Regression estimators for the tail index

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Abstract

We propose a class of weighted least squares estimators for the tail index of a distribution function with a regularly varying tails. Our approach is based on the method developed by Holan and McElroy (2010) for the Parzen tail index. We prove asymptotic normality and consistency for the estimators under suitable assumptions. These and earlier estimators are compared in various models through a simulation study using the mean squared error as criterion. The results show that the weighted least squares estimator has good performance.

1 Introduction and main result

Let $X_1, X_2, \dots$ be independent random variables with a common right-continuous distribution function F, and for each $n \in \mathbb{N}$, let $X_{1,n} \le \cdots \le X_{n,n}$ denote the order statistics pertaining to the sample $X_1, \dots, X_n$. We assume either

$$1 - F(x) = x^{-1/\alpha_1} \ell_1(x), \quad 1 \le x < \infty, \tag{1}$$

or

$$F(-x) = x^{-1/\alpha_2} \ell_2(x), \quad 1 \le x < \infty, \tag{2}$$

where ℓ_1 and ℓ_2 are positive functions on the half line $[1, \infty)$, slowly varying at infinity and $\alpha_1, \alpha_2 > 0$ are fixed unknown parameters to be estimated. Introducing the quantile function Q of F defined as

$$Q(s) := \inf \{x : F(x) \ge s\}, \quad 0 < s \le 1, \quad Q(0) := Q(0+),$$

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it is well known that (1) holds if and only for some function L_1 slowly varying at zero,

$$Q(1-s) = s^{-\alpha_1} L_1(s), \quad 0 < s < 1, \tag{3}$$

and (2) holds if and only for some function L_2 slowly varying at zero,

$$Q(s) = -s^{-\alpha_2} L_2(s), \quad 0 < s < 1. \tag{4}$$

Several estimators exist for the tail index α_1 among which Hill's estimator is the most classical. Hill (1975) proposed the following estimator for α_1:

$$\widehat{\alpha}_n^{(H)} = \frac{1}{k_n} \sum_{j=1}^{k_n} \log X_{n-j+1,n} - \log X_{n-k_n,n},$$

where the k_n are positive integers, which in theoretical asymptotic considerations will satisfy the conditions

$$1 \leq k_n < n, \quad k_n \to \infty \quad \text{and} \quad k_n/n \to 0 \quad \text{as } n \to \infty.$$

The asymptotic normality of $\widehat{\alpha}_n^{(H)}$ was first considered by Hall (1982) in the following submodel of $\mathcal{R}_\alpha$:

$$1 - F(x) = x^{-1/\alpha_1} C_1 [1 + C_2 x^{-\beta/\alpha} \{1 + o(1)\}], \quad \text{as } x \to \infty,$$

for some constants $C_1 > 0$ and $C_2 \neq 0$. This is equivalent to

$$Q(1-s) = s^{-\alpha_1} D_1 [1 + D_2 s^{\beta} \{1 + o(1)\}], \quad s \to 0, \tag{5}$$

where $D_1 = C_1^{\alpha_1}$ and $D_2 = C_2/C_1^{\beta}$. Other estimators were proposed by Pickands (1975), and Dekkers et al. (1989), to name a few.

Assuming that F is absolutely continuous with density function f, Parzen (2004) studied the following alternative model for the tails of the distribution:

$$fQ(s) := f(Q(s)) = s^{\nu_1} \widetilde{L}_1(s), \quad s \in (1/2, 1],$$
$$fQ(s) = (1-s)^{\nu_2} \widetilde{L}_2(1-s), \quad s \in (1/2, 1],$$

where $\nu_1, \nu_2 > 0$ are finite constants and $\widetilde{L}_1, \widetilde{L}_2$ are slowly varying at zero. The parameters ν_1 and ν_2 are called the Parzen tail indices of the density-quantile function $fQ(\cdot)$.

Based on orthogonal series expansions, Holan and McElroy (2010) introduced a regression estimator for the Parzen tail index using ordinary least squares. AL-Najafi and Viharos (2021) obtained a more general class of estimators for ν using weighted least squares. We adopt this method to estimate the classical tail index α. Following the idea of Holan and McElroy (2010), we assume that the slowly varying function L in (3) admits the truncated orthogonal series expansion

$$L_i(s) = \exp\left\{\theta_{i,0} + 2\sum_{k=1}^{p_i} \theta_{i,k}\cos(2\pi ks)\right\}, \quad i = 1, 2$$

where $p_i > 0$ are fixed integers, and $\theta_{i,0}, \dots, \theta_{i,p}$ are unknown parameters. We suppose that $p_i \le \widetilde{p}_i$, where $\widetilde{p}_i$ are prespecified integers. The knowledge of p_i is not assumed; condition $p_i \le \widetilde{p}_i$ gives only an upper bound for p_i. It follows that

$$\begin{aligned} \log Q(1-s) &= -\alpha_1 \log s + \theta_{1,0} + 2\sum_{k=1}^{p_2} \theta_{1,k}\cos(2\pi ks), \\ \log\big(-Q(s)\big) &= -\alpha_2 \log s + \theta_{2,0} + 2\sum_{k=1}^{p_2} \theta_{2,k}\cos(2\pi ks). \end{aligned} \tag{6}$$

Let Q_n be the empirical quantile function defined as

$$Q_n(s) = X_{k,n} \qquad \text{if} \quad \frac{k-1}{n} < s \le \frac{k}{n}, \quad k = 1, 2, \dots, n.$$

Based on the representations in (6), we obtain the regression equations

$$\begin{aligned} \log Q_n(1-s_j) &= -\alpha_1 \log s_j + \theta_{1,0} + 2\sum_{k=1}^{\widetilde{p}_1} \theta_{1,k}\cos(2\pi ks_j) + \varepsilon(s_j), \\ \log\big(-Q_n(s_j)\big) &= -\alpha_2 \log s_j + \theta_{2,0} + 2\sum_{k=1}^{\widetilde{p}_2} \theta_{2,k}\cos(2\pi ks_j) + \varepsilon(1-s_j), \end{aligned}$$

where

$$\varepsilon(s) = \log\big(Q_n(1-s)/Q(1-s)\big) \tag{7}$$

is the residual process, $s_j = j/n$, $j = \lceil na_i \rceil, \dots, \lfloor nb_i \rfloor$, $a_i < b_i$ are fixed constants taken from the interval (0,1), and $\theta_{i,k} = 0$ for $k > p_i$, $i = 1, 2$. The value $\widetilde{p}_i$ is chosen by the statistician. We propose a class of estimators for α_i using weighted least squares. We choose some nonnegative weights of the form $w_{j,n} = R_i(s_j)$ with some weight functions R_i. Set $y_j := \log Q_n(1-s_j)$ or $y_j := \log(-Q_n(s_j))$ for F belonging to (1) or (2) respectively, and let

$$\underline{y}_i := (y_{\lceil na \rceil}, \dots, y_{\lfloor nb \rfloor})',$$
$$W_i := \mathrm{diag}(w_{\lceil na \rceil, n}, \dots, w_{\lfloor nb \rfloor, n}),$$
$$X := [G^*, G_0, 2G_1, \dots, 2G_{\widetilde{p}}],$$

where

$$G^* = \big(-\log(s_{\lceil na \rceil}), \dots, -\log(s_{\lfloor nb \rfloor}) \big)',$$
$$G_k = \big(\cos(2\pi k s_{\lceil na \rceil}), \dots, \cos(2\pi k s_{\lfloor nb \rfloor}) \big)', \quad k = 0, \dots, \widetilde{p}.$$

Set $\beta_{\widetilde{p}_i} := (\alpha_i, \theta_{i,0}, \theta_{i,1}, \dots, \theta_{i,\widetilde{p}_i})'$. By minimizing the weighted sum of squares

$$\sum_{\lceil na \rceil}^{\lfloor nb \rfloor} w_{j,n} \big(y_j + \alpha_i \log s_j - \theta_{i,0} - 2 \sum_{k=1}^{\widetilde{p}} \theta_{i,k} \cos(2\pi k s_j) \big)^2,$$

we obtain the following estimator of $\beta_{\widetilde{p}_i}$:

$$\widehat{\beta}_{\widetilde{p}_i} = (X' W_i X)^{-1} X' W_i \, \underline{y}_i$$

whenever X has full rank and W_i is positive definite implying the invertibility of $X' W_i X$. Then the weighted least squares estimator of α can be written in the form

$$\widehat{\alpha}_{n,i}^{(W)} := e_1' \widehat{\beta}_{\widetilde{p}_i} = e_1' (X' W_i X)^{-1} X' W_i \, \underline{y}_i,$$

where e_1 is the $\widetilde{p}_i + 2$ dimensional vector defined as $e_1 = (1, 0, 0, \dots, 0)'$.

We assume the following conditions on the underlying distribution:

- (Q_1) The distribution function F is continuous and twice differentiable on (a^*, b^*), where $a^* = \sup\{x : F(x) = 0\}$, $b^* = \inf\{x : F(x) = 1\}$, $-\infty \le a^* < b^* \le \infty$ and $f(x) := F'(x) \ne 0$ on (a^*, b^*).
- (Q_2) $\sup_{a^* < x < b^*} F(x)(1 - F(x)) |f'(x)/f^2(x)| < \infty$.
- (Q_3) $\sup_{1-b \le s \le 1-a} 1/|Q(s)| < \infty$, $\sup_{1-b \le s \le 1-a} 1/fQ(s) < \infty$ and $\sup_{1-b \le s \le 1-a} 1/|fQ(s)Q(s)| < \infty$.

We will show that the limit matrix $M(a,b,R_i) := \lim_{n\to\infty} n^{-1}X'W_iX$ exists (see the proof of Theorem 1 in Section 4). Let $(v^*, v_0, \ldots, v_{\widetilde{p}_i})$ be the first row of $M(a,b,R_i)^{-1}$, and set $G_{R_i}(u) := R_i(u)\big(-v^*\log u + v_0 + 2\sum_{k=1}^{\widetilde{p}_i} v_k\cos(2\pi k u)\big)$ for $u \in (0,1)$.

Moreover, we suppose the following conditions:

- (R) The weight function R_i is nonnegative and Riemann integrable on $[a_i, b_i]$, $i = 1,2$.
- (M) The matrix $M(a_i, b_i, R_i)$ is invertible, $i = 1,2$.

Now we state our main result for the estimators $\widehat{\alpha}_{n,i}^{(W)}$. Throughout, $\xrightarrow{D}$ denotes convergence in distribution, $\xrightarrow{P}$ denotes convergence in probability, and limiting and order relations are always meant as $n \to \infty$ if not specified otherwise.

Theorem 1. *Assume that the conditions $(Q_1)-(Q_3)$ are satisfied for the underlying distribution and suppose that the quantile function Q admits the representation (6). Moreover, assume the conditions (R) and (M), and assume also that the percentiles s_j are chosen from a closed set $U_i = [a_i, b_i]$, $0 < a_i < b_i < 1$, such that $s_j = j/n$, $j = \lceil na_i\rceil, \ldots, \lfloor nb_i\rfloor$, and $p \le \widetilde{p}_i$, $i = 1,2$. Then*

$$\sqrt{n}\big(\widehat{\alpha}_{n,i}^{(W)} - \alpha_i\big) \xrightarrow{D} N(0, V_i), \quad i = 1,2, \tag{8}$$

where

$$V_1 = \int_{a_1}^{b_1}\int_{a_1}^{b_1} \frac{G_{R_1}(s)G_{R_1}(t)(s\wedge t - st)}{Q(1-s)Q(1-t)fQ(1-s)fQ(1-t)}dsdt, \tag{9}$$

and

$$V_2 = \int_{a_2}^{b_2}\int_{a_2}^{b_2} \frac{G_{R_2}(s)G_{R_2}(t)(s\wedge t - st)}{Q(s)Q(t)fQ(s)fQ(t)}dsdt.$$

We prove the theorem in Section 4.

2 Asymptotics for $\widetilde{p} \to \infty$

The estimation method proposed in Section 1 is heavily based on the assumption $p_i \le \widetilde{p}_i$. However, choosing $\widetilde{p}_i < p_i$ inflicts a bias. To overcome this difficulty, we adjust our method to study asymptotics when $\widetilde{p}_i = \widetilde{p}_i(n) \to \infty$ as $n \to \infty$. In this section our investigation is based on the following series expansion:

$$\log L_i(s) \sim \sum_{k=0}^{\infty} \theta_{i,k} \varphi_k(s), \quad s \in [a, b],$$

where

$$\begin{aligned}
\varphi_0(s) &= \frac{1}{\sqrt{(b_i - a_i)R(s)}}, \\
\varphi_k(s) &= \cos\left(\pi k \frac{s - a_i}{b_i - a_i}\right) \frac{1}{\sqrt{(b_i - a_i)R_i(s)/2}}, \quad k = 1, 2, \dots,
\end{aligned}$$

and $\theta_{i,k} = \int_{a_i}^{b_i} \log L_i(x) \varphi_k(x) R_i(x) dx$. We assume that $R_i > 0$ on $[a_i, b_i]$, $i = 1, 2$. The sequence $\varphi_k \sqrt{R_i}$, $k = 0, 1, \dots,$ is a complete orthonormal system in $L^2[a_i, b_i]$. For convenience, in this section we use the percentiles $s_j = a_i + j \frac{b_i - a_i}{n}$, $j = 0, \dots, n-1$, $i = 1, 2$. Similarly as in Section 1, with $y_j := \log Q_n(1 - s_{i,j})$ or $y_j := \log(-Q_n(s_j))$ and $w_{j,n} = R_i(s_{i,j})$ define

$$\underline{y}_i := (y_0, \dots, y_{n-1})',$$

$$W_i := \operatorname{diag}(w_{0,n}, \dots, w_{n-1,n}),$$

and let $X := [G^*, G_0, G_1, \dots, G_{\widetilde{p}}]$, where

$$\begin{aligned}
G^* &= \big(-\log s_0, \dots, -\log s_{n-1}\big)', \\
G_k &= \big(\varphi_k(s_0), \dots, \varphi_k(s_{n-1})\big)', \quad k = 0, \dots, \widetilde{p}_i.
\end{aligned} \tag{10}$$

Set

$$b_{\widetilde{p}_i}(s) := \log L_i(s) - \sum_{k=0}^{\widetilde{p}_i} \theta_{i,k} \varphi_k(s). \tag{11}$$

In view of (7) we obtain the regression equations

$$\log Q_n(1-s_j) = -\alpha_1 \log s_j + \sum_{k=0}^{\widetilde{p}_i} \theta_{1,k}\varphi_k(s_j) + b_{\widetilde{p}_i}(s_j) + \varepsilon(s_j),$$
$$\log\left(-Q_n(s_j)\right) = -\alpha \log s_j + \sum_{k=0}^{\widetilde{p}_i} \theta_{2,k}\varphi_k(s_j) + b_{\widetilde{p}_i}(s_j) + \varepsilon(1-s_j).$$

By minimizing the weighted sum of squares

$$\sum_{\lceil na \rceil}^{\lfloor nb \rfloor} w_{j,n}\bigl(y_j + \alpha_i \log s_j - \sum_{k=0}^{\widetilde{p}_i} \theta_{i,k}\varphi_k(s_j)\bigr)^2,$$

we obtain the following estimator of α_i:

$$\widehat{\alpha}_{n,i}^{(W)} = e_1'(X'W_iX)^{-1}X'W_i\,\underline{y}_i.$$

In order to formulate the result for $\widehat{\alpha}_{n,i}^{(W)}$, we need the series expansion of the $-\log(\cdot)$ function:

$$-\log s \sim \sum_{j=0}^{\infty} c_j\varphi_j(s), \tag{12}$$

where $c_j = \int_a^b (-\log x)\varphi_j(x)R_i(x)dx$. We assume the following conditions on the sequences $\widetilde{p}_i$, θ_n and c_n:

(P_1) $\widetilde{p}_i \to \infty$ and $\widetilde{p}_i/n \to 0$.
(P_2) $2(\widetilde{p}_i+1)/n < 1$.
(P_3) $n\sum_{i=\widetilde{p}_i+1}^{\infty} c_i^2 \to \infty$.
(P_4) $\theta_{i,n}/c_n \to 0$.

Theorem 2. *Suppose the conditions* (P_1)*–*(P_4) *are satisfied. Then* $\widehat{\alpha}_{n,i}^{(W)} \xrightarrow{P} \alpha_i$*,* $i = 1,2$.

3 Simulation results

Simulations are presented to evaluate the performance of the weighted least squares (WLS) estimators. Due to symmetry considerations, we study only the right tail estimation. The simulations were performed for widely used heavy tailed models and distributions, such as the strict Pareto distribution $L \equiv 1$, the Hall model (5), the Burr, Fréchet, log-logistic and generalized extreme value (GEV) distributions. The tail indices were chosen between 0.5 and 20. We compare the WLS estimator to earlier estimators such as the ordinary least squares (OLS), Hill, Pickands and DEdH (Dekkers, Einmahl and de Haan) estimators through the simulated mean square errors (MSE).

We tested various a and b parameters and weight functions R to see what difference they make. We found that the WLS estimator had fairly good performance for $R(s) = s$. We experimented with several combinations of a and b values to find an appropriate range, and we stopped when we obtained reasonable stability of the WLS estimator $\widehat{\alpha}_n^{(W)}$. Figure 1 shows the average simulated tail index estimates $\widehat{\alpha}_n^{(W)}$ for $\widetilde{p} = 3$, $R(s) = s$, $b = 0.45$ and different values of a for the Pareto distribution with $\alpha = 1.8$ and $\alpha = 5$. The simulated $\widehat{\alpha}_n^{(W)}$ estimates for other α values and all the distributions investigated gave fairly similar results. As the parameters a and b move away from the values $a = 0.0001$ and $b = 0.45$, the estimates become less accurate.

Tables 1–12 contain the empirical MSE values and average simulated estimates (mean) of the estimates. We assumed $\widetilde{p} = 1, 2, 3$ for the truncation parameter, $n = 5000$ for the sample size, $k_n = 50, 100, 500$ and 1000 for the sample fraction size for the Hill, Pickands and DEdH estimators and $a = 0.0001, b = 0.45$ for the OLS and WLS estimators. All the simulations were repeated 1000 times.

Tables 1 and 2 contain the empirical MSE values and average simulated estimates for the strict Pareto distribution. We conclude that the WLS estimator performs better than the OLS estimator. Moreover, the WLS estimator is better than the Hill estimator for $k_n = 50$ and $k_n = 100$. For α values between 0.5 and 1.2 the WLS estimator is competitive with the Hill estimator when $k_n = 500$ and $k_n = 1000$. For thinner tails ($0.5 \leq \alpha \leq 1.2$) the WLS estimator is competitive with the Pickands and the DEdH estimators for all k_n investigated. We also see that the WLS estimator is better than the Pickands and DEdH estimators when $k_n = 50$ and $k_n = 100$ for all the examined α values. The good performance of the Hill estimator is not surprising since the Hill estimator was obtained in the special case of (1) when the slowly varying function $\ell_1(x)$ is constant for all $x \geq x_{\alpha_1}$, for some threshold x_{α_1}. The Pickands and DEdH estimators tend to have good estimates.

Tables 3 and 4 present the simulation results for the Hall model. Specifically, we used the parameters $D_1 = 0.4, D_2 = 1$, and $\beta = 1$. In the Hall model in most of the cases the WLS estimator is better than the OLS estimator and superior to the other estimators when $k_n = 50, 100$. Moreover, the WLS estimator is better than the Pickands estimator ($k_n = 500, 1000$) for $\alpha < 5$. The Hill and DEdH estimators are better than the Pickands estimator; this is also seen from Figures 1 and 2.

Similarly, Tables 5–12 display the results of the empirical MSE and average simulated estimates for the Burr, Fréchet, log-logistic and GEV distributions. We conclude that the WLS estimator is better than the Hill and DEdH estimators when $k_n = 50, 100$, the WLS is also better than the Pickands estimator when $k_n = 50, 100, 500$. In addition to that, the WLS estimator is competitive with the following estimators: the Hill and DEdH estimators when $k_n = 500, 1000$, and the Pickands estimator when $k_n = 1000$ and $0.5 \leq \alpha \leq 1.2$.

Additionally, we visually investigated the MSE and the mean of the tail index estimates for the Hill, Pickands and DEdH estimators as a function of k_n in the range $[50, 1000]$ for the strict Pareto, Burr, Fréchet, log-logistic and the GEV distributions and the Hall model with $\alpha = 1.8$ and 10; these two α values represents well the low and high tail indices. We see from Figure 2 that as the value of k_n increases the value of the mean is close to the true value for the Pareto model. This is not the situation for the Hall model (Figure 3). For the Burr distribution the Pickands estimator has good performance for low and high α values, and all the three estimatators perform well for high α and large k_n values (see Figure 4). Similarly, Figures 5–7 display average simulated tail index estimates for the Fréchet, log-logistic and GEV distributions.

Figures 8–13 depict the MSE values of the tail index estimates for $\alpha = 1.8, 10$. We see that the Hill, Pickands and DEdH estimatators perform weakly when $k < 300$ for the strict Pareto, Hall and Burr models. Comparing the MSE values in Figures 8–10 with the MSE values in Tables 1,3 and 5 we conclude that the WLS estimator with $\widetilde{p} = 1$ is competitive with the Hill, Pickands and DEdH estimators for the strict Pareto, Hall and Burr models. Finally, Figures 11–13 contain the MSE values for the Fréchet, GEV and log-logistic distributions. In these models the Hill and DEdH estimators perform better than the Pickands estimator. The latter has good performance only for $200 \leq k_n \leq 500$. We see again that the WLS estimator is competitive with these estimators.

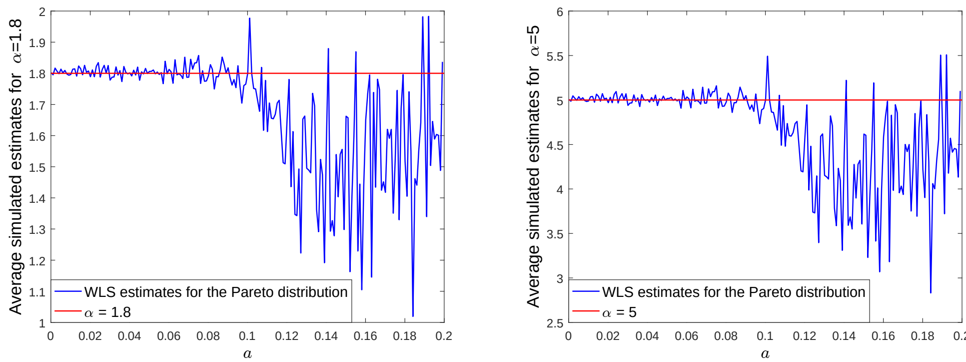

Figure 1: Average WLS estimates for the strict Pareto distribution with $\alpha = 1.8$ (left), $\alpha = 5$ (right).

Table 1: MSE of tail index estimates for the strict Pareto distribution with $a = 0.0001$, $b = 0.45$ and $n = 5000$.

MSE – strict Pareto distribution																		
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.00049	0.00061	0.00092	0.00077	0.00117	0.00171	0.00529	0.00246	0.00045	0.00024	0.07840	0.03688	0.00792	0.00375	0.02834	0.01380	0.00245	0.00118
0.8	0.00120	0.00142	0.00221	0.00194	0.00290	0.00416	0.01311	0.00681	0.00130	0.00064	0.08665	0.04234	0.00841	0.00409	0.03714	0.01696	0.00339	0.00170
1	0.00179	0.00213	0.00356	0.00280	0.00427	0.00625	0.02085	0.01095	0.00215	0.00108	0.09435	0.05010	0.00903	0.00434	0.04734	0.02204	0.00430	0.00215
1.2	0.00282	0.00352	0.00513	0.00417	0.00633	0.00916	0.02828	0.01439	0.00282	0.00139	0.10382	0.05746	0.01058	0.00542	0.05063	0.02479	0.00486	0.00242
1.5	0.00452	0.00560	0.00821	0.00661	0.00991	0.01418	0.04753	0.02051	0.00453	0.00224	0.11273	0.05945	0.01238	0.00620	0.07147	0.03358	0.00652	0.00321
1.8	0.00611	0.00728	0.01107	0.00882	0.01315	0.01894	0.06558	0.03225	0.00610	0.00302	0.13275	0.07084	0.01303	0.00720	0.08659	0.04223	0.00823	0.00394
2	0.00762	0.00967	0.01446	0.01245	0.01864	0.02667	0.08362	0.03983	0.00748	0.00412	0.14250	0.07676	0.01586	0.00735	0.10782	0.05118	0.00965	0.00499
3	0.01787	0.02184	0.03417	0.02765	0.04158	0.05980	0.18202	0.08690	0.01792	0.00893	0.24295	0.13102	0.02522	0.01189	0.20101	0.09752	0.02007	0.01000
4	0.02929	0.03684	0.05754	0.04829	0.07310	0.10550	0.31260	0.16700	0.03460	0.01687	0.38934	0.17584	0.04254	0.01811	0.34092	0.17943	0.03738	0.01802
5	0.04843	0.05722	0.09331	0.07460	0.11225	0.16249	0.47418	0.24533	0.04980	0.02544	0.56605	0.29024	0.05643	0.02717	0.50877	0.25746	0.05069	0.02613
5.5	0.05552	0.06927	0.10458	0.08481	0.12834	0.18596	0.56165	0.26945	0.05495	0.02781	0.65113	0.32892	0.06243	0.03432	0.58573	0.27649	0.05676	0.02864
6	0.07229	0.09246	0.14084	0.10552	0.15610	0.22045	0.70099	0.34494	0.07245	0.03680	0.78123	0.39024	0.07812	0.04152	0.71702	0.35250	0.07341	0.03774
10	0.19441	0.23431	0.37002	0.27091	0.39673	0.55880	1.90544	1.02085	0.20394	0.10620	2.22547	1.05137	0.20832	0.11536	1.93590	1.02353	0.20502	0.10672
15	0.40458	0.49402	0.77201	0.68487	1.04374	1.50364	4.83328	2.31597	0.44962	0.22565	4.73527	2.51673	0.47819	0.26020	4.89123	2.33761	0.45289	0.22676
20	0.79872	0.98042	1.55130	1.15213	1.70514	2.43309	7.90576	4.15925	0.78874	0.38327	8.37301	4.58678	0.86216	0.39111	7.93263	4.16714	0.79090	0.38320

Table 2: Mean of tail index estimates for the strict Pareto model with $a = 0.0001$, $b = 0.45$ and $n = 5000$.

Mean – strict Pareto distribution																		
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.50107	0.50122	0.50257	0.50710	0.50957	0.51237	0.5046	0.5021	0.4995	0.5003	0.5001	0.5070	0.5026	0.5006	0.4495	0.4768	0.4974	0.4982
0.8	0.80206	0.80246	0.80387	0.81101	0.81467	0.81870	0.8008	0.7995	0.8001	0.8003	0.7957	0.8021	0.8018	0.8002	0.7378	0.7689	0.7934	0.7968
1	1.00151	1.00202	1.00263	1.01282	1.01749	1.02261	1.0021	1.0047	1.0028	1.0021	1.0179	0.9962	1.0030	0.9996	0.9392	0.9712	0.9958	0.9989
1.2	1.20180	1.20193	1.20301	1.21512	1.22056	1.22669	1.1985	1.1984	1.1971	1.1992	1.2003	1.1998	1.2028	1.2009	1.1526	1.1738	1.1927	1.1960
1.5	1.50230	1.50315	1.50300	1.52064	1.52817	1.53636	1.4975	1.5001	1.4992	1.4999	1.5116	1.5118	1.5023	1.4994	1.4421	1.4687	1.4935	1.4967
1.8	1.80549	1.80805	1.80907	1.82479	1.83284	1.84104	1.7995	1.8024	1.8004	1.7986	1.8155	1.8069	1.7954	1.8022	1.7477	1.7734	1.7948	1.7963
2	2.00636	2.00823	2.01326	2.03106	2.04112	2.05196	1.9998	2.0018	2.0031	2.0044	2.0163	2.0168	2.0079	2.0032	1.9399	1.9700	1.9970	2.0010
3	3.00562	3.00363	3.00783	3.03677	3.04956	3.06492	3.0302	3.0091	3.0020	2.9992	2.9991	3.0191	2.9982	3.0007	2.9754	2.9855	2.9987	2.9983
4	4.00679	4.00993	4.01810	4.05741	4.07807	4.10056	3.9781	3.9790	3.9869	3.9976	4.0096	4.0035	4.0116	3.9979	3.9077	3.9447	3.9786	3.9914
5	5.00923	5.00772	5.01343	5.05987	5.08090	5.10539	4.9540	4.9872	5.0052	4.9960	5.0506	5.0441	4.9898	4.9986	4.8946	4.9519	4.9973	4.9933
5.5	5.52263	5.52378	5.53535	5.59078	5.61863	5.65001	5.5184	5.4919	5.4977	5.5001	5.5152	5.5176	5.5091	5.5014	5.4701	5.4717	5.4932	5.4979
6	6.01076	6.01978	6.03576	6.08161	6.11061	6.14072	6.0004	6.0013	6.0022	5.9987	6.0353	6.0246	5.9991	5.9967	5.9430	5.9720	5.9960	5.9959
10	10.03592	10.04538	10.04856	10.13609	10.17803	10.22187	9.9857	10.0019	10.0194	10.0229	10.1016	10.0219	10.0296	9.9837	9.9233	9.9701	10.0112	10.0184
15	15.00283	15.01652	15.05731	15.19094	15.26788	15.35117	15.0080	15.0477	15.0386	15.0307	15.2085	15.1189	15.0340	15.0071	14.9579	15.0195	15.0332	15.0283
20	20.05247	20.05602	20.09777	20.26885	20.35852	20.46002	19.9968	20.0406	20.0234	20.0020	20.1981	20.0786	19.9893	19.9716	19.9432	20.0109	20.0172	19.9994

Table 3: MSE of tail index estimates for the Hall model with $a = 0.0001$, $b = 0.45$ and $n = 5000$

MSE – Hall model																		
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.00188	0.00212	0.00130	0.00106	0.00124	0.00160	0.00522	0.00246	0.00265	0.00798	0.07938	0.03984	0.07754	0.52951	0.02805	0.01352	0.00221	0.00096
0.8	0.00246	0.00288	0.00249	0.00207	0.00286	0.00400	0.01305	0.00684	0.00342	0.00835	0.08678	0.04263	0.01159	0.01698	0.03717	0.01701	0.00374	0.00284
1	0.00303	0.00373	0.00386	0.00286	0.00421	0.00604	0.02076	0.01085	0.00401	0.00847	0.09435	0.05010	0.00903	0.00434	0.04745	0.02216	0.00488	0.00408
1.2	0.00401	0.00507	0.00545	0.00416	0.00625	0.00900	0.02821	0.01440	0.00516	0.00919	0.10370	0.05737	0.01130	0.00901	0.05073	0.02494	0.00585	0.00537
1.5	0.00558	0.00724	0.00845	0.00638	0.00971	0.01393	0.04742	0.02047	0.00666	0.00995	0.11235	0.05903	0.01558	0.01932	0.07167	0.03387	0.00775	0.00700
1.8	0.00684	0.00866	0.01110	0.00845	0.01300	0.01875	0.06542	0.03213	0.00811	0.01093	0.13215	0.07045	0.01963	0.02757	0.08678	0.04247	0.00956	0.00841
2	0.00834	0.01102	0.01437	0.01190	0.01847	0.02650	0.08345	0.03970	0.00923	0.01099	0.14179	0.07598	0.02164	0.03164	0.10806	0.05147	0.01093	0.00915
3	0.01824	0.02486	0.03412	0.02734	0.04259	0.06061	0.18143	0.08654	0.01965	0.01661	0.24217	0.12967	0.03673	0.04965	0.20095	0.09756	0.02143	0.01549
4	0.02943	0.04058	0.05737	0.04775	0.07502	0.10741	0.31248	0.16710	0.03759	0.02473	0.38782	0.17513	0.05237	0.06220	0.34143	0.18008	0.04041	0.02502
5	0.04758	0.06330	0.09461	0.07482	0.11678	0.16668	0.47418	0.24513	0.05101	0.03348	0.56286	0.28666	0.07201	0.07319	0.50927	0.25789	0.05226	0.03312
5.5	0.05372	0.07491	0.10321	0.08389	0.13225	0.19016	0.56094	0.26920	0.05687	0.03511	0.64881	0.32658	0.07354	0.07962	0.58551	0.27662	0.05869	0.03503
6	0.07109	0.10184	0.13787	0.10564	0.16160	0.22557	0.70037	0.34441	0.07388	0.04430	0.77809	0.38736	0.09149	0.08908	0.71698	0.35253	0.07508	0.04448
10	0.18769	0.25725	0.36751	0.27388	0.41439	0.57550	1.90466	1.01983	0.20337	0.10903	2.21858	1.04682	0.21307	0.16810	1.93575	1.02316	0.20509	0.10980
15	0.40046	0.53141	0.79384	0.70361	1.10252	1.55863	4.83165	2.31363	0.44663	0.22651	4.72359	2.50437	0.47962	0.30069	4.89012	2.33588	0.45041	0.22776
20	0.78408	1.08216	1.55822	1.18085	1.80120	2.52382	7.90379	4.15631	0.78665	0.38894	8.35936	4.57430	0.87284	0.44662	7.93124	4.16485	0.78938	0.38900

Table 4: Mean of tail index estimates for the Hall model with $a = 0.0001$, $b = 0.45$ and $n = 5000$

Mean – Hall model																		
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.46165	0.46069	0.47852	0.48176	0.48983	0.49915	0.49952	0.49217	0.45270	0.41178	0.52103	0.54966	0.76236	1.22430	0.44965	0.47684	0.49782	0.50029
0.8	0.76269	0.76197	0.77984	0.78580	0.79515	0.80569	0.79573	0.78952	0.75318	0.71182	0.80074	0.81230	0.85498	0.91280	0.73605	0.76526	0.77584	0.76358
1	0.96203	0.96111	0.97884	0.98763	0.99805	1.00979	0.99702	0.99476	0.95578	0.91350	1.01793	0.99624	1.00302	0.99959	0.93672	0.96632	0.97218	0.95423
1.2	1.16250	1.16143	1.17909	1.19011	1.20138	1.21402	1.19340	1.18845	1.15042	1.11105	1.19702	1.19328	1.17086	1.13900	1.14965	1.16806	1.16531	1.14408
1.5	1.46299	1.46199	1.47942	1.49575	1.50919	1.52405	1.49244	1.49010	1.45222	1.41139	1.50511	1.49893	1.44104	1.38428	1.43879	1.46212	1.46205	1.43714
1.8	1.76632	1.76734	1.78502	1.79997	1.81392	1.82866	1.79445	1.79241	1.75338	1.71016	1.80697	1.78997	1.71638	1.65593	1.74408	1.76620	1.76064	1.73189
2	1.96693	1.96840	1.98949	2.00638	2.02268	2.03983	1.99476	1.99188	1.95626	1.91595	2.00677	1.99794	1.92037	1.84274	1.93619	1.96263	1.96173	1.93409
3	2.96580	2.96315	2.98504	3.01232	3.03179	3.05362	3.02507	2.99910	2.95497	2.91072	2.98690	2.99489	2.88852	2.80366	2.97112	2.97717	2.95939	2.92408
4	3.96767	3.97037	3.99528	4.03377	4.06147	4.09014	3.97304	3.96907	3.94004	3.90915	3.99628	3.97732	3.89339	3.78738	3.90325	3.93595	3.93758	3.91373
5	4.96931	4.96789	4.99006	5.03613	5.06460	5.09531	4.94899	4.97731	4.95818	4.90762	5.03688	5.01691	4.86812	4.78198	4.89010	4.94297	4.95486	4.91345
5.5	5.48290	5.48427	5.51346	5.56764	5.60339	5.64090	5.51329	5.48195	5.45078	5.41169	5.50130	5.49019	5.38597	5.28300	5.46547	5.46264	5.45051	5.41717
6	5.97209	5.98158	6.01358	6.05871	6.09524	6.13120	5.99532	5.99136	5.95519	5.91020	6.02130	5.99698	5.87538	5.77721	5.93840	5.96282	5.95278	5.91454
10	9.99728	10.00679	10.02239	10.11439	10.16488	10.21437	9.98061	9.99198	9.97229	9.93413	10.08742	9.99401	9.90377	9.76171	9.91851	9.96068	9.96646	9.93392
15	14.96323	14.98370	15.03649	15.17210	15.26120	15.34867	15.00298	15.03774	14.99160	14.94211	15.19433	15.09092	14.90830	14.78471	14.95305	15.00989	14.98775	14.94260
20	20.01166	20.02243	20.07849	20.25102	20.35461	20.46044	19.99178	20.03066	19.97630	19.91346	20.18388	20.05060	19.86385	19.74957	19.93829	20.00115	19.97125	19.91298

Table 5: MSE of tail index estimates for the Burr distribution with $a = 0.0001$, $b = 0.45$ and $n = 5000$.

MSE – Burr distiribution																		
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.00713	0.02882	0.00824	0.00854	0.01069	0.00732	0.00749	0.00587	0.01979	0.05256	0.07840	0.03688	0.00792	0.00375	0.02825	0.01395	0.00465	0.00688
0.8	0.00179	0.01300	0.00364	0.00368	0.00615	0.00570	0.01347	0.00739	0.00777	0.02417	0.08665	0.04234	0.00841	0.00409	0.03691	0.01679	0.00441	0.00597
1	0.00218	0.00867	0.00404	0.00333	0.00604	0.00695	0.02099	0.01126	0.00546	0.01514	0.09435	0.05010	0.00903	0.00434	0.04725	0.02197	0.00499	0.00553
1.2	0.00381	0.00745	0.00531	0.00422	0.00740	0.00943	0.02832	0.01446	0.00398	0.00896	0.10382	0.05746	0.01058	0.00542	0.05061	0.02476	0.00510	0.00443
1.5	0.00630	0.00780	0.00841	0.00635	0.01058	0.01417	0.04755	0.02053	0.00492	0.00557	0.11273	0.05945	0.01238	0.00620	0.07145	0.03355	0.00659	0.00427
1.8	0.00788	0.00891	0.01113	0.00840	0.01361	0.01879	0.06558	0.03226	0.00624	0.00432	0.13275	0.07084	0.01303	0.00720	0.08659	0.04223	0.00825	0.00441
2	0.00923	0.01100	0.01438	0.01184	0.01908	0.02645	0.08362	0.03983	0.00758	0.00514	0.14250	0.07676	0.01586	0.00735	0.10782	0.05118	0.00968	0.00541
3	0.01845	0.02230	0.03418	0.02712	0.04188	0.05963	0.18202	0.08690	0.01793	0.00897	0.24295	0.13102	0.02522	0.01189	0.20101	0.09752	0.02007	0.01003
4	0.02934	0.03717	0.05743	0.04785	0.07339	0.10535	0.31260	0.16700	0.03460	0.01686	0.38934	0.17584	0.04254	0.01811	0.34092	0.17943	0.03738	0.01801
5	0.04836	0.05735	0.09327	0.07439	0.11241	0.16240	0.47418	0.24533	0.04980	0.02544	0.56605	0.29024	0.05643	0.02717	0.50877	0.25746	0.05069	0.02613
5.5	0.05533	0.06950	0.10445	0.08459	0.12852	0.18586	0.56165	0.26945	0.05495	0.02781	0.65113	0.32892	0.06243	0.03432	0.58573	0.27649	0.05676	0.02864
6	0.07222	0.09259	0.14075	0.10539	0.15621	0.22038	0.70099	0.34494	0.07245	0.03680	0.78123	0.39024	0.07812	0.04152	0.71702	0.35250	0.07341	0.03774
10	0.19440	0.23432	0.37001	0.27091	0.39673	0.55880	1.90544	1.02085	0.20394	0.10620	2.22547	1.05137	0.20832	0.11536	1.93590	1.02353	0.20502	0.10672
15	0.40458	0.49402	0.77201	0.68487	1.04374	1.50364	4.83328	2.31597	0.44962	0.22565	4.73527	2.51673	0.47819	0.26020	4.89123	2.33761	0.45289	0.22676
20	0.79872	0.98042	1.55130	1.15213	1.70514	2.43309	7.90576	4.15925	0.78874	0.38327	8.37301	4.58678	0.86216	0.39111	7.93263	4.16714	0.79090	0.38320

Table 6: Mean of tail index estimates for the Burr distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	Mean – Burr distiribution																	
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.57906	0.66663	0.58291	0.58774	0.59727	0.57481	0.54117	0.55526	0.63841	0.72831	0.50014	0.50704	0.50256	0.50063	0.46215	0.49516	0.54425	0.57413
0.8	0.81688	0.90609	0.83283	0.84232	0.85831	0.84281	0.81238	0.81965	0.87949	0.95293	0.79569	0.80211	0.80178	0.80015	0.74282	0.77779	0.82839	0.86352
1	0.99027	1.07916	1.01221	1.02539	1.04507	1.03404	1.00716	1.01499	1.05660	1.11807	1.01793	0.99624	1.00302	0.99959	0.94162	0.97628	1.02241	1.05615
1.2	1.17284	1.26037	1.20132	1.21573	1.23826	1.23109	1.20069	1.20348	1.23288	1.28638	1.20031	1.19982	1.20278	1.20091	1.15375	1.17655	1.21204	1.24296
1.5	1.46012	1.54443	1.49311	1.51219	1.53805	1.53599	1.49812	1.50182	1.51865	1.55703	1.51162	1.51185	1.50231	1.49940	1.44248	1.46974	1.50512	1.53072
1.8	1.75933	1.83941	1.79704	1.81297	1.83914	1.83916	1.79966	1.80296	1.81078	1.83534	1.81552	1.80692	1.79540	1.80217	1.74781	1.77374	1.80138	1.81983
2	1.96075	2.03517	2.00138	2.01873	2.04616	2.04986	1.99984	2.00203	2.00981	2.03170	2.01626	2.01676	2.00786	2.00324	1.93996	1.97020	2.00147	2.01912
3	2.97596	3.01974	3.00011	3.02856	3.05246	3.06352	3.03018	3.00907	3.00277	3.00527	2.99908	3.01908	2.99816	3.00070	2.97535	2.98547	2.99929	3.00282
4	3.99166	4.01982	4.01360	4.05339	4.07991	4.09977	3.97813	3.97905	3.98693	3.99886	4.00956	4.00346	4.01158	3.99785	3.90768	3.94465	3.97868	3.99245
5	5.00217	5.01324	5.01086	5.05806	5.08191	5.10494	4.95400	4.98725	5.00526	4.99622	5.05064	5.04415	4.98981	4.99856	4.89459	4.95190	4.99731	4.99348
5.5	5.51794	5.52775	5.53345	5.58960	5.61935	5.64968	5.51836	5.49188	5.49771	5.50025	5.51516	5.51758	5.50907	5.50142	5.47007	5.47167	5.49324	5.49796
6	6.00764	6.02261	6.03436	6.08083	6.11112	6.14048	6.00039	6.00134	6.00222	5.99872	6.03529	6.02460	5.99911	5.99668	5.94304	5.97196	5.99596	5.99594
10	10.03581	10.04552	10.04847	10.13606	10.17805	10.22186	9.98566	10.00193	10.01935	10.02288	10.10161	10.02190	10.02959	9.98369	9.92330	9.97013	10.01121	10.01839
15	15.00282	15.01652	15.05731	15.19093	15.26788	15.35117	15.00799	15.04767	15.03860	15.03066	15.20854	15.11895	15.03395	15.00711	14.95789	15.01949	15.03321	15.02830
20	20.05247	20.05602	20.09777	20.26885	20.35852	20.46002	19.99684	20.04065	20.02340	20.00199	20.19813	20.07862	19.98935	19.97160	19.94321	20.01089	20.01719	19.99938

Table 7: MSE of tail index estimates for the Fréchet distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	MSE – Frechet distribution																	
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.00052	0.00102	0.00098	0.00083	0.00135	0.00179	0.00532	0.00249	0.00062	0.00107	0.07829	0.03671	0.01293	0.07468	0.02842	0.01387	0.00251	0.00125
0.8	0.00127	0.00249	0.00236	0.00212	0.00334	0.00437	0.01315	0.00686	0.00179	0.00275	0.08661	0.04227	0.00948	0.02768	0.03713	0.01694	0.00343	0.00195
1	0.00192	0.00373	0.00378	0.00306	0.00494	0.00658	0.02091	0.01108	0.00306	0.00457	0.09434	0.05010	0.00905	0.01113	0.04729	0.02200	0.00446	0.00297
1.2	0.00301	0.00580	0.00538	0.00453	0.00728	0.00961	0.02835	0.01449	0.00372	0.00594	0.10390	0.05754	0.01090	0.00557	0.05058	0.02473	0.00505	0.00373
1.5	0.00483	0.00927	0.00863	0.00723	0.01149	0.01495	0.04765	0.02068	0.00618	0.00957	0.11307	0.05999	0.01530	0.01352	0.07133	0.03342	0.00703	0.00613
1.8	0.00650	0.01304	0.01180	0.00972	0.01537	0.01998	0.06577	0.03255	0.00860	0.01324	0.13343	0.07171	0.01976	0.04018	0.08645	0.04211	0.00924	0.00889
2	0.00807	0.01671	0.01541	0.01363	0.02162	0.02810	0.08384	0.04016	0.01080	0.01809	0.14342	0.07832	0.02955	0.06587	0.10761	0.05099	0.01119	0.01244
3	0.01901	0.03564	0.03615	0.02991	0.04733	0.06263	0.18302	0.08785	0.02511	0.03800	0.24473	0.13555	0.07162	0.27067	0.20118	0.09773	0.02479	0.02986
4	0.03119	0.06297	0.06124	0.05273	0.08459	0.11108	0.31298	0.16741	0.04400	0.06776	0.39357	0.18180	0.14967	0.56477	0.34002	0.17856	0.04293	0.05434
5	0.05146	0.09580	0.09858	0.08063	0.12791	0.17015	0.47439	0.24675	0.07033	0.10430	0.57598	0.30678	0.21459	0.94154	0.50767	0.25708	0.06567	0.08833
5.5	0.05867	0.12282	0.11192	0.09422	0.15185	0.19740	0.56388	0.27123	0.07713	0.12623	0.65975	0.34420	0.27633	1.16623	0.58656	0.27702	0.07400	0.10853
6	0.07669	0.15392	0.15003	0.11519	0.18111	0.23228	0.70318	0.34784	0.10056	0.15309	0.79339	0.40941	0.32546	1.38880	0.71742	0.35347	0.09539	0.13350
10	0.20564	0.41359	0.39522	0.29741	0.46409	0.58968	1.91017	1.02915	0.29044	0.46010	2.26720	1.10113	0.96774	3.87544	1.93737	1.02819	0.27966	0.42213
15	0.43320	0.83865	0.81957	0.74279	1.19649	1.57917	4.84728	2.34051	0.65255	1.01262	4.83864	2.66987	2.15824	8.91936	4.90116	2.35706	0.63968	0.95938
20	0.84280	1.63709	1.65125	1.25782	1.97492	2.56411	7.92852	4.20083	1.12130	1.69086	8.53657	4.81397	3.65813	15.41931	7.94944	4.20156	1.10162	1.61959

Table 8: Mean of tail index estimates for the Fréchet distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	Mean – Fréchet distribution																	
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.50001	0.51934	0.50554	0.51071	0.51626	0.51533	0.50589	0.50470	0.51278	0.52851	0.49485	0.49612	0.51278	0.23317	0.44949	0.47675	0.49724	0.49751
0.8	0.80033	0.83148	0.80859	0.81676	0.82536	0.82341	0.80287	0.80355	0.82137	0.84550	0.79356	0.79761	0.82137	0.64577	0.73855	0.77039	0.80120	0.81306
1	0.99955	1.03840	1.00843	1.02010	1.03088	1.02851	1.00461	1.00985	1.02939	1.05864	1.01786	0.99593	1.02939	0.91700	0.94039	0.97375	1.00886	1.02656
1.2	1.19916	1.24544	1.21000	1.22373	1.23658	1.23374	1.20153	1.20453	1.22880	1.26680	1.20225	1.20361	1.22880	1.18746	1.15433	1.17735	1.21097	1.23469
1.5	1.49927	1.55779	1.51158	1.53152	1.54825	1.54518	1.50134	1.50776	1.53904	1.58481	1.51656	1.52173	1.53904	1.58436	1.44465	1.47379	1.51991	1.55259
1.8	1.80182	1.87369	1.81955	1.83784	1.85695	1.85162	1.80410	1.81159	1.84827	1.90020	1.82336	1.82271	1.84827	1.98031	1.75100	1.77998	1.82914	1.86901
2	2.00221	2.08081	2.02511	2.04553	2.06787	2.06377	2.00485	2.01190	2.05603	2.11728	2.02593	2.03635	2.05603	2.24064	1.94369	1.97758	2.03655	2.08498
3	2.99956	3.11202	3.02541	3.05844	3.08953	3.08260	3.03795	3.02440	3.08176	3.16879	3.01783	3.05726	3.08176	3.50680	2.98178	2.99822	3.06504	3.13895
4	3.99806	4.15501	4.04158	4.08618	4.13153	4.12416	3.98845	3.99950	4.09264	4.22343	4.03684	4.05866	4.09264	4.73683	3.91664	3.96249	4.07097	4.18824
5	4.99836	5.18901	5.04293	5.09581	5.14766	5.13483	4.96669	5.01273	5.13825	5.27804	5.08602	5.11610	5.13825	5.95287	4.90597	4.97477	5.11684	5.24648
5.5	5.51185	5.72309	5.56795	5.63069	5.69207	5.68244	5.53250	5.51988	5.64356	5.81083	5.55435	5.59716	5.64356	6.56002	5.48290	5.49711	5.62564	5.77963
6	5.99820	6.23783	6.07132	6.12508	6.19103	6.17627	6.01580	6.03203	6.16170	6.33775	6.07845	6.11217	6.16170	7.15622	5.95712	6.00005	6.14197	6.30603
10	10.01521	10.40982	10.10737	10.20845	10.31191	10.28056	10.01124	10.05293	10.28534	10.59000	10.17458	10.16922	10.28534	11.93099	9.94755	10.01854	10.26370	10.55645
15	14.97201	15.56123	15.14773	15.29998	15.46911	15.44045	15.04602	15.12401	15.43693	15.87897	15.31803	15.34101	15.43693	17.93024	14.99462	15.09325	15.41810	15.84769
20	20.01108	20.78027	20.21707	20.41333	20.62558	20.57809	20.04805	20.14306	20.55584	21.13322	20.34459	20.37478	20.55584	23.86306	19.99311	20.11070	20.53614	21.10165

Table 9: MSE of tail index estimates for the log-logistic distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	MSE – log-logistic distribution																		
	WLS			OLS			Hill				Pickands				DEdH				
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.00090	0.00226	0.00104	0.00058	0.00164	0.00189	0.00535	0.00253	0.00117	0.00366	0.07824	0.03677	0.02454	0.16518	0.02849	0.01395	0.00258	0.00132
0.8	0.00228	0.00570	0.00253	0.00142	0.00406	0.00461	0.01320	0.00694	0.00323	0.00936	0.08658	0.04225	0.01079	0.02753	0.03712	0.01693	0.00360	0.00282
1	0.00333	0.00867	0.00404	0.00218	0.00604	0.00695	0.02099	0.01126	0.00546	0.01514	0.09435	0.05010	0.00903	0.00434	0.04725	0.02197	0.00499	0.00553
1.2	0.00490	0.01287	0.00569	0.00340	0.00884	0.01012	0.02844	0.01466	0.00673	0.02056	0.10399	0.05768	0.01360	0.02697	0.05053	0.02469	0.00593	0.00842
1.5	0.00785	0.02051	0.00913	0.00543	0.01406	0.01581	0.04781	0.02096	0.01116	0.03282	0.11348	0.06080	0.02900	0.12764	0.07120	0.03331	0.00899	0.01605
1.8	0.01061	0.02978	0.01262	0.00724	0.01900	0.02118	0.06600	0.03302	0.01592	0.04627	0.13425	0.07317	0.05022	0.29698	0.08632	0.04207	0.01275	0.02565
2	0.01478	0.03717	0.01644	0.00894	0.02635	0.02970	0.08412	0.04071	0.02001	0.06024	0.14454	0.08077	0.07904	0.43812	0.10742	0.05092	0.01601	0.03568
3	0.03222	0.07916	0.03849	0.02126	0.05692	0.06589	0.18415	0.08927	0.04568	0.13058	0.24725	0.14322	0.23504	1.39793	0.20142	0.09827	0.03878	0.09373
4	0.05711	0.14252	0.06546	0.03519	0.10303	0.11738	0.31358	0.16867	0.07686	0.23122	0.39937	0.19417	0.48909	2.67258	0.33928	0.17833	0.06635	0.17647
5	0.08678	0.21755	0.10486	0.05775	0.15429	0.17900	0.47494	0.24949	0.12809	0.35875	0.58857	0.33424	0.74548	4.28575	0.50684	0.25777	0.11073	0.29254
5.5	0.10329	0.27769	0.11997	0.06440	0.18858	0.21014	0.56653	0.27462	0.14400	0.43785	0.67158	0.37273	0.94849	5.23765	0.58773	0.27886	0.12811	0.36435
6	0.12484	0.33630	0.16014	0.08565	0.22181	0.24578	0.70587	0.35267	0.18216	0.52322	0.80944	0.44460	1.11928	6.22083	0.71823	0.35605	0.16223	0.44228
10	0.32381	0.93181	0.42364	0.22689	0.57486	0.62516	1.91626	1.04280	0.52593	1.52568	2.32006	1.19591	3.26747	17.26809	1.94004	1.03766	0.48859	1.37924
15	0.80177	1.93354	0.87537	0.49588	1.44711	1.66543	4.86429	2.37703	1.18987	3.39158	4.96709	2.92554	7.28308	39.28978	4.91389	2.38768	1.13879	3.17908
20	1.36367	3.62455	1.76511	0.93154	2.41712	2.71394	7.95673	4.26399	2.05073	5.82460	8.74490	5.22308	12.55848	68.88259	7.97140	4.25644	1.97947	5.54155

Table 10: Mean of tail index estimates for the log-logistic distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	Mean – log-logistic distribution																	
	WLS			OLS			Hill				Pickands				DEdH			
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.49529	0.53964	0.50750	0.51332	0.52334	0.51811	0.5072	0.5073	0.5263	0.5582	0.4897	0.4857	0.3697	0.0977	0.4495	0.4767	0.4972	0.4973
0.8	0.79272	0.86401	0.81170	0.82093	0.83669	0.82784	0.8049	0.8077	0.8431	0.8930	0.7916	0.7937	0.7497	0.6462	0.7393	0.7719	0.8093	0.8307
1	0.99027	1.07916	1.01221	1.02539	1.04507	1.03404	1.0072	1.0150	1.0566	1.1181	1.0179	0.9962	1.0030	0.9996	0.9416	0.9763	1.0224	1.0562
1.2	1.18770	1.29421	1.21457	1.22994	1.25356	1.24036	1.2046	1.2107	1.2612	1.3377	1.2044	1.2081	1.2534	1.3459	1.1561	1.1809	1.2298	1.2759
1.5	1.48524	1.61902	1.51711	1.53942	1.56954	1.55345	1.5052	1.5155	1.5798	1.6740	1.5217	1.5323	1.6268	1.8473	1.4472	1.4789	1.5470	1.6119
1.8	1.78494	1.94724	1.82640	1.84732	1.88253	1.86155	1.8087	1.8208	1.8973	2.0070	1.8314	1.8393	1.9904	2.3370	1.7543	1.7866	1.8645	1.9460
2	1.98339	2.16214	2.03297	2.05602	2.09623	2.07487	2.0099	2.0221	2.1102	2.2359	2.0358	2.0568	2.2488	2.6550	1.9475	1.9852	2.0772	2.1738
3	2.97156	3.23347	3.03699	3.07415	3.13191	3.09921	3.0457	3.0398	3.1634	3.3469	3.0368	3.0964	3.4538	4.1751	2.9882	3.0110	3.1331	3.2873
4	3.95996	4.31761	4.05704	4.10700	4.18823	4.14633	3.9988	4.0200	4.2009	4.4607	4.0644	4.1149	4.6625	5.6264	3.9256	3.9804	4.1656	4.3956
5	4.95067	5.39222	5.06240	5.12176	5.21847	5.16249	4.9794	5.0383	5.2744	5.5743	5.1216	5.1891	5.8229	7.0597	4.9174	4.9978	5.2393	5.5131
5.5	5.46078	5.94641	5.58955	5.65967	5.76993	5.71291	5.5467	5.5480	5.7929	6.1372	5.5938	5.6778	6.4338	7.7766	5.4958	5.5227	5.7613	6.0762
6	5.94160	6.48220	6.09483	6.15662	6.27630	6.20968	6.0312	6.0629	6.3250	6.6939	6.1219	6.2009	7.0113	8.4807	5.9712	6.0283	6.2916	6.6324
10	9.92094	10.81825	10.14607	10.26088	10.45390	10.33567	10.0369	10.1042	10.5576	11.1858	10.2479	10.3181	11.7349	14.1324	9.9719	10.0672	10.5223	11.1223
15	14.83099	16.17163	15.20821	15.37916	15.68248	15.52440	15.0841	15.2007	15.8447	16.7701	15.4281	15.5653	17.5854	21.2334	15.0314	15.1674	15.8122	16.7090
20	19.82303	21.59181	20.29650	20.51804	20.90876	20.68908	20.0994	20.2460	21.1009	22.3216	20.4918	20.6740	23.3888	28.2605	20.0431	20.2110	21.0675	22.2602

Table 11: MSE of tail index estimates for the GEV distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	MSE – GEV distribution																		
	WLS			OLS			Hill				Pickands				DEdH				
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.00138	0.06661	0.00441	0.00528	0.01501	0.00698	0.00763	0.00624	0.02610	0.08309	0.07829	0.03671	0.01293	0.07468	0.02833	0.01403	0.00490	0.00795
0.8	0.00787	0.04276	0.00283	0.00246	0.00937	0.00565	0.01356	0.00762	0.01276	0.05037	0.08661	0.04227	0.00948	0.02768	0.03691	0.01680	0.00517	0.00993
1	0.01292	0.03502	0.00437	0.00279	0.00894	0.00702	0.02108	0.01150	0.00994	0.03943	0.09434	0.05010	0.00905	0.01113	0.04720	0.02195	0.00601	0.01115
1.2	0.01732	0.03170	0.00606	0.00401	0.01026	0.00955	0.02841	0.01462	0.00756	0.03090	0.10390	0.05754	0.01090	0.00557	0.05056	0.02471	0.00610	0.01088
1.5	0.02101	0.03130	0.00954	0.00626	0.01390	0.01449	0.04767	0.02073	0.00837	0.02654	0.11307	0.05999	0.01530	0.01352	0.07132	0.03341	0.00779	0.01236
1.8	0.02161	0.03374	0.01211	0.00836	0.01748	0.01936	0.06577	0.03257	0.00989	0.02481	0.13343	0.07171	0.01976	0.04018	0.08644	0.04211	0.00979	0.01395
2	0.02191	0.03651	0.01516	0.01176	0.02383	0.02728	0.08385	0.04017	0.01172	0.02763	0.14342	0.07832	0.02955	0.06587	0.10761	0.05099	0.01161	0.01701
3	0.02584	0.05255	0.03569	0.02787	0.04933	0.06183	0.18302	0.08785	0.02527	0.04103	0.24473	0.13555	0.07162	0.27067	0.20118	0.09773	0.02488	0.03173
4	0.03392	0.07794	0.06007	0.05074	0.08663	0.11020	0.31298	0.16741	0.04402	0.06868	0.39357	0.18180	0.14967	0.56477	0.34002	0.17856	0.04294	0.05496
5	0.05238	0.10742	0.09761	0.07934	0.12939	0.16947	0.47439	0.24675	0.07033	0.10457	0.57598	0.30678	0.21459	0.94154	0.50767	0.25708	0.06567	0.08853
5.5	0.05862	0.13352	0.11039	0.09287	0.15338	0.19662	0.56388	0.27123	0.07713	0.12638	0.65975	0.34420	0.27633	1.16623	0.58656	0.27702	0.07400	0.10864
6	0.07697	0.16287	0.14872	0.11422	0.18230	0.23167	0.70318	0.34784	0.10056	0.15316	0.79339	0.40941	0.32546	1.38880	0.71742	0.35347	0.09539	0.13356
10	0.20558	0.41561	0.39484	0.29725	0.46433	0.58953	1.91017	1.02915	0.29044	0.46010	2.26720	1.10113	0.96774	3.87544	1.93737	1.02819	0.27966	0.42213
15	0.43321	0.83883	0.81954	0.74278	1.19651	1.57916	4.84728	2.34051	0.65255	1.01262	4.83864	2.66987	2.15824	8.91936	4.90116	2.35706	0.63968	0.95938
20	0.84280	1.63711	1.65125	1.25782	1.97492	2.56411	7.92852	4.20083	1.12130	1.69086	8.53657	4.81397	3.65813	15.41931	7.94944	4.20156	1.10162	1.61959

Table 12: Mean of tail index estimates for the GEV distribution with $a = 0.0001, b = 0.45$ and $n = 5000$.

	Mean – GEV distribution																		
	WLS			OLS			Hill				Pickands				DEdH				
α	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$\widetilde{p}=1$	$\widetilde{p}=2$	$\widetilde{p}=3$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$	$k=50$	$k=100$	$k=500$	$k=1000$
0.5	0.48180	0.75548	0.55379	0.56652	0.61728	0.57246	0.5426	0.5583	0.6595	0.7874	0.4949	0.4961	0.4278	0.2332	0.4622	0.4953	0.5460	0.5803
0.8	0.72166	1.00174	0.80590	0.82351	0.88119	0.84204	0.8145	0.8240	0.9061	1.0225	0.7936	0.7976	0.7658	0.6458	0.7436	0.7794	0.8382	0.8886
1	0.89751	1.17965	0.98679	1.00863	1.07003	1.03446	1.0098	1.0202	1.0873	1.1953	1.0179	0.9959	0.9922	0.9170	0.9429	0.9789	1.0373	1.0927
1.2	1.08219	1.36562	1.17794	1.20096	1.26535	1.23277	1.2038	1.2097	1.2676	1.3711	1.2022	1.2036	1.2165	1.1875	1.1555	1.1801	1.2319	1.2901
1.5	1.37413	1.65774	1.47247	1.50102	1.56860	1.53963	1.5020	1.5095	1.5604	1.6550	1.5166	1.5217	1.5520	1.5844	1.4450	1.4748	1.5326	1.5940
1.8	1.67846	1.96060	1.77988	1.80559	1.87318	1.84486	1.8043	1.8122	1.8598	1.9467	1.8234	1.8227	1.8796	1.9803	1.7511	1.7803	1.8365	1.8985
2	1.88346	2.16113	1.98677	2.01391	2.08247	2.05702	2.0049	2.0122	2.0636	2.1524	2.0259	2.0363	2.1147	2.2406	1.9437	1.9778	2.0415	2.1082
3	2.91802	3.17084	2.99644	3.03690	3.10009	3.07751	3.0379	3.0244	3.0827	3.1774	3.0178	3.0573	3.2113	3.5068	2.9818	2.9982	3.0657	3.1454
4	3.94875	4.19716	4.02046	4.07363	4.13907	4.12049	3.9885	3.9995	4.0927	4.2255	4.0368	4.0587	4.3220	4.7368	3.9166	3.9625	4.0710	4.1899
5	4.96987	5.21737	5.02796	5.08876	5.15262	5.13225	4.9667	5.0127	5.1383	5.2785	5.0860	5.1161	5.3909	5.9529	4.9060	4.9748	5.1168	5.2469
5.5	5.49063	5.74564	5.55569	5.62550	5.69596	5.68033	5.5325	5.5199	5.6436	5.8111	5.5543	5.5972	5.9555	6.5600	5.4829	5.4971	5.6256	5.7798
6	5.98228	6.25580	6.06125	6.12123	6.19409	6.17455	6.0158	6.0320	6.1617	6.3379	6.0784	6.1122	6.4888	7.1562	5.9571	6.0000	6.1420	6.3061
10	10.01366	10.41222	10.10578	10.20809	10.31228	10.28030	10.0112	10.0529	10.2853	10.5900	10.1746	10.1692	10.8586	11.9310	9.9475	10.0185	10.2637	10.5564
15	14.97192	15.56139	15.14761	15.29996	15.46914	15.44043	15.0460	15.1240	15.4369	15.8790	15.3180	15.3410	16.2748	17.9302	14.9946	15.0933	15.4181	15.8477
20	20.01108	20.78028	20.21706	20.41333	20.62559	20.57809	20.0481	20.1431	20.5558	21.1332	20.3446	20.3748	21.6425	23.8631	19.9931	20.1107	20.5361	21.1016

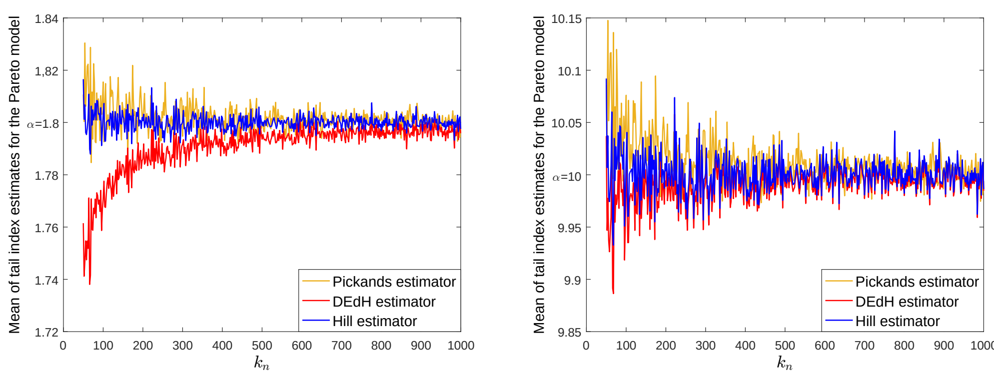

Figure 2: Mean of the Hill, Pickands and DEdH estimates for the strict Pareto distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

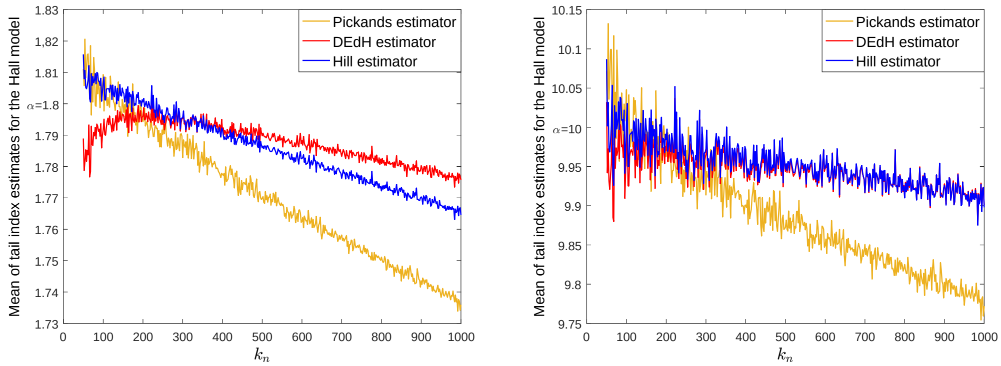

Figure 3: Mean of the Hill, Pickands and DEdH estimates for the Hall model with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

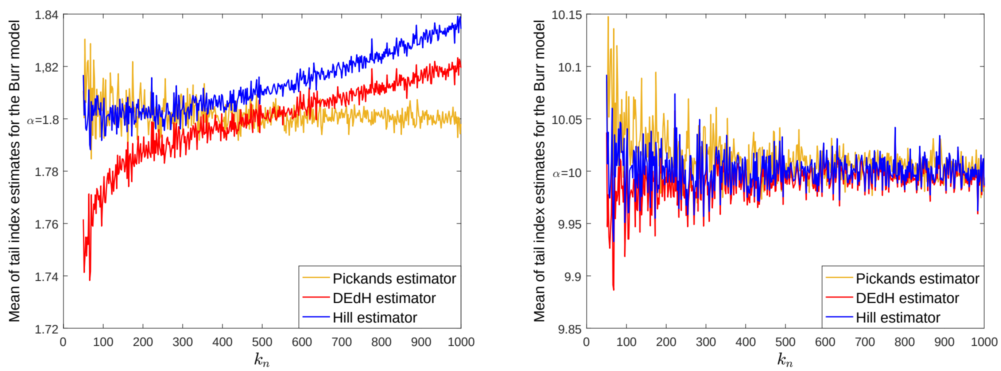

Figure 4: Mean of the Hill, Pickands and DEdH estimates for the Burr distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

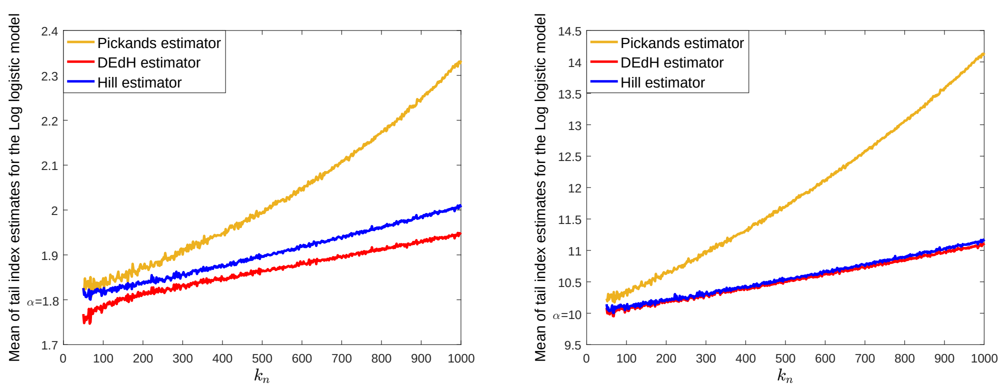

Figure 5: Mean of the Hill, Pickands and DEdH estimates for the log-logistic distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

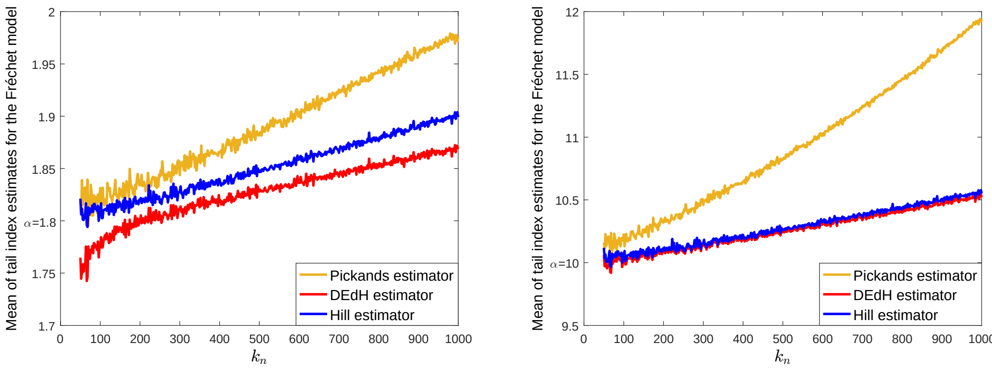

Figure 6: Mean of the Hill, Pickands and DEdH estimates for the Fréchet distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

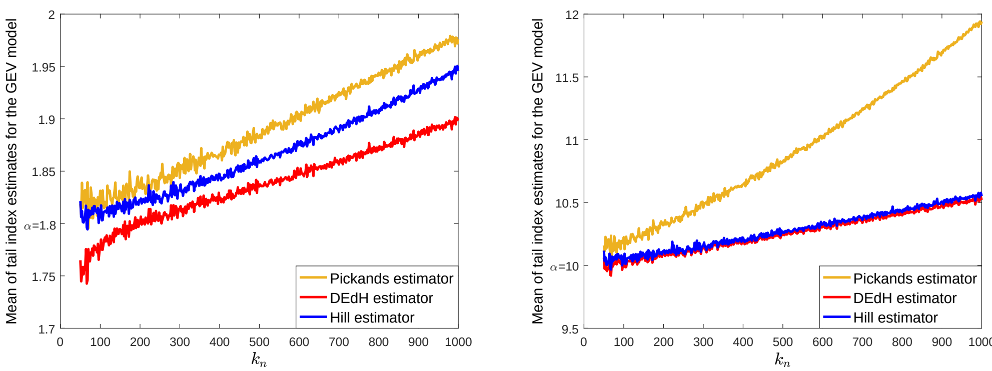

Figure 7: Mean of the Hill, Pickands and DEdH estimates for the GEV distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

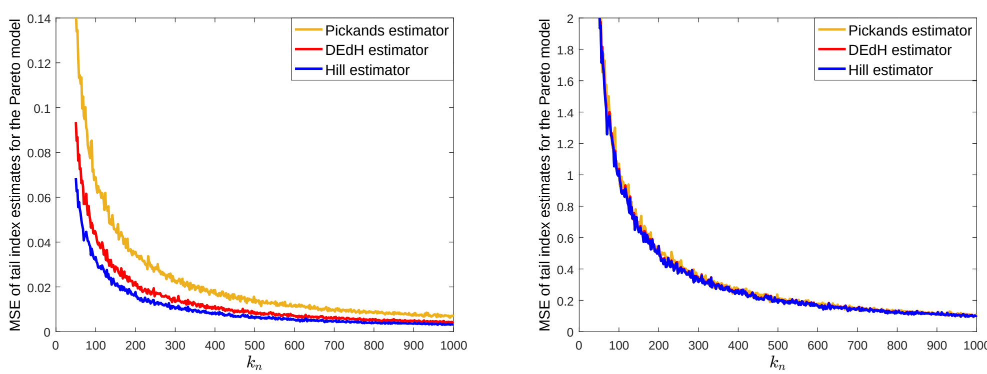

Figure 8: MSE of the Hill, Pickands and DEdH estimates for the strict Pareto distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

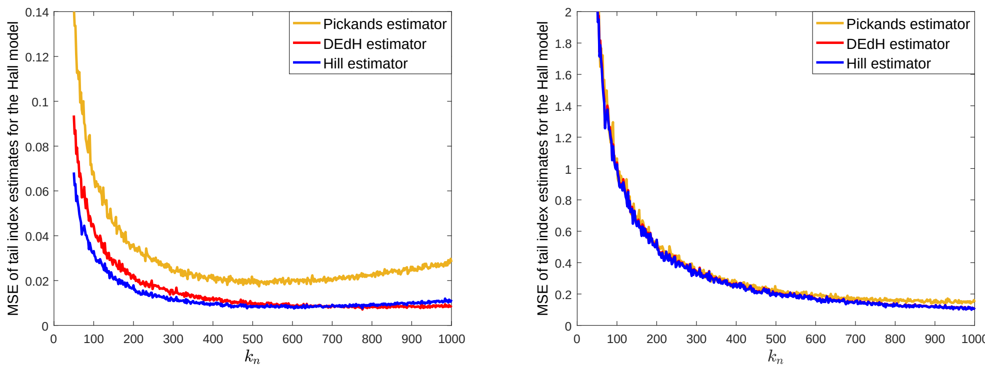

Figure 9: MSE of the Hill, Pickands and DEdH estimates for the Hall model with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

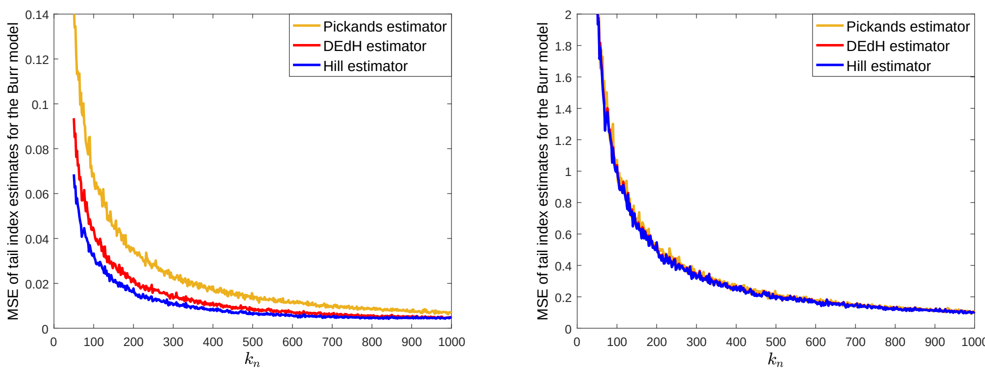

Figure 10: MSE of the Hill, Pickands and DEdH estimates for the Burr distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

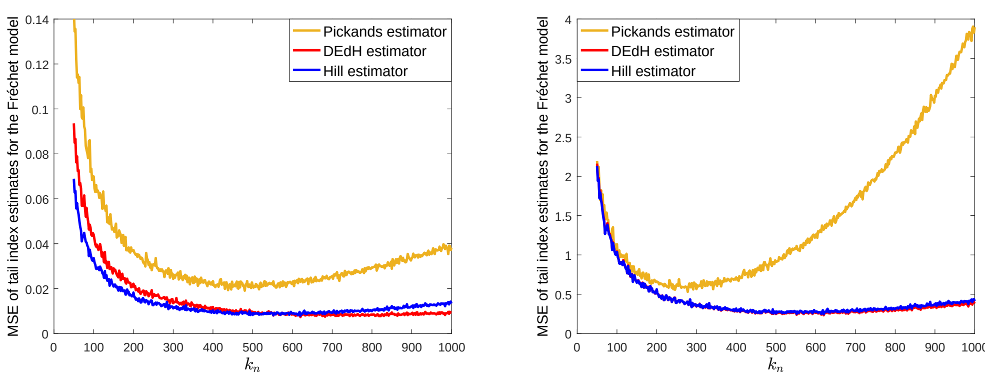

Figure 11: MSE of the Hill, Pickands and DEdH estimates for the Fréchet distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

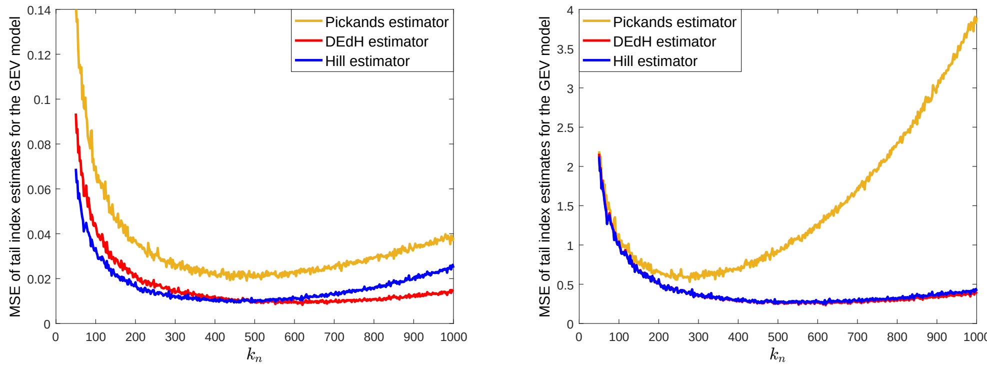

Figure 12: MSE of the Hill, Pickands and DEdH estimates for the GEV distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

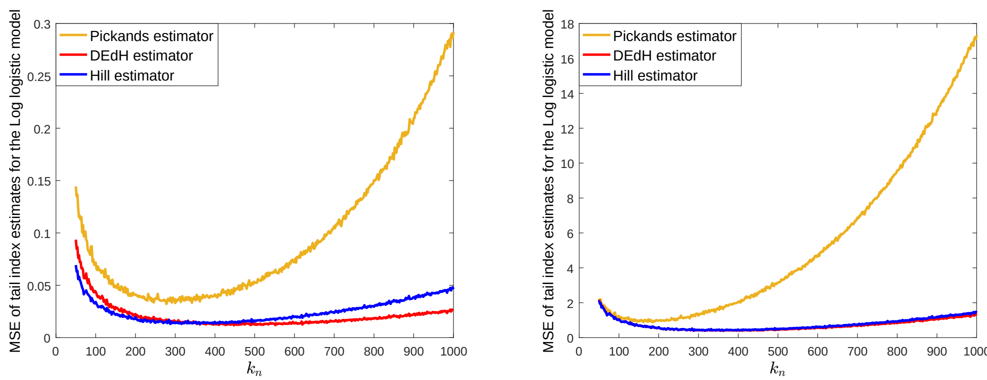

Figure 13: MSE of the Hill, Pickands and DEdH estimates for the log-logistic distribution with $\alpha = 1.8$ (left), $\alpha = 10$ (right).

4 Proofs

We deal only with the right tail, the proof for the left tail is similar. To ease notation we suppress the subscript 1 referring to the right tail. Let $q_n(s)$ be the quantile process defined as

$$q_n(s) = \sqrt{n}(Q_n(s) - Q(s)), \quad 0 < s < 1.$$

The proof of Theorem 1 is based on the strong approximation of the quantile process.

Theorem 3. (Csörgő and Révész (1978), [2, Theorem 6]). *Suppose that the conditions (Q_1) and (Q_2) are satisfied. Then on some probability space one can define a sequence $\{B_n(t) : 0 \le t \le 1\}_{n=1}^{\infty}$ of Brownian bridges such that*

$$\sup_{\delta_n \le s \le 1-\delta_n} \big| fQ(s) q_n(s) - B_n(s) \big| \overset{a.s.}{=} O(n^{-1/2} \log n),$$

where $\delta_n = 25 n^{-1} \log\log n$.

Proof of Theorem 1. We assume that the random variables $X_1, X_2, \dots$ are defined on the probability space given in Theorem 3. By a simple calculation,

$$X'WX = \begin{bmatrix} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \log^2 s_j R(s_j) & -\sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \log s_j R(s_j) & -2 \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \log s_j \cos(2\pi s_j) R(s_j) \dots \\ -\sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \log s_j R(s_j) & \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} R(s_j) & 2 \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \cos(2\pi s_j) R(s_j) \dots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

By means of Riemann sum approximation, we get

$$\lim_{n\to\infty} n^{-1} X'WX = M(a,b,R) \tag{13}$$

$$:= \begin{bmatrix} \int_a^b \log^2 u\, R(u) du & -\int_a^b \log u\, R(u) du & -2\int_a^b \log u \cos(2\pi u) R(u) du \dots \\ -\int_a^b \log u\, R(u) du & \int_a^b R(u) du & 2\int_a^b \cos(2\pi u) R(u) du \dots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

Set $\underline{\varepsilon} := \big(\varepsilon(s_{\lceil na \rceil}), \dots, \varepsilon(s_{\lfloor nb \rfloor})\big)'$ and

$$y^* := \big(\log Q(1 - s_{\lceil na \rceil}), \dots, \log Q(1 - s_{\lfloor nb \rfloor}))\big).$$

Then we have $y^* = X\beta_{\widetilde{p}}$, $\beta_{\widetilde{p}} = (X'WX)^{-1} X'Wy^*$ and hence $\alpha = e_1'\beta_{\widetilde{p}} = e_1'(X'WX)^{-1}X'Wy^*$. It follows that $\underline{\varepsilon} = y - y^*$ and

$$\sqrt{n}(\widehat{\alpha}_n^{(W)} - \alpha) = \frac{1}{\sqrt{n}} e_1' (n^{-1} X'WX)^{-1} X'W\underline{\varepsilon} = Y_n + A_n,$$

where $Y_n = n^{-1/2} e_1' M(a,b,R)^{-1} X'W\underline{\varepsilon}$ and

$$A_n = n^{-1/2} e_1' \big((n^{-1}X'WX)^{-1} - M(a,b,R)^{-1} \big) X'W\underline{\varepsilon}.$$

A straightforward calculation yields

$$Y_n = \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) G_R(s_j). \tag{14}$$

The main point of the proof is to show that

$$\frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) G_R(s_j) \xrightarrow{D} N(0, V). \tag{15}$$

With $\gamma_n(s) := (Q_n(1-s) - Q(1-s))/Q(1-s)$, the residual process can be written as

$$\varepsilon(s) = \log(1 + \gamma_n(s)). \tag{16}$$

Set $\eta(x) := \log(1+x) - x$, and let C and δ be some constants such that $\eta(x) \leq Cx^2$, if $|x| \leq \delta$. Then we obtain $Y_n = Y_{n,1} + A_{n,1}$, where

$$Y_{n,1} = \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \gamma_n(s_j) G_R(s_j), \quad A_{n,1} = \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \eta(\gamma_n(s_j)) G_R(s_j).$$

First we show that $A_{n,1} = o_P(1)$. On the event

$$E_n := \left\{ \max_{\lceil na \rceil \leq j \leq \lfloor nb \rfloor} |\gamma_n(s_j)| \leq \delta \right\},$$

we have

$$|A_{n,1}| \leq C\sqrt{n} \max_{\lceil na \rceil \leq j \leq \lfloor nb \rfloor} \gamma_n^2(s_j) \, \frac{1}{n} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} |G_R(s_j)|.$$

With $\kappa_1 := \sup_{1-b \leq s \leq 1-a} 1/|Q(s)|$, we obtain

$$\max_{\lceil na \rceil \leq j \leq \lfloor nb \rfloor} \gamma_n^2(s_j) \leq \kappa_1^2 \sup_{1-b \leq s \leq 1-a} (Q_n(s) - Q(s))^2.$$

Set $e_n(s) := fQ(s)q_n(s) - B_n(s)$. With the Brownian bridges in Theorem 3 and $\kappa_2 := \sup_{1-b \leq s \leq 1-a} 1/fQ(s)$ we get

$$\begin{aligned} \sup_{1-b \leq s \leq 1-a} |Q_n(s) - Q(s)| &= \frac{1}{\sqrt{n}} \sup_{1-b \leq s \leq 1-a} \frac{|e_n(s) + B_n(s)|}{fQ(s)} \\ &\leq \frac{\kappa_2}{\sqrt{n}} \sup_{1-b \leq s \leq 1-a} (|e_n(s)| + |B_n(s)|). \end{aligned}$$

It follows that

$$\sqrt{n} \max_{\lceil na\rceil \le j \le \lfloor nb\rfloor} \gamma_n^2(s_j) \le \frac{\kappa_1^2\kappa_2^2}{\sqrt{n}} \left(\sup_{1-b\le s\le 1-a} |e_n(s)| + \sup_{1-b\le s\le 1-a} |B_n(s)| \right)^2.$$

Applying Theorem 3, we obtain $\sqrt{n}\max_{\lceil na\rceil \le j \le \lfloor nb\rfloor} \gamma_n^2(s_j) = o_P(1)$. This, in combination with $P(E_n) \to 1$ and $\frac{1}{n}\sum_{j=\lceil na\rceil}^{\lfloor nb\rfloor} |G_R(s_j)| \to \int_a^b |G_R(s)ds$ implies $A_{n,1} = o_P(1)$.

Now we decompose $Y_{n,1}$ as $Y_{n,1} = Y_{n,2} + A_{n,2}$, where

$$Y_{n,2} = \frac{1}{n} \sum_{j=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{B_n(1-s_j)G_R(s_j)}{fQ(1-s_j)Q(1-s_j)},$$

$$A_{n,2} = \frac{1}{n} \sum_{j=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{e_n(1-s_j)}{fQ(1-s_j)Q(1-s_j)} G_R(s_j).$$

To prove that $A_{n,2} = o_P(1)$, we use the inequality

$$A_{n,2} \le \kappa_3 \sup_{1-b\le s\le 1-a} |e_n(s)| \, \frac{1}{n} \sum_{j=\lceil na\rceil}^{\lfloor nb\rfloor} |G_R(s_j)|,$$

where

$$\kappa_3 = \sup_{1-b\le s\le 1-a} 1/|fQ(s)Q(s)|.$$

By Theorem 3 we have $A_{n,2} = o_P(1)$. We prove that the limit of $Y_{n,2}$ is $N(0,V)$ given in (8). By the distributional equality

$$Y_{n,2} \overset{D}{=} \frac{1}{n} \sum_{j=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{B(1-s_j)G_R(s_j)}{fQ(1-s_j)Q(1-s_j)}, \quad n = 1, 2, \ldots,$$

where $B(\cdot)$ is a Brownian bridge process, we obtain

$$Y_{n,2} \xrightarrow{D} \int_a^b \frac{B(1-s)G_R(s)}{fQ(1-s)Q(1-s)} ds.$$

The variance of the limit random variable is described in (9).

The last step is to prove that $A_n = o_P(1)$. Let $(v_n^*, v_{0,n}, \dots, v_{\widetilde{p},n})$ be the first row of $(n^{-1}X'WX)^{-1} - M(a,b,R)^{-1}$. Using statement (13), we have $(v_n^*, v_{0,n}, \dots, v_{\widetilde{p},n}) \to \mathbf{0}$. Set

$$G^{(n)}(u) := R(u)\big(- v_n^* \log u + v_{0,n} + 2\sum_{k=1}^{\widetilde{p}} v_{k,n} \cos(2\pi k u)\big), \quad u \in (0,1).$$

Similarly as in (14),

$$\begin{aligned} A_n &= \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) G^{(n)}(s_j) \\ &= -v_n^* \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) R(s_j) \log s_j + v_{0,n} \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) R(s_j) \\ &\quad + 2\sum_{k=1}^{\widetilde{p}} v_{k,n} \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) R(s_j) \cos(2\pi k s_j). \end{aligned}$$

Each term in the last sum tends to zero, e.g., in the first term $v_n^* \to 0$ and applying (15), in which $G_R(s_j)$ is replaced by $R(s_j)\log s_j$, the sequence $\frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(s_j) R(s_j) \log s_j$ has a weak limit.

□

To prove Theorem 2, we need three lemmas.

Lemma 1. *With G_k given in* (10) *for $i, j \geq 0$ we have*

$$h_n(i,j) := \frac{b-a}{n} G_i' W G_j = \begin{cases} \frac{1}{n} + \frac{1}{n}(-1)^{i+j+1}, & \text{if } i \neq j, \\ 1 + \frac{1}{2n} - \frac{(-1)^j}{n}, & \text{if } i = j. \end{cases} \tag{17}$$

Proof. If $1 \le i < j$ then

$$\begin{aligned}
h_n(i,j) &= \frac{2}{n}\sum_{k=0}^{n-1}\cos\left(\pi i\frac{s_k-a}{b-a}\right)\cos\left(\pi j\frac{s_k-a}{b-a}\right)\\
&= \frac{2}{n}\sum_{k=0}^{n-1}\cos\left(\pi i\frac{k}{n}\right)\cos\left(\pi j\frac{k}{n}\right)\\
&= \frac{1}{n}\sum_{k=0}^{n-1}\Big(\cos\left(\pi(i+j)\frac{k}{n}\right)+\cos\left(\pi(i-j)\frac{k}{n}\right)\Big)\\
&= \frac{1}{2n}\Big(\frac{\sin((n-\frac{1}{2})(\pi\frac{i+j}{n}))}{\sin(\pi\frac{i+j}{2n})}+1\Big)+\frac{1}{2n}\Big(\frac{\sin((n-\frac{1}{2})(\pi\frac{i-j}{n}))}{\sin(\pi\frac{i-j}{2n})}+1\Big)\\
&= \frac{1}{n}+\frac{\sin((1-\frac{1}{2n})\pi(i+j))}{2n\sin(\pi\frac{i+j}{2n})}+\frac{\sin((1-\frac{1}{2n})\pi(i-j))}{2n\sin(\pi\frac{i-j}{2n})}.
\end{aligned}$$

Using the identity $\sin(m\pi - z) = (-1)^{m+1}\sin z$, $m \in \mathbb{Z}$, we have

$$h_n(i,j) = \frac{1}{n}+\frac{1}{2n}\big((-1)^{i+j+1}+(-1)^{i-j+1}\big) = \frac{1}{n}+\frac{1}{n}(-1)^{i+j+1}.$$

The proof for the remaining cases is similar, therefore, we omit it. □

Lemma 2. *If $R(\cdot) \in C^2[a,b]$, for the expansion* (12),

$$c_j = O(j^{-2}) \quad \textit{as } j \to \infty.$$

Proof. With $g(z) = \log(a+z(b-a))\sqrt{R(a+b(z-a))}\sqrt{2(b-a)}$,

$$\begin{aligned}
-c_j &= \int_a^b (\log x)\sqrt{R(x)}\cos(j\pi\frac{x-a}{b-a})\frac{\sqrt{2}}{\sqrt{b-a}}dx\\
&= \int_0^1 \log(a+z(b-a))\sqrt{R(a+b(z-a))}\sqrt{2(b-a)}\cos(j\pi z)dz\\
&= \int_0^1 g(z)\cos(j\pi z)dz.
\end{aligned}$$

Then, integrating by parts, we have

$$\begin{aligned}
-c_j &= \frac{1}{j\pi}\sin(j\pi z)g(z)\Big|_{z=0}^{1} - \int_0^1 \frac{1}{j\pi}\sin(j\pi z)g'(z)dz \\
&= -\frac{1}{j\pi}\int_0^1 \sin(j\pi z)g'(z)dz \\
&= -\frac{1}{j\pi}[\frac{1}{j\pi}(-\cos(j\pi z))g'(z)\Big|_{z=0}^{1} + \int_0^1 \frac{1}{j^2\pi^2}\cos(j\pi z)g''(z)dz] \\
&= \frac{1}{j^2\pi^2}[(-1)^j g'(1) - g'(0) + \int_0^1 \cos(j\pi z)g''(z)dz].
\end{aligned}$$

It follows that

$$|c_j| \le \frac{1}{j^2\pi^2}[|g'(1)| + |g'(0)| + \max_{0\le x\le 1}|g''(x)|] = O(j^{-2}).$$

□

From Lemma 2 we obtain that the series $\sum_{j=0}^{\infty} c_j\varphi_j(s)$ converges uniformly on $[a, b]$, and hence,

$$-\log s = \sum_{j=0}^{\infty} c_j\varphi_j(s), \quad s \in [a, b]. \tag{18}$$

Lemma 3. *If* $\theta_n/c_n \to 0$ *then*

$$\frac{\sum_{j=n}^{\infty}|c_j\theta_j|}{\sum_{j=n}^{\infty}c_j^2} \to 0.$$

Proof. Fix $\varepsilon > 0$ and choose N such that $|\theta_n/c_n| < \varepsilon$ is satisfied for all $n > N$. Then for all $n > N$,

$$\frac{\sum_{j=n}^{\infty}|c_j\theta_j|}{\sum_{j=n}^{\infty}c_j^2} \le \frac{\sum_{j=n}^{\infty}\varepsilon c_j^2}{\sum_{j=n}^{\infty}c_j^2} = \varepsilon.$$

□

Proof of Theorem 2. The proof is inspired by the proof of [6, Theorem 3]. Recall (7) and (11) and set

$$\underline{\varepsilon} := \big(\varepsilon(s_0), \ldots, \varepsilon(s_{n-1})\big)', \quad \underline{b_{\widetilde{p}}} := \big(b_{\widetilde{p}}(s_0), \ldots, b_{\widetilde{p}}(s_{n-1})\big)'. \tag{19}$$

Similarly as in the proof of Theorem 1, we have

$$\widehat{\alpha}_n^{(W)} - \alpha = \frac{b-a}{n} e_1' M_n^{-1} X' W (\underline{\varepsilon} + \underline{b_{\widetilde{p}}}), \tag{20}$$

where

$$M_n = \frac{b-a}{n} X' W X. \tag{21}$$

Now the matrix M_n can be written as

$$M_n = \begin{bmatrix} m_n & r_n' \\ r_n & H_n \end{bmatrix},$$

where $m_n = \frac{b-a}{n} G_*' W G_*$,

$$r_n = \frac{b-a}{n} (G_*' W G_0, \ldots, G_*' W G_{\widetilde{p}})', \tag{22}$$

and H_n is a $\widetilde{p}+1 \times \widetilde{p}+1$ matrix with elements $h_n(i,j) = \frac{b-a}{n} G_i' W G_j$, $0 \leq i, j \leq \widetilde{p}$. The inverse of M_n is given by

$$M_n^{-1} = \begin{bmatrix} S^{-1} & -S^{-1} r_n' H_n^{-1} \\ -H_n^{-1} r_n S^{-1} & H_n^{-1} + H_n^{-1} r_n S^{-1} r_n' H_n^{-1} \end{bmatrix},$$

where

$$S = m_n - r_n' H_n^{-1} r_n \tag{23}$$

(see e.g. Seber [9, p. 293]). It follows that

$$e_1' M_n^{-1} X' W = (S^{-1}, -S^{-1} r_n' H_n^{-1}) X' W = S^{-1} (G_*' W - r_n' H_n^{-1} R_n), \tag{24}$$

where R_n is the last $\widetilde{p}+1$ rows of $X'W$. Let f and g be real functions defined on the interval $[a,b]$ and set $\langle f | g \rangle_n := \frac{b-a}{n} \sum_{j=0}^{n-1} f(s_j) g(s_j)$. Then $h_n(i,j) = \langle \varphi_i | R \varphi_j \rangle_n$, and using (18)

$$\frac{b-a}{n} G_*' W G_j = \langle -\log | R\varphi_j \rangle_n = \sum_{i=0}^{\infty} c_i \langle \varphi_i | R \varphi_j \rangle_n = \sum_{i=0}^{\infty} c_i h_n(i,j). \tag{25}$$

Thus r_n can be written as

$$r_n = \Big(\sum_{i=0}^{\infty} c_i h_n(i,0), \ldots, \sum_{i=0}^{\infty} c_i h_n(i,\widetilde{p}) \Big)'. \tag{26}$$

Define the vectors

$$\underline{c}(\widetilde{p}) = (c_0, \ldots, c_{\widetilde{p}})', \quad \underline{d}(\widetilde{p}) = (c_{\widetilde{p}}, c_{\widetilde{p}+1}, \ldots)',$$

and let $\widetilde{H}_n$ be the $\widetilde{p}+1 \times \infty$ matrix with elements $h_n(i,j)$, $0 \le i \le \widetilde{p}$, $\widetilde{p} < j$. Equations (22) and (25) yield

$$r_n' = \underline{c}(\widetilde{p})' H_n + \underline{d}(\widetilde{p})' \widetilde{H}_n'. \tag{27}$$

It follows that

$$G_*' W - r_n' H_n^{-1} R_n = G_*' W - \underline{c}(\widetilde{p})' R_n - T_n.$$

where

$$T_n = \underline{d}(\widetilde{p})' \widetilde{H}_n' H_n^{-1} R_n. \tag{28}$$

Again by (18), $G_*' W = \sum_{j=0}^{\infty} c_j G_j' W$, and by a routine calculation $c(\widetilde{p})' R_n = \sum_{j=0}^{\widetilde{p}} c_j G_j' W$. Therefore, we obtain

$$G_*' W - r_n' H_n^{-1} R_n = \sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' W - T_n. \tag{29}$$

Next we examine H_n^{-1}. Let $I_{\widetilde{p}}$ be the $\widetilde{p}+1 \times \widetilde{p}+1$ identity matrix and set $O_n = I_{\widetilde{p}} - H_n$. For an $m \times n$ matrix A with elements a_{ij} define

$$\|A\|_\infty = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} |a_{ij}|.$$

By Lemma 1, we have $\|O_n\|_\infty \le 2/n$. This implies that if $2(\widetilde{p}+1)/n < 1$, then $H_n^{-1} = I_{\widetilde{p}} + \sum_{k=1}^{\infty} O_n^k$. By induction, $\|O_n^k\|_\infty \le (\widetilde{p}+1)^{k-1}(2/n)^k$. It follows that for $\widetilde{O} := \sum_{k=1}^{\infty} O_n^k$,

$$\|\widetilde{O}\|_\infty \le \frac{2}{n} \frac{1}{1 - (\widetilde{p}+1)\frac{2}{n}}.$$

Recall (28). We show that $\|T_n\|_\infty = O(1/n)$. Using the decomposition $H_n^{-1} = I_{\widetilde{p}} + \widetilde{O}$, we have

$$T_n = \underline{d}(\widetilde{p})'\widetilde{H}_n' R_n + \underline{d}(\widetilde{p})'\widetilde{H}_n'\widetilde{O}R_n. \tag{30}$$

Let $(\underline{d}(\widetilde{p})'\widetilde{H}_n')_k$ denote the k-th component of the vector $\underline{d}(\widetilde{p})'\widetilde{H}_n'$. $k = 0, \dots, \widetilde{p}$. Applying Lemmas 1 and 2, for some $K_1 > 0$,

$$|(\underline{d}(\widetilde{p})'\widetilde{H}_n')_k| = \Big| \sum_{j=\widetilde{p}+1}^{\infty} h_n(k,j)c_j \Big| \le \frac{2}{n} \sum_{j=\widetilde{p}+1}^{\infty} |c_j| \le \frac{K_1}{n} \sum_{j=\widetilde{p}+1}^{\infty} \frac{1}{j^2} \le \frac{K_1}{n\widetilde{p}}.$$

Furtheremore, letting K_2 be an upper bound of $\varphi_k(\cdot)R(\cdot)$, for some $K_3 > 0$ and $k = 0, \dots, n-1$ we have

$$\begin{aligned} |(\underline{d}(\widetilde{p})'\widetilde{H}_n' R_n)_k| &= \Big| \sum_{i=0}^{\widetilde{p}} \Big(\sum_{j=\widetilde{p}+1}^{\infty} h_n(k,j)c_j \Big) \varphi_i(s_k)R(s_k) \Big| \\ &\le K_2 \sum_{i=0}^{\widetilde{p}} \sum_{j=\widetilde{p}+1}^{\infty} |h_n(k,j)c_j| \le K_2 \sum_{i=0}^{\widetilde{p}} \frac{K_1}{n\widetilde{p}} \le \frac{K_3}{n}. \end{aligned} \tag{31}$$

Thus we have $\|\underline{d}(\widetilde{p})'\widetilde{H}_n' R_n\|_\infty = O(1/n)$. A similar argument yields $\|\underline{d}(\widetilde{p})'\widetilde{H}_n'\widetilde{O}R_n\|_\infty = O(1/n^2)$. We then have

$$\|T_n\|_\infty = O(1/n). \tag{32}$$

Next, we turn to examine S in (23). The order of the Riemann sum approximation yields

$$m_n = \frac{b-a}{n} \sum_{j=0}^{n-1} R(s_j) \log^2 s_j = \int_a^b R(x) \log^2 x dx + O\Big(\frac{1}{n}\Big). \tag{33}$$

Applying (27),

$$r_n' H_n^{-1} r_n = \underline{c}(\widetilde{p})' r_n + \underline{d}(\widetilde{p})'\widetilde{H}_n' H_n^{-1} r_n =: r_n^{(1)} + r_n^{(2)}. \tag{34}$$

For the first term, using (26), we have

$$\begin{aligned} r_n^{(1)} &= \sum_{j=0}^{\widetilde{p}} c_j \sum_{k=0}^{\infty} c_k h_n(k,j) = \sum_{j=0}^{\widetilde{p}} c_j \sum_{k=0}^{\widetilde{p}} c_k h_n(k,j) + \sum_{j=0}^{\widetilde{p}} c_j \sum_{k=\widetilde{p}+1}^{\infty} c_k h_n(k,j) \\ &=: t_n^{(1)} + t_n^{(2)}. \end{aligned} \tag{35}$$

A similar argument as in (31) yields

$$t_n^{(2)} = O\Big(\frac{1}{n\widetilde{p}}\Big). \tag{36}$$

Letting δ_{ij} denote the Kronecker delta, we have

$$t_n^{(1)} = \sum_{j=0}^{\widetilde{p}} c_j^2 + \sum_{j=0}^{\widetilde{p}} \sum_{k=0}^{\widetilde{p}} c_j c_k (h_n(j,k) - \delta_{jk}) =: \sum_{j=0}^{\widetilde{p}} c_j^2 + t_n^{(3)}. \tag{37}$$

By Lemmas 1 and 2, we have

$$|t_n^{(3)}| \leq \frac{3}{n}\Big(\sum_{j=0}^{\widetilde{p}} c_j\Big)^2 = O\Big(\frac{1}{n}\Big). \tag{38}$$

To treat $r_n^{(2)}$ in (34) write

$$r_n^{(2)} = \underline{d}(\widetilde{p})'\widetilde{H}_n' r_n + \underline{d}(\widetilde{p})'\widetilde{H}_n'\widetilde{O} r_n.$$

Using again Lemmas 1 and 2, we obtain

$$\underline{d}(\widetilde{p})'\widetilde{H}_n' r_n = O\Big(\frac{1}{n\widetilde{p}}\Big) \ \text{ and } \ \|\underline{d}(\widetilde{p})'\widetilde{H}_n'\widetilde{O}\|_\infty = O\Big(\frac{1}{n^2}\Big). \tag{39}$$

The sequence r_n is bounded, since

$$|(r_n)_k| = \Big|\sum_{i=0}^{\infty} c_i h_n(i,k)\Big| \leq 3\sum_{i=0}^{\infty} |c_i|.$$

By (39) it follows that $\underline{d}(\widetilde{p})'\widetilde{H}_n'\widetilde{O} r_n = O(\widetilde{p}/n^2)$ and hence

$$r_n^{(2)} = O\Big(\frac{1}{n\widetilde{p}}\Big). \tag{40}$$

Equations (23)-(40) above imply

$$S = \int_a^b R(x)\log^2 x dx - \sum_{j=0}^{\widetilde{p}} c_j^2 + O\Big(\frac{1}{n}\Big).$$

By Parseval's equality, S can be written as

$$S = \sum_{\widetilde{p}+1}^{\infty} c_j^2 + O\Big(\frac{1}{n}\Big). \tag{41}$$

Recall (19) and (28). By (20), (24) and (29),

$$\begin{aligned}
\widehat{\alpha}_n^{(W)} - \alpha =& \frac{1}{nS}\Big(\sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' W - T_n\Big)(\underline{\varepsilon} + \underline{b}_{\widetilde{p}}) \\
=& \frac{1}{nS}\Big(\sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W}\Big)\sqrt{W}\underline{\varepsilon} - \frac{1}{nS} T_n\,\underline{\varepsilon} \\
&+ \frac{1}{nS}\Big(\sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' W\Big)\underline{b}_{\widetilde{p}} - \frac{1}{nS} T_n\,\underline{b}_{\widetilde{p}} \\
=:& \,\xi_n^{(1)} + \xi_n^{(2)} + \xi_n^{(3)} + \xi_n^{(4)}.
\end{aligned} \tag{42}$$

where T_n satisfies $\|T_n\|_\infty = O(1/n)$.

Let $\langle x|y\rangle = x'x$ be the inner product of the vectors $x, y \in \mathbb{R}^n$ and let $\|x\|_2 = \sqrt{\langle x|x\rangle}$ denote the induced norm of x. Using the representation (16) and Theorem 3, one can obtain $\sup_{a\le s\le b} |\varepsilon(s)| = O_P(1/\sqrt{n})$, from which it follows that $\big\|\sqrt{W}\underline{\varepsilon}\big\|_2 = O_P(1)$.

Next we determine the order of $\sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W}$. Then we have

$$\begin{aligned}
\Big\|\sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W}\Big\|_2^2 &= \Big\langle \sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W} \Big| \sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W}\Big\rangle \\
&= \sum_{i=\widetilde{p}+1}^{\infty}\sum_{j=\widetilde{p}+1}^{\infty} c_i c_j \langle G_i'\sqrt{W} | G_j'\sqrt{W}\rangle \\
&= \sum_{i=\widetilde{p}+1}^{\infty}\sum_{j=\widetilde{p}+1}^{\infty} c_i c_j \sum_{m=0}^{n-1} \varphi_i(s_m)\varphi_j(s_m) R(s_m) \\
&= \frac{n}{b-a}\sum_{i=\widetilde{p}+1}^{\infty}\sum_{j=\widetilde{p}+1}^{\infty} c_i c_j h_n(i,j) \\
&= \frac{n}{b-a}\sum_{i=\widetilde{p}+1}^{\infty}\sum_{j=\widetilde{p}+1}^{\infty} c_i c_j (h_n(i,j) - \delta_{ij}) + \frac{n}{b-a}\sum_{i=\widetilde{p}+1}^{\infty} c_i^2.
\end{aligned}$$

By Lemmas 1 and 2, for some $K > 0$,

$$\Big\| \sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W} \Big\|_2^2 \le K \sum_{i=\widetilde{p}+1}^{\infty} \sum_{j=\widetilde{p}+1}^{\infty} \frac{1}{i^2} \frac{1}{j^2} + \frac{n}{b-a} \sum_{i=\widetilde{p}+1}^{\infty} c_i^2 \le \frac{K}{\widetilde{p}^2} + \frac{n}{b-a} \sum_{i=\widetilde{p}+1}^{\infty} c_i^2.$$

Therefore, using the Cauchy-Schwarz inequality, for $\xi_n^{(1)}$ in (42) we have

$$\begin{aligned} |\xi_n^{(1)}| &\le \frac{1}{n|S|} \Big\| \sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' \sqrt{W} \Big\|_2 \|\sqrt{W} \underline{\varepsilon}\|_2 \\ &\le \frac{O_P(1)}{\left| n \sum_{\widetilde{p}+1}^{\infty} c_j^2 + O(1) \right|} \sqrt{\frac{K}{\widetilde{p}^2} + \frac{n}{b-a} \sum_{i=\widetilde{p}+1}^{\infty} c_i^2}. \end{aligned} \tag{43}$$

Hence, by condition (P_3) it follows that

$$|\xi_n^{(1)}| = o_P(1). \tag{44}$$

For $\xi_n^{(2)}$, by (32) and condition (P_3), we have

$$|\xi_n^{(2)}| = \frac{1}{n|S|} \Big| \sum_{m=0}^{n-1} (T_n)_m \varepsilon(s_m) \Big| \le \frac{1}{n|S|} O\Big(\frac{1}{n}\Big) n O_P\Big(\frac{1}{\sqrt{n}}\Big) = \frac{1}{n|S|} O_P\Big(\frac{1}{\sqrt{n}}\Big). \tag{45}$$

Next we turn to examine $\xi_n^{(3)}$:

$$\begin{aligned} \Big(\sum_{j=\widetilde{p}+1}^{\infty} c_j G_j' W \Big) \underline{b_{\widetilde{p}}} &= \sum_{j=\widetilde{p}+1}^{\infty} c_j \sum_{m=0}^{n-1} \varphi_j(s_m) R(s_m) b_{\widetilde{p}}(s_m) \\ &= \sum_{j=\widetilde{p}+1}^{\infty} c_j \sum_{m=0}^{n-1} \varphi_j(s_m) R(s_m) \sum_{k=\widetilde{p}+1}^{\infty} \theta_k \varphi_k(s_m) \\ &= \frac{n}{b-a} \sum_{j=\widetilde{p}+1}^{\infty} \sum_{k=\widetilde{p}+1}^{\infty} c_j \theta_k h_n(k,j) \\ &= \frac{n}{b-a} \sum_{j=\widetilde{p}+1}^{\infty} \sum_{k=\widetilde{p}+1}^{\infty} c_j \theta_k (h_n(k,j) - \delta_{kj}) + \frac{n}{b-a} \sum_{j=\widetilde{p}+1}^{\infty} c_j \theta_j \\ &=: \eta_n^{(1)} + \frac{n}{b-a} \sum_{j=\widetilde{p}+1}^{\infty} c_j \theta_j. \end{aligned}$$

By Lemmas 1 and 2, for some $K > 0$,

$$|\eta_n^{(1)}| \leq \frac{3}{b-a} \sum_{j=\widetilde{p}+1}^{\infty} \sum_{k=\widetilde{p}+1}^{\infty} |c_j \theta_k| \leq K \sum_{j=\widetilde{p}+1}^{\infty} \frac{1}{j^2} \sum_{k=\widetilde{p}+1}^{\infty} |\theta_k| \leq \frac{K}{\widetilde{p}} \sum_{k=\widetilde{p}+1}^{\infty} |\theta_k|.$$

Thus

$$|\xi_n^{(3)}| \leq \frac{1}{n|S|} \frac{K}{\widetilde{p}} \sum_{k=\widetilde{p}+1}^{\infty} |\theta_k| + \frac{1}{(b-a)|S|} \sum_{j=\widetilde{p}+1}^{\infty} |c_j \theta_j|.$$

Condition (P_3) and Lemma 3 implies

$$\xi_n^{(3)} \to 0. \tag{46}$$

Finally, we examine $\xi_n^{(4)}$. We have

$$T_n \, \underline{b}_{\widetilde{p}} = \sum_{m=0}^{n-1} (T_n)_m b_{\widetilde{p}}(s_m) = \sum_{m=0}^{n-1} (T_n)_m \sum_{k=\widetilde{p}+1}^{\infty} \theta_k \varphi_k(s_m)$$

Applying (32) and condition (P_3), for some $K > 0$ we obtain

$$|\xi_n^{(4)}| \leq \frac{K}{n^2|S|} \sum_{m=0}^{n-1} \sum_{k=\widetilde{p}+1}^{\infty} |\theta_k| = \frac{K}{n|S|} \sum_{k=\widetilde{p}+1}^{\infty} |\theta_k| \to 0. \tag{47}$$

Equations (44)-(47) imply the statement of the theorem. □

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