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JORDAN ALGEBRAS IN ANALYSIS, OPERATOR THEORY, AND QUANTUM MECHANICS

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Contents

4. Geometry of Jordan Structures and Quantum Mechanics

The expectations that Jordan theory could provide the *algebraic* foundations of quantum mechanics did not fully materialize since the only Jordan structures totally unrelated to associative structures are finite dimensional. Some physicists propose a role of $\mathscr{H}_3(\mathbf{O})$ in elementary particle physics **[40]** but the major applications of operator algebras to quantum (statistical) mechanics came from C^*-algebra theory **[18, 129]**. On the other hand, the recent results **[35, 38, 62]** about the *geometry* of JB-algebras and JB^*-triples have interesting interpretations in terms of quantum mechanical postulates concerning "states" and "filter projections". From this point of view, the geometry underlying C^*-algebras seems to be unnecessarily restrictive. In §10, we show that also in *analysis* there exist promising applications of Jordan theory to quantum mechanics.

For a deeper understanding of operator algebras, E. Effros **[31]** proposed a systematic study of the following three "metric" categories of Banach spaces.

(i) *Complex Banach spaces* Z, endowed with a fixed norm $\|\cdot\|$. Morphisms are the contractive linear mappings.

(ii) *Real order-unit Banach spaces* (X, Ω, e), endowed with the open positive cone Ω and the order-unit $e \in \Omega$. Morphisms are the positive unital linear mappings.

(iii) Complex *matrix ordered* involutive Banach spaces (cf. **[31**, §2**]**). Morphisms are the *completely positive* linear mappings.

The *algebraic models* for the categories are (i) JB^*-*triples* Z with the JB^*-norm $\|\cdot\|$, (ii) JB-*algebras* X with the open positive cone Ω and the unit element e and (iii) C^*-*algebras* with the canonical matrix order.

A basic problem is the *characterization of the algebraic models* in their respective category, either directly using the geometry of the underlying Banach space or "dually" using a suitable *state space* of linear functionals. We will now describe the recent progress made on this question in the "Jordan theoretic" categories (i) and (ii). For (iii), cf. **[2, 100]**.

The main results of §§2 and 3 can be reformulated as follows (for more details, cf. **[59, 61, 19]**).

4.1 THEOREM.

(i) *A complex Banach space* $(Z, \|\cdot\|)$ *is a* JB^*-*triple if and only if its open unit ball* D *is a (bounded) symmetric domain (i.e., homogeneous under the biholomorphic automorphism group* $\mathrm{Aut}(D)$).

(ii) *A real order-unit Banach space* (X, Ω, e) *is a* JB-*algebra with unit* e *if and only if its right half-plane* $D_\Omega := \Omega + iX$ *is symmetric, i.e., biholomorphically equivalent to a bounded symmetric domain. In this case,* Ω *is a "symmetric cone", i.e., the group*

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{GL}(X) : g(\Omega) = \Omega\}$$

acts transitively on Ω, *and* $x \mapsto x^{-1}$ *is a symmetry of* Ω *about* e.

In any of the categories described above an idempotent morphism P (satisfying $P^2 = P$) will be called a *projection*. Somewhat surprisingly (and with possible implications to the foundations of quantum mechanics where projections are used to describe "yes-no" filter experiments), the class of algebraic models is stable under projections. More precisely, we have

4.2 THEOREM.

(i) *The range of a contractive projection* P *on a* JB^*-*triple* Z *is a* JB^*-*triple under the triple product* $(u, v, w) \mapsto P\{uv^*w\}$.

(ii) *The range of a positive unital projection* P *on a* JB-*algebra* X *is a* JB-*algebra under the algebra product* $(x, y) \mapsto P(x \circ y)$

(iii) *The range of a completely positive unital projection* P *on a* C^*-*algebra* A *is a* C^*-*algebra under the algebra product* $(a, b) \mapsto P(ab)$.

Part (iii) is due to Choi-Effros **[21]** whereas part (ii) was shown by Effros-Størmer **[32]**. Part (i), for the slightly smaller class of JC^*-triples (Example 1.8) was obtained by Friedman-Russo **[35]**. The proof presented here, due to W. Kaup **[62]** and L. Stachó **[103]**, works for JB^*-triples in general and demonstrates the usefulness of holomorphic methods in functional analysis.

Since we will use complete holomorphic vector fields and their analytic flow, we first study the behavior of solutions of ordinary differential equations on the unit sphere $S = \{a \in E : \|a\| = 1\}$ of a real Banach space E with dual space $E^t := \mathscr{L}(E, \mathbf{R})$.

4.3 LEMMA. *For* $a \in S$, *a vector* $x \in E$ *satisfies*

$$\lim_{t \to 0} \frac{1}{t}(\|a + tx\| - 1) = 0 \tag{4.1}$$

if and only if

$$\lambda(x) = 0 \quad \text{for all } \lambda \in E^t, \ \|\lambda\| = 1 = \lambda(a). \tag{4.2}$$

PROOF. For $\lambda \in E^t$ with $\|\lambda\| = 1 = \lambda(a)$, we have

$$\lambda(x) = \frac{\lambda(a + tx) - \lambda(a)}{t} \leqslant \frac{\|a + tx\| - 1}{t}$$

whenever $t > 0$. Hence $\lambda(x) \leqslant 0$ if x satisfies (4.1). Since $-x$ satisfies (4.1) as well, $\lambda(x) = 0$. Conversely, suppose (4.1) does not hold. Then there exists $\varepsilon > 0$ such that

$$\frac{1}{t_n}\big|\ \|a + t_n x\| - 1\big| \geqslant \varepsilon$$

for some sequence $t_n > 0$ converging to 0. If $(1/t_n)(\|a + t_n x\| - 1) \leqslant -\varepsilon$ for a subsequence (t_n), we have

$$\frac{1}{t_n}\big(\|a - t_n x\| - 1\big) \geqslant \varepsilon$$

since $2 = 2\|a\| \leqslant \|a + t_n x\| + \|a - t_n x\|$. Hence we may assume

$$\frac{1}{t_n}\big(\|a + t_n x\| - 1\big) \geqslant \varepsilon$$

for all n. By the Hahn-Banach Theorem, choose $\lambda_n \in E^t$ with $\|\lambda_n\| = 1$ and $\lambda_n(a + t_n x) = \|a + t_n x\|$. Then $\lambda_n(a + t_n x) \geqslant 1 + \varepsilon t_n$ and therefore

$$1 \geqslant \lambda_n(a) \geqslant 1 + t_n(\varepsilon - \lambda_n x).$$

This implies $\lambda_n(x) \geqslant \varepsilon$ and $\lim_n \lambda_n(a) = 1$. A cluster point $\lambda \in E^t$ of the sequence (λ_n) satisfies $\|\lambda\| = 1 = \lambda(a)$ and $\lambda(x) \geqslant \varepsilon$, showing that (4.2) is not satisfied. Q.E.D.

The closed linear subspace

$$T_a(S) := \bigcap\{\mathrm{Ker}(\lambda) : \|\lambda\| = 1 = \lambda(a)\}$$

will be called the *tangent space* of S at a.

4.4 LEMMA. *Suppose $f\colon E \to E$ is locally Lipschitz and satisfies*

(4.3) $\qquad f(b) \in T_b(S)\quad$ *for all* $b \in S$.

Let $u\colon I \to E$ be a solution of the initial value problem $u'(t) = f(u(t))$, $u(0) = a \in S$. Here I is an interval about $0 \in \mathbf{R}$. Then $u(I) \subset S$.

PROOF. The mapping $E\setminus\{0\} \ni x \mapsto \hat{x} := x/\|x\| \in S$ is locally Lipschitz and we may assume that there exists a solution $v\colon I \to E\setminus\{0\}$ of the initial value problem $v'(t) = f(\hat{v}(t))$, $v(0) = a \in S$. By the uniqueness of solutions it suffices to show $v(I) \subset S$. To this end, put $n(t) := \|v(t)\|$. Since

$$v(t) = a + \int_0^t f(\hat{v}(h))\,dh,$$

we have

$$\begin{aligned} n(t+s) - n(t) &= \|v(t+s)\| - \|v(t)\| \\ &= \left\| v(t) + \int_t^{t+s} f(\hat{v}(h))\,dh \right\| - \|v(t)\| \\ &\leqslant \|v(t) + s\cdot f(\hat{v}(t))\| + \left\| \int_t^{t+s} \big(f(\hat{v}(h)) - f(\hat{v}(t))\big)\,dh \right\| - \|v(t)\|. \end{aligned}$$

Now (4.3) and Lemma 4.3 imply

$$\begin{aligned} &\|v(t) + s\cdot f(\hat{v}(t))\| - \|v(t)\| \\ &= \|v(t)\|\left(\left\|\hat{v}(t) + \frac{s}{\|v(t)\|} f(\hat{v}(t))\right\| - 1\right) = o(s) \end{aligned}$$

as $s \to 0$. On the other hand,

$$\left\| \int_t^{t+s} \big(f(\hat{v}(h)) - f(\hat{v}(t))\big)\,dh \right\| \leqslant \text{const} \cdot \int_t^{t+s} (h - t)\,dh = o(s)$$

since $f \circ \hat{v}$ is locally Lipschitz. This shows that $t \mapsto n(t)$ is differentiable with derivative $n'(t) = 0$. Hence $n(t) = n(0) = \|a\| = 1$. Q.E.D.

PROOF OF THEOREM 4.2(i). Suppose Z is a JB^*-triple with open unit ball D and let P be a contractive projection on Z with range U. Put $S := \partial D$. For any $c \in U$, the polynomial vector field

$$X_c := \big(c - \{zc^*z\}\big)\frac{\partial}{\partial z} = f(z)\frac{\partial}{\partial z}$$

is complete on D. Hence, for $a \in S \cap U$, we have

$$\exp(tX_c)(a) = a + tf(a) + o(t) \in S$$

showing that $f(a) \in T_a(S)$. Since $P(T_a(S)) \subset T_a(S \cap U)$ by Lemma 4.3, applied to the real Banach spaces underlying Z and U, Lemma 4.4 shows that the polynomial vector field

$$Y_c := \big(c - P\{uc^*u\}\big)\frac{\partial}{\partial u} = P(f(u))\frac{\partial}{\partial u}$$

is complete on $D \cap U$, having integral curves respecting $S \cap U$. By the inverse mapping theorem [**124**, Theorem 1.23], the real-analytic mapping $U \ni c \mapsto \exp(Y_c)(0) \in D \cap U$ covers a neighborhood of $0 \in D \cap U$. Hence Lemma 2.17 implies that $D \cap U$ is homogeneous, hence symmetric. Now apply Theorem 2.18. Q.E.D.

The categories (i), (ii), and (iii) allow biduals in a natural way. It turns out that in each case, the class of algebraic models is closed under taking biduals. For C^*-algebras this is classical, the case of JB-algebras was settled in [**99, 4**] and the result for JB^*-triples is due to S. Dineen [**26**]: As a consequence of the "principle of local reflexivity" [**10**], it follows that any class $\mathscr{E}$ of (complex) Banach spaces which is closed under taking ultrapowers ($z \in \mathscr{E} \Rightarrow Z^{I,\omega} \in \mathscr{E}$) and ranges of contractive projections is also closed under taking biduals. Here $Z^{I,\omega} := L^\infty(I, Z)/N_\omega$, where

$$N_\omega := \left\{(z_i) \in L^\infty(I, Z) : \lim_\omega \|z_i\| = 0\right\}$$

and ω is an ultrafilter on the index set I. Now if Z is a JB^*-triple, N_ω is a closed Jordan triple ideal and, by [**61**, p. 523], $Z^{I,\omega}$ is again a JB^*-triple. Together with the projection theorem 4.2(i), the assertion follows.

4.5 DEFINITION. A C^*-algebra $E = A$, a JB-algebra $E = X$ or a JB^*-triple $E = Z$ are called a W^*-algebra, a JBW-algebra or a JBW^*-triple, respectively, if E has a unique Banach space predual E_t such that the algebraic operations on E are $\sigma(E, E_t)$-continuous in each variable separately.

The results of [**99, 47, 10**] show that, in each class, the biduals are of the type described above.

We will now discuss the "dual" characterization of the respective algebraic models in terms of "state spaces". The state space Σ_Z of a complex Banach space $(Z, \|\cdot\|)$ is its dual unit ball, consisting of all morphisms $(Z, \|\cdot\|) \to (\mathbf{C}, |\cdot|)$. For a real order-unit Banach space (X, Ω, e), the state space Σ_X consists of all morphisms $(X, \Omega, e) \to (\mathbf{R}, \mathbf{R}^+, 1)$. (For matrix ordered Banach spaces it is more natural to consider "operator states", cf. [**31**].)

4.6 EXAMPLE. Let E be an infinite-dimensional Hilbert space over $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ and consider the JC-algebra

$$X := \{\alpha \cdot \mathrm{id}_E + x \colon \alpha \in \mathbf{R},\ x \in \mathscr{H}(E) \text{ compact}\}.$$

Then

$$\Sigma_X = \{u \in \mathscr{H}(E) \colon u \geqslant 0 \text{ of trace class, } \mathrm{trace}(u) \leqslant 1\}$$

consists of all "density matrices", with the pairing

$$(u, \alpha \cdot \mathrm{id}_E + x) \mapsto \alpha + \mathrm{trace}(ux).$$

The pure states are the "vector states" $x \mapsto (\xi | x\xi)$, for $\xi \in \mathbf{S}(E)$ of norm 1, and the functional $\varepsilon_\infty \colon \alpha \cdot \mathrm{id}_E + x \mapsto \alpha$. It follows that

$$\partial_{\mathrm{ex}} \Sigma_X \setminus \{\varepsilon_\infty\} = \mathbf{S}(E)/U(\mathbf{K}) =: \mathbf{P}(E)$$

is the projective space of all "rays" in E, the concept used in traditional quantum theory.

4.7 EXAMPLE. Let M be a locally compact noncompact space and consider the abelian JC-algebra

$$X = \{\alpha \cdot 1_M + f \colon \alpha \in \mathbf{R},\ f \in \mathscr{C}_0(M, \mathbf{R}) \text{ vanishing at } \infty\}.$$

Then

$$\Sigma_X = \{\text{positive Radon measures } \mu \text{ on } M,\ \mu(M) \leqslant 1\}$$

under the pairing

$$(\mu, \alpha 1_M + f) \mapsto \alpha + \mu(f) = \alpha + \int_M f(m)\, d\mu(m).$$

The pure states are the "Dirac measures" $f \mapsto f(m)$ for $m \in M$ and the Dirac functional $\alpha \cdot 1_M + f \mapsto \alpha$ at ∞. Hence $\partial_{\mathrm{ex}} \Sigma_X$ is the 1-point compactification of M.

4.8 EXAMPLE. Let $X = \mathbf{R} \oplus Y$ be the algebra spin factor (cf. Example 1.3). Then

$$\Sigma_X = \{u \in Y \colon (u|u) \leqslant 1\}$$

is the unit ball of Y ("algebra spin ball") under the pairing $(u, \alpha + y) \mapsto \alpha + (u|y)$. The pure states form the sphere of Y: $\partial_{\mathrm{ex}} \Sigma_X = \mathbf{S}(Y)$.

4.9 EXAMPLE. Let Z be the triple spin factor, with conjugation $z \mapsto \bar{z}$ (cf. Example 1.9). Then the dual unit ball Σ_Z is called the "triple spin ball".

Suppose that E is a Banach dual space in category (i) or (ii), with "normal" state space $N_E := \Sigma_E \cap E_t$. Any (non-unital) weak* continuous projection P on E induces a mapping

$$P_t \colon N_E \to \mathbf{K} \cdot N_E \tag{4.4}$$

($\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$) defined by $P_t\phi := \phi \circ P$. In this situation, there are two natural problems:

(A) Define a class of "filter projections" P on E using (i) geometric or (ii) order-theoretic properties.

(B) Characterize the (normal) state spaces of the algebraic models in each category by means of axioms for filter projections, possibly with physical interpretation.

For the algebraic models, there exist natural candidates for "filter projections": If Z is a JBW^*-triple, we consider the Peirce projections

$$P_j = P_j(c) \colon Z \to Z_j(c) \tag{4.5}$$

onto the Peirce spaces $Z_j(c) := \{z \in Z \colon \{cc^*z\} = jz\}$ associated with a tripotent $c \in Z$ (cf. (3.12) and (3.13)). As shown in [**38**], the P_j can be recovered via the norm structure from the isometric "Peirce reflection" [**81**, Theorem 5.6]

$$S_c := P_1 - P_{1/2} + P_0.$$

If X is a JBW-algebra with unit e, we consider the "quadratic representations" (cf. Definition 6.2)

$$P_1 := P(c) := 2M(c)^2 - M(c^2) \tag{4.6}$$

and $P_0 := P(e - c)$, associated with a projection $c \in X$. The corresponding reflection $S_c = 2(P_1 + P_0) - \mathrm{id}$ is positive and unital. In the special case where X is the self-adjoint part of a W^*-algebra we have

$$P_1 x = cxc, \qquad P_0 x = (e - c)x(e - c) \quad \text{and} \quad S_c x = (2c - e)x(2c - e).$$

Let us first describe the solution of problems (A) and (B) in the setting of ordered Banach spaces, due to Alfsen-Shultz [**3, 4, 5**]. For a real order-unit Banach dual space (X, Ω, e) with $X_+ := \overline{\Omega}$, the "filter projections" (called P-projections in [**3**]) come in pairs P_0, P_1 of weak* continuous positive contractive projections on X which are *quasicomplementary* (i.e., $X_+ \cap \mathrm{Ker}\, P_j = X_+ \cap \mathrm{Ran}\, P_{1-j}$) and *smooth*:

$$\mathrm{Ker}\, P_j = T(X_+ \cap \mathrm{Ker}\, P_j).$$

Here the *affine tangent space* $T(F)$ of a face F of X_+ is the intersection of all weakly closed affine hyperplanes supporting X_+ and containing F [**3**, Proposition 1.1]. By [**3**, Lemma 1.7], a filter projection P_j determines its "quasicomplement" P_{1-j} uniquely. For a JBW-algebra X, the filter projections have the form (4.6) [**4**, Proposition 3.1].

A filter projection P on X is called minimal if the associated "projective unit" $u := Pe$ is "atomic", i.e., the corresponding face contains only one (normal) state, denoted by u_t:

(4.7) $$\{\phi \in N_X \colon \phi u = 1\} = \{u_t\}.$$

Using this notation, the *pure state properties* of $K := N_X$ are the following postulates:

(4.8) For every pure state $\rho \in \partial_{ex} K$ ("beam of particles in state ρ") there exists a filter projection P which "prepares" ρ (transforms every incoming beam into a multiple of ρ of possibly smaller intensity) via the mapping (4.4):
$$P_t K \subset \mathbf{R} \cdot \rho.$$

(4.9) Filter projections preserve extreme rays:
$$P_t(\partial_{ex} K) \subset \mathbf{R} \cdot \partial_{ex} K.$$

(4.10) For any two atoms $u, v \in X$ with corresponding states u_t, v_t (cf. (4.7)) we have "symmetry of transition probabilities"
$$u_t v = v_t u.$$

Using these properties, the "dual characterization theorem" [**5**, Corollary 7.3] is as follows.

4.10 THEOREM. *A compact convex set K (with order-unit space $X = \mathscr{A}(K)$ consisting of all continuous affine functions on K) is the state space of a JB-algebra if and only if the following conditions hold*:

(4.8)–(4.10) *Pure state properties.*

(4.11) *Every norm-exposed face $F = K \cap H$ of K (H norm-closed supporting hyperplane) is projective, i.e., of the form*:
$$F = \{\phi \in K \colon \phi(Pe) = 1\}$$
for some filter projection P on X.

(4.12) *Each $x \in X$ has a unique "Riesz decomposition" $x = x_+ - x_-$, where $x_\pm \in X_+$ are orthogonal, i.e., $F_0 x_+ = 0$ and $F_1 x_- = 0$ for a pair (P_0, P_1) of filter projections with corresponding faces (F_0, F_1).*

As a consequence of these properties, any two pure states (facially) generate an (algebra) spin ball (cf. Example 4.8) [**5**, Theorem 3.11]. The characterization of C^*-algebra state spaces (in category (ii)) involves the notion of "orientability" and can be found in [**2**].

Y. Friedman and B. Russo have started to study problems (A) and (B) in the category (i) of complex Banach spaces without order [**36, 37, 38**]. Let $(Z, \|\cdot\|)$ be a Banach dual space with predual Z_t and normal state space $K = \Sigma_Z \cap Z_t$. Given an element $c \in Z$ of norm 1, the convex set

(4.13) $$F_c := \{\phi \in K \colon \phi c = 1\},$$

if not empty, is called a *norm-exposed face* of K. The basic assumption, corresponding to (4.11), is now

(4.14) Every norm-exposed face F_c is *symmetric*, i.e., there exists a unique weak* continuous linear isometry S_c of order 2 on Z having fixed point space $\overline{\mathbf{C}\langle F_c\rangle} \oplus F_c^\perp$ (when acting on Z_t). Here
$$F_c^\perp := \{\psi \in Z_t \colon \|\psi \pm \phi\| = \|\psi\| + \|\phi\| \ \forall \phi \in F_c\}$$
is the "orthogonal complement" of F_c.

Using the symmetry S_c and the norm structure, one can construct contractive weak* continuous projections P_1, $P_{1/2}$, P_0 on Z satisfying
$$S_c = P_1 - P_{1/2} + P_0$$
[**38**]. These are the "filter projections"; they correspond to the Peirce projections (4.5) if Z is a JBW^*-triple. We put $P_c := P_1$. (In the "ordered" category (ii), each pair (P_0, P_1) of filter projections on a JBW-algebra with corresponding faces (F_0, F_1) gives rise to a *positive* symmetry $S = 2(P_0 + P_1) - \mathrm{id}$ fixing $\mathrm{co}(F_0 \cup F_1)$ [**5**, Proposition 3.14].)

Every norm-exposed face is of the form F_c, where c is a "generalized tripotent" in Z satisfying $\|c\| = 1$, $S_c c = c$, and $F_c^\perp c = 0$. Assuming (4.14), this relationship gives a 1-1 correspondence between the norm-exposed faces and the generalized tripotents [**38**, Proposition 1.4]. In this case, the "pure state properties" for nonordered Banach spaces are the following.

(4.15) Every extreme point $\phi \in \partial_{ex} K$ is norm-exposed.

It follows that $\{\phi\} = F_c$ for a unique generalized tripotent $c \in Z$. We write $\phi = c_*$. The tripotents obtained this way are called minimal or atomic. The corresponding filter projection $P_c = P_1(c)$ satisfies $P_c(Z) = \mathbf{C} \cdot c$ and hence
$$(P_c)_t K \subset \mathbf{C}\phi,$$
i.e., ϕ is "prepared" by P_c.

(4.16) For each generalized tripotent c, the filter projection P_c preserves extreme rays, i.e.,
$$(P_c)_t(\partial_{ex} K) \subset \mathbf{C} \cdot (\partial_{ex} K).$$

(4.17) For every pair c, e of minimal tripotents we have the "symmetry of transition probabilities"
$$c_* e = \overline{e_* c}.$$

By [36, Propositions 4, 6, and 7], the normal state space of a JBW^*-triple Z satisfies (4.14) and has the pure state properties (4.15)–(4.17). The Riesz decomposition property (4.12) makes no sense in the nonordered setting, but one knows several other physico-geometrical properties of N_Z (e.g., any two extreme points generate a triple spin ball (Example 4.9); the splitting into atomic and nonatomic part [37]). The complete solution of problem (B) in this setting, however, is still open.

5. Derivations and Dynamical Systems

If the states and observables of a quantum mechanical system are described in terms of a C^*-algebra A (this is common, e.g., in quantum statistical mechanics [18]) the time evolution of the system is given by a 1-parameter group of C^*-automorphisms, subject to certain continuity conditions. Similarly, other symmetries of the system correspond to C^*-algebra automorphisms. Already in the classical case of the full operator algebra $A = \mathscr{L}(E)$ over a complex Hilbert space E, Jordan algebra automorphisms also play a certain role: According to Wigner's Theorem, every bijection ϕ of the projective space $\mathbf{P}(E)$ preserving transition probabilities has the form $\mathbf{C}\cdot h \mapsto \mathbf{C}\cdot uh$ $(h \in E\setminus\{0\})$, where u is a unitary or antiunitary operator on E unique up to a scalar of modulus 1. If u is unitary, then $gz := uzu^{-1}$ defines a C^*-algebra automorphism of A, but if u is antiunitary, then the complex linear transformation $gz := uz^*u^{-1}$ is an antiautomorphism. Therefore, in general, ϕ corresponds to a Jordan algebra automorphism of A.

For a (real or complex) Banach space E, consider the Banach Lie group $\mathrm{GL}(E)$ of all invertible bounded linear operators and its Lie algebra $\mathfrak{gl}(E)$ consisting of all bounded linear operators on E, endowed with the commutator product. The exponential mapping $\exp\colon \mathfrak{gl}(E) \to \mathrm{GL}(E)$ is given by the usual exponential series.

5.1 Definition. For a C^*-algebra A, a JB-algebra X and a JB^*-triple Z, define the *automorphism groups*

$$\begin{aligned}
\mathrm{Aut}(A) &:= \{g \in \mathrm{GL}(A)\colon g(ab) = (ga)(gb),\ g(a^*) = (ga)^*\},\\
\mathrm{Aut}(X) &:= \{g \in \mathrm{GL}(X)\colon g(x\circ y) = (gx)\circ(gy)\},\\
\mathrm{Aut}(Z) &:= \{g \in \mathrm{GL}(Z)\colon g\{uv^*w\} = \{(gu)(gv)^*(gw)\}\}
\end{aligned}$$

and the *derivation* (Lie) *algebras*

$$\begin{aligned}
\mathrm{aut}(A) &:= \{\delta \in \mathfrak{gl}(A)\colon \delta(ab) = (\delta a)b + a(\delta b),\ \delta(a^*) = (\delta a)^*\},\\
\mathrm{aut}(X) &:= \{\delta \in \mathfrak{gl}(X)\colon \delta(x\circ y) = (\delta x)\circ y + x\circ(\delta y)\},\\
\mathrm{aut}(Z) &:= \{\delta \in \mathfrak{gl}(Z)\colon \delta\{uv^*w\} = \{(\delta u)v^*w\} + \{u(\delta v)^*w\} + \{uv^*(\delta w)\}\}.
\end{aligned}$$

Via the symplectic structure (10.19), each $X \in \mathfrak{g}$ gives rise to a function $f_X \in \mathscr{C}^\infty(D)$. Viewing the operators d_X^η, for admissible values η, as "quantizations" of f_X it is an interesting problem to give a rigorous meaning to the "correspondence principle", e.g. in the form

$$\lim_{\eta \to 0} \sigma_\eta [d_X^\eta, d_Y^\eta] = i[f_X, f_Y].$$

Here the left-hand side involves the Berezin symbol (10.15) and the commutator of operators, whereas the right-hand side is the Poisson bracket (10.20). For results in this direction, cf. [**98, 123**].

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