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Norms of Symmetrised Two-Sided Multiplication Operators

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Abstract

Let P a projection and Q_k defined from a self-adjoint operator A_k be its canonical representation. We calculate the norm of the symmetrised two-sided multiplication operator $T_{PQ_k}E = PEQ_k + Q_kEP$ defined on a C^*-algebra $C^*(P, Q_k, 1)$ generated by P and Q_k where E is an idempotent related to P and Q_k.

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1 Introduction

The knowledge of structural properties of the underlying C^*-algebra is undoubtedly one of the cornerstones in solving the problem of norms of elementary operators. Equivalently the spectral theorem helps a great deal in understanding Hilbert space theory. The spectral theory gives the unitary invariants of self-adjoint operator A on a Hilbert space H in terms of its multiplicity measure classes $\{\mu_n\}$ and provide unitarily equivalent model operators. Problems concerning pairs of projections play a fundamental role in the theory of operator algebras. A theorem by Pedersen [8], shows that if P and Q are projections then the C^*-algebra generated by P, Q has a concrete realisation as an algebra of 2×2-matrix valued functions on $[0, 1]$, so its representation theory is well understood. Given two pair of projections P and Q_k in terms of a generating self adjoint operator A_k, the C^*-algebra $\mathcal{A}(E)$ generated by an idempotent E is also generated by the range projections P, Q_k of E respectively E^*. The invariants of P, Q_k are therefore unitary invariants as well. These to an extent have been analysed also by [5]. In any C^*-algebra of operators say Ω, we can define a symmetrised two-sided multiplication operator

$$X \to T_{AB}X = AXB + BXA \qquad A, B, X \in M(\Omega), \tag{1.1}$$

where $M(\Omega)$ is the multiplier algebra of Ω. If Ω is a C^*-algebra of $n \times n$ matrices, then $M(\Omega) = \Omega$. The problem of calculating the norm of this operator T_{AB} in equation (1.1) is still unresovled for a general Banach algebra. Several attempts, however, by various authors have been made for example see [3, 6, 7, 10, 11, 12]. The norm of this operator as revealed in many literatures is in the range of 1 and 2, that is, $\|A\|\|B\| \leq \|T_{AB} : B(H)\| \leq 2\|A\|\|B\|$. This implies, with right normalisation, that the optimal norm is 2. However, this maximal value has only been obtained in specific cases with certain restrictions. As our main result, we have obtain this maximal value in (1.1) for a C^*-algebra $\mathcal{A}(P, Q_k, 1)$. These results are outlined in section four. Therefore, in this paper we have calculated the norm of the opearator T_{AB} defined on C^*-algebra $\mathcal{A}(P, Q_k, 1)$ generated by 2×2 projection matrices P, Q_k and 1, where Q_k are determined by various self-adjoint operators A_k. How A_k determines the representation of Q_k will become clear in sections 2 and 3. Thus, we have shown that as $A_k \to 0$, $\|T_{AB}\| \to 2$ for a particular case of $A = P$, $B = Q_k$ and $A = 1 - 2P$, $1 - 2Q_k$. As usual in our workings, we will demand that X be normalized. Although our results are not for a general Banach algebra, it adds some results to the special cases where the optimal upper bound can be obtained. The results are obtained by simple techniques. Our approach rely on the calculation of spectral radius and finally the use of limits. In section 2, we discussed the algebra of projections and section 3 the C^*-algebras generated by projections and their canonical representations and also those generated by idempotents. Section four contains our main results concerning the norm and Hilbert schmidt norm estimates of operator T_{AB} and also in the same section, we have calculated the operator norm and the Hilbert-Schmidt norm of $T_{UV_k} + T_{PQ_k}$ with $U = 1-2P$, $V_k = 1-2Q_k$. A short discussion of the Fredholm properties of $T_{UV_k}E$, $T_{PQ_k}E$ and $T_{UV_k}E + T_{PQ_k}E$ are done in section 5. Thus, the conclusion in this section is that the Fredholm properties of these operators purely depends on the Fredholm properties of A_k. The sections of this paper are therefore organized as follows: section 1; Introduction, section 2; Projections, section 3; C^*-algebras, section 4; Symmetrised Two-Sided Multiplication Operators and finally section 5; Fredholm Properties.

2 Projections

Let H be a Hilbert space and $B(H)$ denote the C^*-algebra of all bounded linear operators on H and $\mathcal{F}(H) \subset B(H)$ the ideal of all finite rank operators.

A sublalgebra $\mathcal{M} \subset B(H)$ is called a standard operator algebra on H if it contains $\mathcal{F}(H)$. It is now clear that $\mathcal{F}(H) \subset \mathcal{M}, B(H)$. We can also define

$$\mathcal{M}_s = \{A \in \mathcal{M} : A = A^*\}.$$

As $\mathcal{M}$ contains all finite rank operators, the set $\mathcal{M}_s$ contains all finite rank self adjoint operators. We can therefore define

$$\mathcal{F}_s(H) = \mathcal{F}(H) \cap \mathcal{M}_s = \mathcal{F}(H) \cap B_s(H)$$

to be the set of all self adjoint finite rank operators on H. We can assume that $\mathcal{M}_s = \mathcal{F}_s(H) = B_s(H)$. Therefore an equivalence relation on $\mathcal{F}_s(H)$ is defined as $A \equiv B$ if and only if $ImA \equiv ImB$. Here Im denotes the image set of the operator. In each equivalence class there exist a unique projection P, which is the orthogonal projection on ImA. A projection of a finite rank can be identified with a finite dimensional subspace of H. Let $x, y \in H$ be unit vectors and t, s, arbitrary non-zero real numbers. If xx^* denotes rank one projection onto the linear span of x, then it follows that $(txx^*)(syy^*)(txx^*) = 0$ if and only if x and y are orthogonal. Let two projections P and Q be defined by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} cos^2\theta & cos\theta sin\theta \\ cos\theta sin\theta & sin^2\theta \end{bmatrix},$$

where $\theta \in (0, \frac{\pi}{2})$, then P and Q are rank one projections and as $\theta \to \frac{\pi}{2}$, P and Q tend to be orthogonal. We also mention in passing that every rank one self adjoint operator is of the form txx^* for some non zero real number t and some unit vector x. It should be understood that this non zero real number t depends on the operator itself. It is therefore straightforward to see that for every rank one operator $A \in \mathcal{F}_s(H)$, there exists some bounded linear operators $T_1, T_2 \in B(H)$ such that $T_1AT_2 = tA$ and $T_2AT_1 = sA$ for some non zero real numbers s and t depending on A. By the same argument, it can be shown that there is some positive real number t such that for the same bounded linear operators T_1 and T_2 we have $T_2T_1AT_2T_1 = tA$ for every $A \in \mathcal{F}_s(H)$. For any pair of operators $A, B \in \mathcal{F}_s(H)$, we can choose any finite rank projection P such that $PAP = A$ and $PBP = B$, then it will also follow that $PT_1AT_2P = T_1AT_2$ and $PT_1BT_2P = T_1BT_2$. In this case one can easily conclude that A and B are constant multiple of each other. It also follow by the same arguments that $T_2PT_1 = sP$ for some non zero real number s depending on P. It is worth noting here that all these triple products are possible because the C^*-algebras $B(H)$ is closed under Jordan triple product. Therefore, we can find some operator $T \in B(H)$ such that for $T^* \in B(H)$, we have $PTAT^*P = sP$ and $PTBT^*P = tP$ for some constants s and t.
As an example let $B(H) = M_2(\mathbb{C})$ and choose $P = \text{diag}(1, 0)$, $A = \text{diag}(q, 0)$, $B = \text{diag}(a, 0)$, then if $T = (r_{ij})$, $(i = 1, 2)$, $a, q \in \mathbb{R}$ and $r_{ij} \in \mathbb{C}$ we have $s =\mid r_{11} \mid^2 q$, $t =\mid r_{11} \mid^2 a$ and $c = \frac{q}{a}$ for $A = cB$.

Now let u and v be real positive numbers. Then there exist unique positive real numbers α and $\beta \in (0,1)$ such that $u+v=\alpha^2$, $u=\beta\alpha^2$ and $v=(1-\beta)\alpha^2$. Therefore we can find rank one projections P, Q, and A such that $PQ=QP=0$, $APA=\beta A$ and $AQA=(1-\beta)A$. If we denote R by $R=P+Q$, then R becomes a rank two projection and $T_1RT_2=T_1PT_2+T_1QT_2$. To show this claim, we only need to find 2×2 matrix projections of rank one as stated earlier on that satisfy the above conditions and elementary matrix computations show that $A=\text{diag}(1,0)$,

$$P=\begin{bmatrix} \beta & \eta \\ \eta & 1-\beta \end{bmatrix},\ Q=\begin{bmatrix} 1-\beta & -\eta \\ -\eta & \beta \end{bmatrix}$$

where $\eta=(\beta(1-\beta))^{\frac{1}{2}}$. With the assumption that $\beta=sin^2 A_k$ for some $A_k \in (0,\frac{\pi}{2})$, then the matrices P and Q above can be computed explicitly in terms of A_k. It is from this background, with this kind of understanding that we develope the norm theory of symmetrised two-sided multiplication operators defined on C^*-algebras generated by two projections and an idempotent all depending on a self adjoint operator defined from the angle between $RanP$ and $RanQ_k$. In developing our C^*-algebra in the next section, we will therefore fix the projection P and vary Q_k over A_k.

3 C^*-algebras

In this section we only highlight the results that the author and Behncke [2] had obtained. These will be stated without proofs. The proofs can be obtained from [2, 5] and the literature stated therein. The C^*-algebras generated by P, Q_k and 1 will form underlying spaces on which the operator T_{PQ_k} will be defined.
An operator A on a Hilbert space $\mathcal{H}$ will be called to be of type I_n, if $\mathcal{A}(A,1)$ only has irreducible representations of dimensions less than n, i.e. $\hat{\mathcal{A}} =_n \hat{\mathcal{A}}$ [[4], ch. 3.6]. Selfadjoint operators are thus of type I_1. In this case there is a unitary map: $U:\mathcal{H}\to\sum^{\oplus}\mathcal{H}_i\otimes\mathbb{C}^i$ so that A becomes the sum of its "simple" multiplicity components

$$UAU^{-1}=\sum\oplus(A_i\otimes 1_i). \tag{3.1}$$

Now let P and Q_k be two orthogonal projections. Then it is well known that $\mathcal{A}=\mathcal{A}(P,Q_k,1)$ is of type I_2, that is, irreducible representations of $\mathcal{A}$ are either one or two dimensional. Moreover the generic two dimensional irreducible representation has the form

$$P=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad Q_k=\begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix},\quad s=\sin\beta, c=\cos\beta, \tag{3.2}$$

where $0 < \beta < \frac{\pi}{2}$. From this the following result can easily be deduced by standard C^*-algebra and von Neumann algebra techniques.

Theorem 3.1 *Let P and Q_k be two orthogonal projections on the Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ has a unique decomposition $\mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2)$ so that*

(i) $\mathcal{H}_1 = \{x \in \mathcal{H} | PQ_k x = Q_k P x\}$ *is a reducing subspace for P and Q_k.*

(ii) $P = P_1 + P_2$, $Q_k = Q_{k1} + Q_{k2}$ *where* $P_1 = P \mid \mathcal{H}_1$, $P_2 = P \mid \mathcal{H}_1^{\perp}$ $Q_{k1} = Q_k \mid \mathcal{H}_1$ *and* $Q_{k2} = Q_k \mid \mathcal{H}_1^{\perp}$. $\mathcal{H}_2 = P_2 \mid \mathcal{H}_1^{\perp}$.

(iii) There exists a selfadjoint operator A_k on $\mathcal{H}_2$ with $\sigma(A_k) \subset \left[0, \frac{\pi}{2}\right]$, and $0, \frac{\pi}{2} \notin \sigma_p(A_k)$ so that P and Q_k are unitarily equivalent to

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } Q_{k2} = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}, \quad c = \cos\ A_k, s = \sin\ A_k \tag{3.3}$$

(iv) The unitary invariants of P and Q_k are those of A and the multiplicities

$$\dim(\mathcal{H}_1 \ominus (P_1 + Q_{k1} - P_1 Q_{k1})\mathcal{H}_1), \quad \dim(P_1 - P_1 Q_{k1}), \tag{3.4}$$

$$\dim(Q_{k1} - P_1 Q_{k1}), \quad \dim(P_1 . Q_{k1}).$$

In the remainder, Theorem 3.1 will be used systematically to obtain information about operators A_k for which P and Q_k are the range projections of A_k respectively A_k^*. As a rule, statements about the commutative parts P_1, Q_{k1} tend to be trivial.

Let E be an idempotent on the Hilbert space H and let P and Q_k be the range projections of E respectively E^*, i.e. $PEE^* = EE^*P$ and $Q_k E^* E = E^* E Q_k$. Then $((1-P)E)(I-P)E)^* = 0$ shows $PE = E$ and by symmetry

$$PEQ_k = E \tag{3.5}$$

As before write $\mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2)$ and $P = P_1 + P_2, Q_k = Q_{k1} + Q_{k2}$ for the decomposition of $\mathcal{H}, P$ and Q_k into its commutative and genuinely type I_2 parts.

Theorem 3.2 *a) A pair of projections P, Q_k is the range projections of E respectively E^* iff $P_1 Q_{k1} = P_1 = Q_{k1}$ and if the operator A_k of Theorem 3.1 satisfies $||A_k|| < \frac{\pi}{2}$.*

b) $\mathcal{A}(P, Q_k, 1) = \mathcal{A}(E, 1)$ and the unitary invariants of E are those of P, Q_k, i.e. dim $P_1 Q_{k1}$, dim $\mathcal{H}_1-$ dim$P_1 Q_{k1}$ and those of A_k.

c) Any idempotent E on $\mathcal{H}$ is unitarily equivalent to an operator of the form

$$E = 0 \ \oplus \ 1 \ \oplus \begin{pmatrix} 1 & tgA_k \\ 0 & 0 \end{pmatrix} \quad on\ (\mathcal{H}_{10} \oplus \mathcal{H}_{11}) \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2) \tag{3.6}$$

where $\mathcal{H}_1 = \mathcal{H}_{10} \oplus \mathcal{H}_{11}$ and where A_k satisfies $A_k = A_k^*, ||A_k|| < \frac{\pi}{2}$ with $0 \notin \sigma_p(A_k)$.

One can thus show easily from (3.6) using spectral mapping theorem that

$$||E||^2 = ||EE^*|| = ||1 + tg^2 A_k|| = \frac{1}{cos^2\|A_k\|} \tag{3.7}$$

and $\sigma(A_k) \subset [0, \beta], \beta < \frac{\pi}{2}$. Theorem 3.2 thus shows that the idempotent E is similar to the projection $P_1 + P_2$. By induction one can then show that a finite family of orthogonal idempotents is similar to a finite family of orthogonal projections.

Now fix P and vary Q_k over A_k and define an idempotent E which is related to P and Q_k. A symmetrised two-sided multiplication operator can thus be defined on the C^*-algebras $\mathcal{A}(P, Q_k, 1) = \mathcal{A}(E, 1)$. For simplicity let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_k = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & \frac{s}{c} \\ 0 & 0 \end{bmatrix}.$$

4 Symmetrised Two-sided Multiplication Operators

Now define a symmetrised two-sided multiplication operator T_{AB} as in equation (1.1), with $A = P$, $B = Q_k$ and without loss of generality take $X = E$ since X will be normalized in order to calculate $\|T_{P,Q_k}\|$. To normalize E, we will tend A_k to zero. Since $\mathcal{A}(P, Q_k, 1) = \mathcal{A}(E, 1)$ from results of Theorem 3.2 and $\mathcal{A}(P, Q_k, 1) = M_2(\mathbb{C})$, this algebra will be the same as its multiplier algebra. Thus,

Theorem 4.1 *Let $T_{PQ_k}E = PEQ_k + Q_kEP$ be a symmetrised two-sided multiplication operator, then $\lim_{A_k \to 0} \|T_{PQ_k}\| = 2$ and $\lim_{A_k \to 0} \|T_{PQ_k}\|_2 = 2$*

Proof A simple matrix computation show that

$$T_{PQ_k}(E) = PEQ_k + Q_kEP = E + Q_kP.$$

Thus the eigenvalues of $T_{PQ_k}E$ are given by

$$\lambda_{1|2} = \frac{1}{2}\{(1 + c_k^2) \pm \sqrt{(1 + c_k^2)^2 + 4s_k^2}\}.$$

As $A_k \to 0$, (3.7) implies that $\|E\| \to 1$ and by spectral radius formula, one has

$$\|T_{PQ_k}\| = \sup_{\|E\|\to 1} \|T_{PQ_k}(E)\| \geq \sup_{\|E\|\to 1} \{|\ \lambda_i\ |: \lambda_i \in \sigma(T_{PQ_k}E)\} = 2$$

With the right normalisation, it easy to show that $\|T_{PQ_k}\| \leq 2$ as $A_k \to 0$. To prove the second claim, use the following explicit formula for calculating Hilbert Schmidt norm

$$\| \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \|^2 = \frac{1}{2}\{|\ a\ |^2 + |\ b\ |^2 + |\ c\ |^2 + \sqrt{(|\ a\ |^2 + |\ b\ |^2 + |\ c\ |^2)^2}\}.$$

Thus one has

$$\|T_{PQ_k}(E)\|_2^2 = \frac{1+3c_k^4}{c_k^2}.$$

The desired results now follow at once.

□

Corollary 4.2 *Let P and Q_k be projections with $P = PQ_k = Q_k$, then there exists a unitary operator W_k with $W_kPW_k^{-1} = Q_k$ and $W_kQ_kW_k^{-1} = P$ and $W_kEW_k^{-1} = E^*$ for the idempotent E associated to P and Q_k, then $\|T_{W_kW_k^*}\| \to 2$ as $A_k \to 0$.*

Proof By functional calculus, it suffices to solve this on $\mathbb{C}^2$. Here W_k is just the reflection on the line halving the angle between PH and Q_kH. Explicitly W_k is given by $W_k = \begin{bmatrix} -c & -s \\ -s & c \end{bmatrix}$. These are the main results of [1]. It follows by direct matrix calculation that $W_kEW_k^{-1} = W_kEW_k^* = E^*$. The last claim now follows from Theorem 4.1 and the fact that $W_k = W_k^{-1} = W_k^*$. □

Remark 4.3 The unitary equivalence problem for operators on a Hilbert space has received a lot of attention over the years. The analogue problem for operators on a Banach space, due to lack of an inner product structure, requires different techniques directly related to the specific setting under consideration. Therefore given a symmetric norm ideal $\Im$ of $B(H)$ which is different from $C_2(H)$ and fixed elements A_n and B_n $(n = 1, 2)$ in $B(H)$, we can let $\delta_n : \Im \to \Im$ be defined by $\delta_n = A_nEB_n$. Then δ_1 is isommetrically equivalent to δ_2 if and only if there exist unitary operators $W_1, W_2 \in B(H)$ such that for some $\lambda \in \mathbb{C}$ we have

$$A_2 = \lambda W_1A_1W_1^*, \qquad B_2 = \frac{1}{\lambda}W_2B_1W_2^*.$$

In this setting, the summands of $T_{PQ_k}E$ can be split such that $\delta_1(E) = PEQ_k$ and $\delta_2(E) = Q_kEP$. With idempotent E being symmetric and making the right substitutions, we have $W_1 = \begin{bmatrix} -c & -s \\ -s & c \end{bmatrix}$ and $W_2 = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$ and $\lambda = 1$. Note that as $A_k \to 0$, not only is δ_1 and δ_2 isometrically equivalent but they also tend to be unitarily equivalent.

Theorem 4.4 *Let $U = 1 - 2P$ and $V_k = 1 - 2Q_k$ be unitaries of order 2. As $A_k \to 0$, we have $\|T_{UV_k}\| \to 2$ and also $\|T_{UV_k}\|_2 \to 2$.*

Proof A quick check shows that $Q_kE = Q_k$, $EQ_k = E$, $PE = E$, $EP = P$ and therefore

$$T_{UV_k}(E) = UEV_k + V_kEU = 2E - 2P - 2Q_k + 4Q_kP.$$

The proofs follow immediately from Theorem 4.1. □

Now define an operator

$$T_{A_1B_1}(X) + T_{A_2B_2}(X) = \sum_{i=1}^{2} A_iXB_i + B_iXA_i, \tag{4.1}$$

where A_2, B_2 somehow are related to A_1 and B_1 respectively in the underlying C^*-algebra. For the sake of this study, call the operator in (4.1) a double symmetrised two-sided multiplication operator. Define the operator in (4.1) with the following assumptions: $A_1 = P$, $B_1 = Q_k$, $A_2 = 1-2P$, $B_2 = 1-2Q_k$ and $X = E$. Then (4.1) reduces to

$$(T_{UV_k} + T_{PQ_k})(E) = UEV_k + V_kEU + PEQ_k + Q_kEP. \tag{4.2}$$

Theorem 4.5 *Assume (4.2) holds, then*

(i) $\lim_{A_k\to 0} \|T_{UV_k} + T_{PQ_k}\| = 4$

(ii) $\lim_{A_k\to 0} \|T_{UV_k} + T_{PQ_k}\|_2 = 4$.

Proof

(i) It follows that $(T_{UV_k} + T_{PQ_k})(E) = 3E - 2P - 2Q_k$ and the eigenvalues of $(T_{UV_k} + T_{PQ_k})(E)$ are given by

$$\lambda_1 = \frac{1}{2}\{\gamma + (\gamma^2 - \beta)^{\frac{1}{2}}\}, \quad \lambda_2 = \frac{1}{2}\{\gamma - (\gamma^2 - \beta)^{\frac{1}{2}}\},$$

where $\gamma = 4c^2 - s^2$, $\beta = 8s^2$. As before we obtain

$$\|T_{UV_k} + T_{PQ_k}\| \geq \frac{1}{2} \sup_{\|E\|\to 1} \{|\, \gamma + (\gamma^2 - \beta)^{\frac{1}{2}}\, |, |\, \gamma - (\gamma^2 - \beta)^{\frac{1}{2}}\, |\}.$$

Similar arguments to those of Theorem 4.1 clears the proof.

(ii) Given any operator A, its Hilbert-Schmidt norm $\|A\|_2$ is defined by $\|A\|_2 = (tr \mid A \mid^2)^{\frac{1}{2}}$. This is easily calculated from eigenvalues. Therefore, with the results in part (i), we have

$$\|(T_{UV_k} + T_{PQ_k})(E)\|_2 = \{(\sum_{i=1}^{2} \mid \lambda_i \mid^2)^{\frac{1}{2}} \mid \|E\|) = 1\} = \frac{1}{\sqrt{2}}\{2\gamma^2 - \beta\}^{\frac{1}{2}}$$

and thus $\lim_{A_k \to 0} \|T_{UV_k} + T_{PQ_k}\|_2 = 4$.

□

5 Fredholm Properties

Theorem 5.1 *The operators $T_{PQ_k}E$, $T_{UV_k}(E)$, and $(T_{UV_k} + T_{PQ_k})(E)$ are Fredholm iff A_k is.*

Proof To show that $T_{PQ_k}E$ is Fredholm, we need to show that the matrix representation of $T_{PQ_k}E$ is invertible and this is so if and only if $-sin^2 A_k$ is invertible, that is, if A_k is invertible. Therefore $T_{PQ_k}E$ is Fredholm if and only if A_k is. Similarly $T_{UV_k}(E)$ is invertible if and only if $-2sin^2 A_k$ is invertible, that is, if A_k is invertible. Hence the implication. Finally $(T_{UV_k} + T_{PQ_k})(E)$ is invertible iff $-11sin^2 A_k$ is invertible implying that only if A_k is invertible. This completes the proof and hence it becomes clear that the Fredholmness of these operators depends on that of A_k. □

Remark 5.2 As proposed in [12] an application of essential numerical range can provide an alternative proof for Theorem 4.1 provided $W_{m,e}(P^*, Q_k^*) \cap W_{m,e}(P, Q_k) \neq \emptyset$. With the assumption that this holds one has $\|T_{PQ_k}\|_{cb} = \|T_{PQ_k}\| = \|PP^*\| + \|Q_k^* Q_k\|$ where the subscript cb denotes completely bounded norm. As $A_k \to 0$, note that $\|P\| = \|Q_k\| = 1$.

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