Then, independently of the choice of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *we get*

$$\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}\lambda_1 = \sum_{i=1}^{3}\Psi(\lambda_i)\lambda_1 + \chi_1(\lambda_2, \lambda_1, \lambda_3) + \chi_1(\lambda_3, \lambda_1, \lambda_2). \tag{5.8}$$

Proposition. 5.9. *An RSD tuple* $\mathbf{\Pi}$ *is affinity invariant if and only if it is of the form* $\mathbf{\Pi} = [\Psi, \Psi, \chi_0, \chi_1]$ *with range shift property,* (5.6) *and*

$$\chi_1(t_2, t_1, t_3) + \chi_1(t_3, t_1, t_2) = t_1 \sum_{(\ell,m,n)\in S_3}\chi_1(t_\ell, t_m, t_n) \quad \big((t_1, t_2, t_3)\in\Delta_3\big). \tag{5.10}$$

Proof. Necessity: Assume the affinity invariance of $\mathbf{\Pi}$. According to 5.5 and Corollary 5.7, we can write $\mathbf{\Pi} = [\Psi, \Psi, \chi_0, \chi_1]$ with

$$\lambda_1 = \sum_{i=1}^{3}\Psi(\lambda_i)\lambda_1 + \chi_1(\lambda_2, \lambda_1, \lambda_3) + \chi_1(\lambda_3, \lambda_1, \lambda_2). \tag{5.10}$$

On the other hand (Remark 5.2(a), Lemma 5.3) $\mathbf{\Pi}$ has range shift property with (5.6) and $1 = \sum_{i=1}^{3}\Psi(\lambda_i) + \chi_0(\lambda_1, \lambda_2, \lambda_3) + \chi_0(\lambda_2, \lambda_3, \lambda_1) + \chi_0(\lambda_3, \lambda_1, \lambda_2)$ by (4.9). Hence, in view of (5.6) we conclude that

$$1 = \sum_{i=1}^{3}\Psi(\lambda_i) + \sum_{(\ell,m,n)\in S_3}\chi_1(\lambda_\ell, \lambda_m, \lambda_n). \tag{5.11}$$

We obtain (5.10) by subtracting (5.10) from $\lambda_1 \cdot (5.11)$.

Sufficiency: We obtain (5.10) i.e. the affinity invariance of $\mathbf{\Pi}$ by adding the equations $0 = \chi_1(\lambda_2, \lambda_1, \lambda_3) + \chi_1(\lambda_3, \lambda_1, \lambda_2) - \lambda_1 \sum_{(\ell,m,n)\in S_3}\chi_1(\lambda_\ell, \lambda_m, \lambda_n)$ and (5.11) multiplied with λ_1. □

Corollary. 5.12. *A tuple* $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_0, \chi_1]$ *with range shift property and such that* $\chi_1 = \chi_0\varphi$ *for some* $\varphi \in \mathcal{C}^1\big(\mathrm{dom}(\chi_0)\big)$ *is affinity invariant if for every* $(t_1, t_2, t_3) \in \Delta_3$ *we have*

$$\begin{aligned} 1 =& \varphi(t_1,t_2,t_3) + \varphi(t_2,t_1,t_3), \\ &\chi_0(t_3,t_1,t_2)\varphi(t_3,t_1,t_2) + \chi_0(t_2,t_1,t_3)\varphi(t_2,t_1,t_3) = \\ &\quad = t_1\Big[\chi_0(t_1,t_2,t_3) + \chi_0(t_2,t_3,t_1) + \chi_0(t_3,t_1,t_2)\Big]. \end{aligned} \tag{5.13}$$

Example. 5.14. The tuple $\mathbf{\Pi} = \big[\Phi, \Phi, \chi_0, \chi_0\varphi\big]$ is affinity invariant with

$$\Phi(t) = t^3(10 - 15t + 6t^2), \quad \chi_0(t_1,t_2,t_3) = 30t_1^2t_2^2t_3, \quad \varphi(t_1,t_2,t_3) = t_2 + t_3/2.$$

Proof. The range shift property of $\mathbf{\Pi}$ is established in Example 4.10. The identities (5.13) follow with straightforward calculation.

Inspired by [3, Section 2], we proceed to estimates for the accuracy of interpolation with affinity invariant RSD operators. Conveniently, without loss of confusion, we write $\|(x_1,\dots,x_N)\| = \big[\sum_{i=1}^N x_i^2\big]^{1/2}$ resp. $\|A\| = \max_{\|\mathbf{e}\|=1} \|\mathbf{e}A\|$ $\big(A = [a_{ij}]_{i=1}^M{}_{j=1}^N \in \mathbb{R}^{M\times N}\big)$ for the Euclidean norm in $\mathbb{R}^N$ spaces and the spectral norm of $M\times N$ matrices, respectively. As previously, $\mathbf{T}$ denotes an arbitrarily fixed nondegenerate triangle in $\mathbb{R}^2$ with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and associated weights $\lambda_i : \mathbb{R}^2 \to \mathbb{R}$ composed in a vector function $\overline{\lambda}(\mathbf{x}) = [\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \lambda_3(\mathbf{x})]$ in terms of the gradient $\nabla f = [D_1 f, D_2 f]$ and the Hessian $\nabla^2 f = \big[D_iD_jf\big]_{i,j=1}^2$. We write

$$\delta_{\mathbf{T}} = \max\nolimits_{i,j=1}^3 \|\mathbf{p}_i - \mathbf{p}_j\|\ , \quad \alpha_{\mathbf{T}} = \mathrm{area}(\mathbf{T})\ , \quad \kappa_{\mathbf{T}} = \delta_{\mathbf{T}}^2/[2\,\alpha_{\mathbf{T}}]$$

for the diameter, the area and the flatness ratio of $\mathbf{T}$.

Notice that, given an RSD tuple $\mathbf{\Pi} = [\Psi_0, \Psi_1, \chi_1, \chi_1]$ along with a family $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subset \mathbb{R}^2$, the mapping $\mathcal{C}^1(\mathbf{T}) \ni f \mapsto \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}[f]$ is a bounded linear operator $\mathcal{C}^1(\mathbf{T}) \to \mathcal{C}^1(\mathbf{T})$ of finite rank. Namely, by setting $f_i = f(\mathbf{p}_i)$ resp. $A_i(\mathbf{p}_j - \mathbf{p}_i) = \big\langle \nabla f(\mathbf{x}) \big| \mathbf{p}_j - \mathbf{p}_i \big\rangle$ in (1.3),

$$\begin{aligned} \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}} = \sum_{(i,j,k)\in \mathrm{S}_3} \big[\mathfrak{A}_{ijk} + \mathfrak{B}_{ijk}\big] \quad \text{with} \quad & \mathfrak{A}_{ijk}[f](\mathbf{x}) = f(\mathbf{p}_i) a_{ijk}\big(\overline{\lambda}(\mathbf{x})\big), \\ \text{and} \quad & \mathfrak{B}_{ijk}[f](\mathbf{x}) = \big\langle \nabla f(\mathbf{p}_i) \big| \mathbf{p}_j - \mathbf{p}_i \big\rangle b_{ijk}\big(\overline{\lambda}(\mathbf{x})\big) \end{aligned} \tag{5.15}$$

as the sum of 12 bounded operators $\mathcal{C}^1(\mathbf{T}) \to \mathcal{C}^1(\mathbf{T})$ of rank one where

$$\begin{aligned} a_{ijk} &: \Delta_3 \ni (t_1,t_2,t_3) \mapsto \frac{1}{2}\Psi_0(t_i) + \frac{G_i\mathbf{u}_j}{G_j\mathbf{u}_j}\chi_0(t_i,t_j,t_k) \quad \text{resp.} \\ b_{ijk} &: \Delta_3 \ni (t_1,t_2,t_3) \mapsto \Psi_1(t_i)t_j + \frac{G_i\mathbf{u}_k}{G_k\mathbf{u}_k}\chi_1(t_i,t_j,t_k)\ . \end{aligned} \tag{5.16}$$

Given any function $f\in\mathcal{C}(\mathbf{T})$ along with a point $\mathbf{p}\in\mathbf{T}$ and a unit vector $\mathbf{e}\in\mathbb{R}^2$, we write $f_{\mathbf{p},\mathbf{e}}$ for the section $t\mapsto f_{\mathbf{p}}+t\mathbf{e}$. According to the Taylor formula with remainder in integral form, $\big|f_{\mathbf{p},\mathbf{e}}(t)-f(\mathbf{p})\big]\le M_1(f)t$ for any $f\in\mathcal{C}^1(\mathbf{T})$, furthermore in the case of $f\in\mathcal{C}^2(\mathbf{T})$ we have

$$\Big|f_{\mathbf{p},\mathbf{e}}(t)-\big[f(\mathbf{p})+t\langle\nabla f(\mathbf{p})|\mathbf{e}\rangle\big]\Big|\le\frac{1}{2}M_2(f)t^2\ \text{ resp. }\ \Big\|\nabla f_{\mathbf{p},\mathbf{e}}(t)-\nabla f(\mathbf{p})\Big\|\le M_2(f)t.$$

For the investigation of affinity invariance, we introduce the operator

$$\mathfrak{L}[f]=\sum_{i=1}^{3}f(\mathbf{p}_i)\lambda_i\qquad\big(f\in\mathcal{C}(\mathbf{T})\big).\tag{5.17}$$

Observe that $\mathfrak{L}$ is a linear projection onto the space of all affine functions on $\mathbf{T}$ spanned by the weigths λ_i, such that $g\in\text{kernel}(\mathfrak{L})$ if and only if $g(\mathbf{p}_i)=0$ $(i=1,2,3)$. The affinity invariance of $\mathfrak{F}$ can algebraically be formulated in terms of the identities

$$\mathfrak{F}\mathfrak{L}=\mathfrak{L},\qquad f-\mathfrak{F}[f]=(f-\mathfrak{L}[f])-\mathfrak{F}(f-\mathfrak{L}[f]).\tag{5.18}$$

Theorem. 5.19. *Let $\mathfrak{F}:\mathcal{C}^1(\mathbf{T})\to\mathcal{C}^1(\mathbf{T})$ be a bounded linear operator leaving all affine functions $\mathbf{T}\to\mathbb{R}$ invariant. Then for any function $f\in\mathcal{C}^2(\mathbf{T})$,*

$$\|f-\mathfrak{F}f\|_1\le M_2(f)\Big(1+\big\|\mathfrak{F}_{|\text{kernel}(\mathfrak{L})}\big\|\Big)\Big(1+\kappa_{\mathbf{T}}+\frac{1}{2}\delta_{\mathbf{T}}\Big)\delta_{\mathbf{T}}\ .$$

Proof. Consider a function $f\in\mathcal{C}^2(\mathbf{T})$ and let $g:=f-\mathfrak{L}f$. According to (5.18), $f-\mathfrak{F}f=g-\mathfrak{F}g$. In particular $\max|f-\mathfrak{F}f|=M_0\big((I-\mathfrak{F})g\big)$ $\max\|\nabla(f-\mathfrak{F}f)\|=M_1\big((I-\mathfrak{F})g\big)$, and hence

$$\begin{aligned}\|f-\mathfrak{F}f\|_1\le\|(I-\mathfrak{F})g\|_1\le\big\|(I-\mathfrak{F})_{|\text{kernel}(\mathfrak{L})}\big\|\|g\|_1\le\\ \le\big(1+\|\mathfrak{F}_{|\text{kernel}(\mathfrak{L})}\|\big)\|g\|_1.\end{aligned}\tag{5.20}$$

Since the Hessian of any affine function vanishes, we have $\nabla^2f=\nabla^2g$ implying that $M_2(g)=M_2(f)$. Thus the statement of the theorem follows from the below lemma of possible independent interest (but containing rough constants in the estimates). □

Lemma. 5.21. *Let $g\in\mathcal{C}^2(\mathbf{T})$ be a function vanishing at the vertices $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ of the triangle $\mathbf{T}$. Then*

$$\max|g|\le\frac{1}{2}M_2(g)\,\delta_{\mathbf{T}}^2\ ,\quad\max\|\nabla g\|\le M_2(g)\Big(1+\kappa_{\mathbf{T}}\Big)\delta_{\mathbf{T}}\ .$$

Proof. Let $\mathbf{p}_*$ be a point where the function $|g|$ assumes it maximal value (well-defined due to the continuity of g and the compactness of $\mathbf{T}$). Disregarding the trivial case $g \equiv 0$, we may assume without loss of generality that $g(\mathbf{p}_*) > 0$. We have the alternatives:

(1) $\mathbf{p}_*$ is an inner point of $\mathbf{T}$;

(2) $\mathbf{p}_*$ is a relatively inner point of some edge $[\mathbf{p}_i, \mathbf{p}_j]$ of $\mathbf{T}$.

In case (1) we have $\nabla g(\mathbf{p}_*) = 0$ while in case of (2) the directional derivative $g'(\mathbf{p}_*)(\mathbf{p}_i - \mathbf{p}_j) = 0$. In both cases we can find a vertex $\mathbf{p}_i$ such that, toward the unit vector $\mathbf{e} = \|\mathbf{p}_i - \mathbf{p}_*\|^{-1}(\mathbf{p}_i - \mathbf{p}_*)$, we have $g'_{\mathbf{p}_*,\mathbf{e}}(0) = 0$. Then, by setting $t = \|\mathbf{p}_i - \mathbf{p}_*\|$,

$$\max|g| = g(\mathbf{p}_*) = -\big[g(\mathbf{p}_i) + g(\mathbf{p}_*)\big] = -\big[g_{\mathbf{p}_*,\mathbf{e}}(t) - g_{\mathbf{p}_*,\mathbf{e}}(0)\big] \le$$
$$\le |g'_{\mathbf{p}_*,\mathbf{e}}(0)| + \frac{1}{2}M_2(g)t^2 = \frac{1}{2}M_2(g)\|\mathbf{p}_i - \mathbf{p}_*\|^2 \le \frac{1}{2}M_2(g)\delta_{\mathbf{T}}^2 \ .$$

For the proof of the second inequality, we may assume that $\mathbf{E} = [\mathbf{p}_2, \mathbf{p}_3]$ is the largest edge of $\mathbf{T}$ (i.e. $\|\mathbf{p}_2 - \mathbf{p}_3\| = \delta_{\mathbf{T}}$ and $\mathbf{q}$ is a point where $|g|$ attains its maximum on $\mathbf{E}$. Define $\mathbf{v} = \nabla g(\mathbf{q})$, $\rho = \|\mathbf{q} - \mathbf{p}_1\|$, $\mathbf{e} = \rho^{-1}(\mathbf{q} - \mathbf{p}_1)$. Since $0 = g'(\mathbf{q})(\mathbf{p}_3 - \mathbf{p}_2) = \big\langle \nabla g(\mathbf{q}) \big| \mathbf{p}_3 - \mathbf{p}_2 \big\rangle$,

$$\mathbf{v} \perp \mathbf{p}_3 - \mathbf{p}_2, \quad \big|\big\langle \|\mathbf{v}\|^{-1}\mathbf{v} \big| \mathbf{p}_1 - \mathbf{q}\big\rangle\big| = \big[\text{ height of } \mathbf{T} \text{ over the edge } \mathbf{E}\big] = \frac{2\,\alpha_{\mathbf{T}}}{\delta_{\mathbf{T}}}.$$

According to the Taylor formula, we can write

$$0 = g(\mathbf{p}_1) = g_{\mathbf{q},\mathbf{e}}(\rho) = g_{\mathbf{q},\mathbf{e}}(0) + \rho g'_{\mathbf{q},\mathbf{e}}(0) + \frac{1}{2}\nu M_2(g)\rho^2 \quad \text{where} \quad |\nu| \le 1.$$

On the other hand, also

$$\big|\rho g'_{\mathbf{q},\mathbf{e}}(0)\big| = \big|\rho\langle\nabla g(\mathbf{q})|\mathbf{e}\rangle\big| = \big|\langle\mathbf{v}|\rho\mathbf{e}\rangle\big| = \|\mathbf{v}\|\big|\langle\|\mathbf{v}\|^{-1}\mathbf{v}|\mathbf{p}_1 - \mathbf{q}\rangle\big| = \|\mathbf{v}\|\frac{2\,\alpha_{\mathbf{T}}}{\delta_{\mathbf{T}}},$$
$$g_{\mathbf{q},\mathbf{e}}(0) = g(\mathbf{q}) = \mu\max|g| = \mu\frac{1}{2}M_2(g)\delta_{\mathbf{T}}^2 \quad \text{where} \quad |\mu| \le 1.$$

That is, since $\nu\rho^2 = \nu\|\mathbf{q} - \mathbf{p}_1\|^2 \le \|\mathbf{q} - \mathbf{p}_1\|^2 \le \delta_{\mathbf{T}}^2$,

$$0 = \frac{\mu}{2}M_2(g)\delta_{\mathbf{T}}^2 + 2\sigma\|\mathbf{v}\|\frac{\alpha_{\mathbf{T}}}{\delta_{\mathbf{T}}} + \frac{\nu'}{2}M_2(g)\delta_{\mathbf{T}}^2; \quad |\mu|, |\nu'| \le 1 = |\sigma|.$$

It follows

$$\|\mathbf{v}\| = -\frac{\mu + \nu'}{4\sigma}M_2(g)\frac{\delta_{\mathbf{T}}^3}{\alpha_{\mathbf{T}}} \le \frac{1}{2}M_2(g)\frac{\delta_{\mathbf{T}}^3}{\alpha_{\mathbf{T}}}.$$

We complete the proof with the argument that given any point $\mathbf{r} \in \mathbf{T}$ we have

$$\|\nabla g(\mathbf{r})\| \le \|\nabla g(\mathbf{q})\| + \|\nabla g(\mathbf{r}) - \nabla g(\mathbf{q})\| =$$
$$= \|\mathbf{v}\| + \|\nabla g(\mathbf{r}) - \nabla g(\mathbf{q})\| \le \|\mathbf{v}\| + M_2(g)\mathrm{diam}(\mathbf{T}). \qquad \square$$

Corollary. 5.22. *Let $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}$ be an affinity invariant RSD operator given in the form (5.15 – 16) with the ratios $G_i\mathbf{u}_j/G_j\mathbf{u}_j$ in terms of the coefficients α_i in Lemma 4.3. Then for any function $f \in \mathcal{C}^2(\mathbf{T})$ we have*

$$M_0\big(f - \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}[f]\big) \le M_2(f)\delta_{\mathbf{T}}^2\Big[\tfrac{1}{2} + \big(1+\kappa_{\mathbf{T}}\big)C_0^{\mathbf{\Pi},\mathbf{U}}\Big],$$

$$M_1\big(f - \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}[f]\big) \le M_2(f)\delta_{\mathbf{T}}\big(1+\kappa_{\mathbf{T}}\big)\Big[1 + \kappa_{\mathbf{T}}C_1^{\mathbf{\Pi},\mathbf{U}}\Big]$$

with the constants $C_0^{\mathbf{\Pi},\mathbf{U}} = 6\Big[\max_{\tau\in[0,1]}|\Psi_1(\tau)| + \Big(\tfrac{1}{2} + \max_{i=1}^{3}|\alpha_i|\Big)\max_{\mathbf{t}\in\Delta_3}|\chi_1(\mathbf{t})|\Big]$ *and* $C_1^{\mathbf{\Pi},\mathbf{U}} = 6\Big[\max_{\tau\in[0,1]}|\Psi_1'(\tau)| + \max_{\tau\in[0,1]}|\Psi_1(\tau)| + \Big(\tfrac{1}{2} + \max_{i=1}^{3}|\alpha_i|\Big)\sum_{m=1}^{3}\max_{\mathbf{t}\in\Delta_3}|D_m\chi(\mathbf{t})|\Big]$.

Proof. We can refine the arguments of the proof of Theorem 5.19 relying upon the specific form (5.15-16) of the operator $\mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}$ as follows. Consider any function $f \in \mathcal{C}^2(\mathbf{T})$. Using the abbreviations $g = f - \mathfrak{L}[f]$, $\mathfrak{F} = \mathfrak{F}^{\mathbf{\Pi}}_{\mathbf{P},\mathbf{U}}$, $M = M_2(f) = M_2(g)$, $\delta = \delta_{\mathbf{T}}$, $\kappa = \kappa_{\mathbf{T}}$, we know that

$$M_k(f - \mathfrak{F}[f]) = M_k(g - \mathfrak{F}[g]) \le M_k(g) + M_k(\mathfrak{F}[g]) \quad (k = 0, 1);$$

$$M_0(f - \mathfrak{F}[f]) \le M\frac{\delta^2}{2} + \max\big|\mathfrak{F}[g]\big|, \quad M_1(f - \mathfrak{F}[f]) \le M\delta\,(1+\kappa) + \max\big\|\nabla\mathfrak{F}[g]\big\|.$$

We estimate the values $|\mathfrak{F}[g](\mathbf{x})|$ resp. $\|\nabla\mathfrak{F}[g](\mathbf{x})\|$ ($\mathbf{x} \in \mathbf{T}$) as follows. Since $g(\mathbf{p}_i) = 0$ ($i = 1, 2, 3$) by construction, we have the reduced formulas

$$\mathfrak{F}[g] = \sum_{(i,j,k)\in \mathrm{S}_3}\mathfrak{B}_{ijk}[g] = \sum_{(i,j,k)\in \mathrm{S}_3}\big\langle\nabla g(\mathbf{p}_i)\big|\mathbf{p}_j - \mathbf{p}_i\big\rangle\Big[\Psi_1\big(\lambda_i\big)\lambda_j + \frac{G_i\mathbf{u}_k}{G_k\mathbf{u}_k}\chi_1\big(\overline{\lambda}_{ijk}\big)\Big],$$

$$\nabla\mathfrak{F}[g] = \sum_{(i,j,k)\in \mathrm{S}_3}\big\langle\nabla g(\mathbf{p}_i\big|\mathbf{p}_j - \mathbf{p}_i\big\rangle\ \nabla\Big[\Psi_1\big(\lambda_i\big)\lambda_j + \frac{G_i\mathbf{u}_k}{G_k\mathbf{u}_k}\chi_1\big(\overline{\lambda}_{ijk}\big)\Big]$$

where $\overline{\lambda}_{ijk} : \mathbf{x} \mapsto \big[\lambda_i(\mathbf{x}), \lambda_j(\mathbf{x}), \lambda_k(\mathbf{x})\big]$. By Lemma 5.21, $\|\nabla g(\mathbf{p}_i)\| \le M_1(g) \le M\delta(1+\kappa)$ implying that $\big|\big\langle\nabla g(\mathbf{p}_i)\big|\mathbf{p}_j - \mathbf{p}_i\big\rangle\big| \le M\delta^2(1+\kappa)$ for any pair of indices i, j. Also $\big|G_i\mathbf{u}_i/G_j\mathbf{u}_j\big| = \big|\tfrac{1}{2} \pm \alpha_j\big| \le \tfrac{1}{2} + \max_j|\alpha_j|$ with $i \ne j$ in all the 6 terms expressing $\mathfrak{F}[g]$ resp. $\nabla\mathfrak{F}[g]$. Since for $\mathbf{x} \in \mathbf{T}$ we have $|\lambda_j(\mathbf{x}))| \le 1$ and $\overline{\lambda}_{ijk}(\mathbf{x}) \in \Delta_3$, the estimate for $M_0(f)$ is immediate.

To estimate $M_1(f)$, we proceed analogously. Recall from plain geometry [1] that, in terms of the rotation, $R : [\xi_1, \xi_2] \mapsto [-\xi_2, \xi_1]$ of $\mathbb{R}^2$, for any triple $(i, j, k) \in \mathrm{S}_3$ we can write $\nabla\lambda_k(\mathbf{x}) \equiv \mathbf{g}_k^*$ with the vector $\mathbf{g}_k^* = \big\langle R(\mathbf{p}_j - \mathbf{p}_i)\big|\mathbf{p}_k - \mathbf{p}_i\big\rangle^{-1}R(\mathbf{p}_j - \mathbf{p}_i)$ and $\|\mathbf{g}_k^*\| = [2\,\mathrm{area}(\mathbf{T})]^{-1}\|\mathbf{p}_j - \mathbf{p}_k\| \le \kappa/\delta$ independently of the location $\mathbf{x}$. Hence

$$\nabla\big[\Psi_1\big(\lambda_i(x)\big)\lambda_j(\mathbf{x})\big] = \lambda_j(\mathbf{x})\Psi_1'\big(\lambda_i(\mathbf{x})\big)\mathbf{g}_i^* + \Psi_1\big(\lambda_i(\mathbf{x})\big)\mathbf{g}_j^*,$$

$$\nabla\chi_1\big(\overline{\lambda}_{i_1,i_2,,i_3}(\mathbf{x})\big) = \sum_{m=1}^{3}\big[D_\ell\chi_1\big(\overline{\lambda}_{i_1,i_2,,i_3}(\mathbf{x})\big)\big]\mathbf{g}_{i_m}^*$$

implying $\|\nabla[\Psi_1(\lambda_i(x))\lambda_j(\mathbf{x})]\| \leq \max_{\tau\in[0,1]} \big(|\Psi_1'(\tau)| + |\Psi_1(\tau)|\big)\kappa/\delta \quad (i \neq j,\ \mathbf{x} \in \mathbf{T})$

respectively $\|\nabla\chi_1(\overline{\lambda}_{ijk}(\mathbf{x}))\| \leq \sum_{m=1}^{3} \max_{\mathbf{t}\in\Delta_3} |D_m\chi_1(\mathbf{t})|\kappa/\delta \quad ((i,j,k) \in \mathrm{S},\ \mathbf{x} \in \mathbf{T})$

which completes the proof. □

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