Bull. London Math. Soc. 35 (2003) 218–224
DOI: 10.1112/S0024609302001856

BANACH SPACES WHOSE ALGEBRAS OF OPERATORS ARE UNITARY: A HOLOMORPHIC APPROACH

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Abstract

An element u of a norm-unital Banach algebra A is said to be unitary if u is invertible in A and satisfies $\|u\| = \|u^{-1}\| = 1$. The norm-unital Banach algebra A is called unitary if the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. If X is a complex Hilbert space, then the algebra $BL(X)$ of all bounded linear operators on X is unitary by the Russo–Dye theorem. The question of whether this property characterizes complex Hilbert spaces among complex Banach spaces seems to be open. Some partial affirmative answers to this question are proved here. In particular, a complex Banach space X is a Hilbert space if (and only if) $BL(X)$ is unitary and, for Y equal to X, X^* or X^{**}, there exists a biholomorphic automorphism of the open unit ball of Y that cannot be extended to a surjective linear isometry on Y.

1. *Introduction*

An element u of a norm-unital Banach algebra A is said to be *unitary* if u is invertible in A and satisfies $\|u\| = \|u^{-1}\| = 1$. The norm-unital Banach algebra A is called unitary if the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. If X is a complex Hilbert space, then the algebra $BL(X)$ of all bounded linear operators on X is unitary (by the Russo–Dye theorem [**10**, Theorem 30.2]). The question of whether the above fact characterizes complex Hilbert spaces among complex Banach spaces seems to be open (see [**12**], [**18**] and [**31**]). In this paper we prove some partial affirmative answers to the question just quoted. Indeed, a complex Banach space X is a Hilbert space if (and only if) $BL(X)$ is unitary and, for Y equal to X, X^* or X^{**}, there exists a biholomorphic automorphism of the open unit ball of Y that cannot be extended to a surjective linear isometry on Y (see Theorems 2.2, 2.5 and 2.10, respectively).

The proof of our results involves deep facts taken from the theory of the infinite-dimensional holomorphy. The reader is referred to the survey paper of J. Arazy [**1**], as well as to the Arazy–Solel paper [**2**, Section 2], for a comprehensive view of the part of that theory involved in our arguments. Actually, we have had to develop, slightly, the theory of circular homogeneous domains on complex Banach spaces. Indeed, as a key tool for the proof of Theorem 2.10, we prove in Proposition 2.9 that, if X is a complex Banach space, and if the orbit of zero under the group of all biholomorphic automorphisms of the open unit ball of X^{**} contains the open unit ball of X, then such an orbit is in fact the whole open unit ball of X^{**} (that is, X^{**} is a JB*-triple; see [**21**, **22**]).

Received 12 October 2001; revised 24 May 2002.

2000 *Mathematics Subject Classification* 46B04, 46B10, 46B20.

Partially supported by Junta de Andalucía grant FQM 0199 and Acción Integrada HB1999-0052.

2. *The results*

Throughout this paper, $\mathbb{K}$ will mean the field of real or complex numbers. Let X be a Banach space over $\mathbb{K}$. We denote by S_X, B_X, Δ_X, and X^* the unit sphere, the closed unit ball, the open unit ball, and the (topological) dual, respectively, of X. Given (x,f) in $X \times X^*$, the value of f at x will be denoted by $\langle f,x\rangle$. The symbol $\mathscr{G}_X$ will stand for the group of all surjective linear isometries from X to X. We note that $\mathscr{G}_X$ is nothing but the set of all unitary elements of the norm-unital Banach algebra $BL(X)$. For a subset A of X, $\overline{\text{co}}A$ will mean the (norm-) closed convex hull of A in X.

Lemma 2.1. *For a Banach space X over $\mathbb{K}$, consider the following conditions.*
(1) $BL(X)$ *is unitary.*
(2) *For every α in $S_{X^{**}}$ we have*
$$\overline{\text{co}}\{T^{**}(\alpha) : T \in \mathscr{G}_X\} \supseteq B_X.$$
(3) *For every f in S_{X^*} we have*
$$\overline{\text{co}}\{T^*(f) : T \in \mathscr{G}_X\} = B_{X^*}.$$
(4) *For every x in S_X we have*
$$\overline{\text{co}}\{T(x) : T \in \mathscr{G}_X\} = B_X.$$
Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2). Let x and α be in B_X and $S_{X^{**}}$, respectively. Fix $\varepsilon > 0$, take f in B_{X^*} such that $|1-\langle\alpha,f\rangle| < \varepsilon/2$, and denote by $x\otimes f$ the operator on X defined by $(x\otimes f)(y) := \langle f,y\rangle x$ for every y in X. If we assume that Condition 1 holds, there exists F in the convex hull of $\mathscr{G}_X$ satisfying $\|x\otimes f - F\| < \varepsilon/2$. Therefore we have
$$\begin{aligned}\|x - F^{**}(\alpha)\| &\leqslant \|x - \langle\alpha,f\rangle x\| + \|\langle\alpha,f\rangle x - F^{**}(\alpha)\| \\ &= \|(1-\langle\alpha,f\rangle)x\| + \|(x\otimes f - F)^{**}(\alpha)\| \\ &< \varepsilon.\end{aligned}$$

(2) $\Rightarrow$ (3). As a consequence of assuming Condition 2, for every α in $S_{X^{**}}$ the convex hull of $\{T^{**}(\alpha) : T \in \mathscr{G}_X\}$ is w^*-dense in $B_{X^{**}}$. Then Condition 3 follows from the Hahn–Banach theorem.

(3) $\Rightarrow$ (4). Condition 3 implies that for every f in S_{X^*}, the convex hull of $\{T^*(f) : T \in \mathscr{G}_X\}$ is w^*-dense in B_{X^*}. By another application of the Hahn–Banach theorem, Condition 4 holds. □

A Banach space X is said to be *convex-transitive* if it satisfies Condition 4 in Lemma 2.1. Thus Condition 3 in the above lemma is a stronger form of the convex transitivity for X^*. The implications (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) in Lemma 2.1 were first proved in [**12**]. For convex-transitive Banach spaces the reader is referred to [**4**, **5**, **6**, **7**, **8**, **9**, **11**, **12**, **20**, **24**, **30**].

Let X be a complex Banach space. Then Δ_X is invariant under $\mathscr{G}_X$, and hence $\mathscr{G}_X$ can be seen as a subgroup of the group of all biholomorphic automorphisms of Δ_X. According to [**1**, Theorem 3.6 and Main Lemma 4.2], the orbit of zero under the group of all biholomorphic automorphisms of Δ_X becomes the open unit ball of a closed subspace of X, which is called the *symmetric part of* X, and is denoted by X_s. The possibility that $X = X_s$ has been deeply studied by many authors since the fundamental work of W. Kaup (see [**21**, **22**]), who proved that such an equality is equivalent to the fact that X is a JB*-triple. We recall that the complex Banach space X is said to be a *JB*-triple* if it is endowed with a continuous triple product $\{\ldots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies the following conditions.

(1) For all x in X, the mapping $y \to \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.

(2) The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.

(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

JB*-triples that are dual Banach spaces are called JBW*-triples.

THEOREM 2.2. *Let X be a nonzero complex Banach space. Then X is a Hilbert space if (and only if) $\mathrm{BL}(X)$ is unitary and there exists a biholomorphic automorphism of Δ_X that does not lie in $\mathcal{G}_X$.*

Proof. The 'only if' part is well known. Indeed, every complex Hilbert space is in fact a JB*-triple under the triple product

$$\{xyz\} := \tfrac{1}{2}((x|y)z + (z|y)x).$$

Assume that $\mathrm{BL}(X)$ is unitary, and that there exists a biholomorphic automorphism of Δ_X that is not in $\mathcal{G}_X$. By the second assumption and [**1**, Lemma 2.1], X_s is nonzero. Since X_s is invariant under $\mathcal{G}_X$, the first assumption, together with the implication (1) $\Rightarrow$ (4) in Lemma 2.1, gives $X = X_s$; that is, X becomes a JB*-triple. By [**15**], X^{**} is a JBW*-triple.

Now X^* is the predual of a JBW*-triple, as well as a convex-transitive Banach space (a consequence of the implication (1) $\Rightarrow$ (3) in Lemma 2.1), and moreover the closed unit ball of X^* has extreme points (by the Krein–Milman theorem). It follows from [**6**, Theorem 3.1] that X^* (and hence X) is a Hilbert space. □

Banach spaces X that are ranges of a linear projection P on X^{**} such that $1 - 2P$ is an isometry have been considered in [**17**].

PROPOSITION 2.3. *Let X be a Banach space over $\mathbb{K}$. Assume that $\mathrm{BL}(X)$ is unitary, and that X is the range of a linear projection P on X^{**} such that $1-2P$ is an isometry. Then X is reflexive.*

Proof. Let T be in $\mathcal{G}_X$. Then $Q := (T^{**})^{-1}PT^{**}$ is a projection on X^{**} satisfying $Q(X^{**}) = X$ and such that $1 - 2Q$ is an isometry. By the comments after [**8**, Theorem 3.5], we have $Q = P$ (that is, P commutes with T^{**}), and hence $(1-P)(X^{**})$ is invariant under T^{**}. Now note that T is arbitrary in $\mathcal{G}_X$ and that, by the implication (1) $\Rightarrow$ (2) in Lemma 2.1, every nonzero closed subspace of X^{**} that is invariant under $\{T^{**} : T \in \mathcal{G}_X\}$ must contain X. It follows that if X were not reflexive, then we would have the contradiction $(1-P)(X^{**}) \supseteq X = P(X^{**})$. □

Let X be a Banach space over $\mathbb{K}$. An *L-projection on X* is a linear projection (say, π) on X satisfying

$$\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$$

for every x in X. The Banach space X is said to be *L-embedded* if it is the range of an L-projection on X^{**}. For the theory of L-embedded Banach spaces, the reader is referred to [**19**]. The next corollary follows straightforwardly from Proposition 2.3.

COROLLARY 2.4. *Let X be an L-embedded Banach space over $\mathbb{K}$ such that $\mathrm{BL}(X)$ is unitary. Then X is reflexive.*

THEOREM 2.5. *Let X be a nonzero complex Banach space. Then X is a Hilbert space if (and only if) $\mathrm{BL}(X)$ is unitary and there exists a biholomorphic automorphism of Δ_{X^*} that does not lie in $\mathcal{G}_{X^*}$.*

Proof. Assume that $\mathrm{BL}(X)$ is unitary, and that there exists a biholomorphic automorphism of Δ_{X^*} that is not in $\mathcal{G}_{X^*}$. By the second assumption and [**1**, Lemma 2.1], $(X^*)_s$ is nonzero. Since $(X^*)_s$ is invariant under $\mathcal{G}_{X^*}$, the first assumption, together with the implication (1) $\Rightarrow$ (3) in Lemma 2.1, gives $X^* = (X^*)_s$; that is, X^* becomes a JBW*-triple. Then, by [**3**, Proposition 3.4], X is an L-embedded Banach space.

Now X is the predual of a JBW*-triple, as well as a convex-transitive Banach space (by the implication (1) $\Rightarrow$ (4) in Lemma 2.1), and moreover the closed unit ball of X has extreme points (because, by Corollary 2.4, X is reflexive). It follows from [**6**, Theorem 3.1] that X is a Hilbert space. □

REMARK 2.6. The Banach space X in Proposition 2.3 and Corollary 2.4, as well as its dual X^*, is in fact superreflexive and almost transitive. This is so because, by Lemma 2.1, X and X^* are convex transitive, and, since they are reflexive, [**4**, Corollary 3.3] applies. We recall that almost transitivity of a Banach space Y means that, for every y in S_Y, $\mathcal{G}_Y(y)$ is dense in S_Y. Since superreflexive almost transitive Banach spaces are uniformly smooth (see [**16**]; see also [**14**, Corollary IV.5.7]), the concluding paragraph in the proof of Theorem 2.5 can be replaced with an application of the refined version of Corollary 2.4 just noted, keeping in mind either that smooth preduals of JBW*-triples are Hilbert spaces [**6**, Proposition 2.4], or Tarasov's theorem [**28**] that smooth JB*-triples are Hilbert spaces.

The next lemma is a non-linear generalization of [**10**, Theorem 17.2]. Given a Banach space X over $\mathbb{K}$, we denote by $\Pi(X)$ the set of those elements (x, f) in $S_X \times S_{X^*}$ such that $\langle f, x\rangle = 1$.

LEMMA 2.7. *Let X be a Banach space over $\mathbb{K}$, let (f, α) be an element of $\Pi(X^*)$, and let Λ be a bounded function from S_{X^*} to X^*, continuous at f. Then $\langle \alpha, \Lambda(f)\rangle$ belongs to the closure in $\mathbb{K}$ of the set*

$$\{\langle \Lambda(h), x\rangle : (x, h) \in \Pi(X)\}.$$

More precisely, for every positive number ρ, $\langle \alpha, \Lambda(f) \rangle$ lies in the closure in $\mathbb{K}$ of the set

$$\{\langle \Lambda(h), x \rangle : (x,h) \in \Pi(X), \|f-h\| < \rho\}.$$

Proof. Fix the positive number ρ in the statement, and let $\varepsilon > 0$. Since Λ is continuous at f, there exists

$$0 < \delta < \min\{1, \rho, \varepsilon\}$$

such that $\|\Lambda(g) - \Lambda(f)\| < \varepsilon$ whenever g is in S_{X^*} and $\|g - f\| < \delta$. Since B_X is w^*-dense in $B_{X^{**}}$, there exists $y \in B_X$ satisfying

$$|\langle \alpha - y, \Lambda(f) \rangle| < \varepsilon \quad \text{and} \quad |1 - \langle f, y \rangle| = |\langle \alpha - y, f \rangle| < \delta^2/4.$$

By the Bishop–Phelps–Bollobás theorem [**10**, Theorem 16.1], there exists $(x,h) \in \Pi(X)$ such that $\|y - x\| < \delta$, and $\|f - h\| < \delta$. Now we have $\|f-h\| < \rho$ and

$$|\langle \alpha, \Lambda(f) \rangle - \langle \Lambda(h), x \rangle| \leqslant |\langle \alpha - y, \Lambda(f) \rangle| + |\langle \Lambda(f) - \Lambda(h), y \rangle| + |\langle \Lambda(h), y - x \rangle|$$
$$< \varepsilon(2 + \|\Lambda\|).$$

□

Let X be a complex Banach space. We recall that a holomorphic vector field on Δ_X is nothing other than a holomorphic mapping from Δ_X to X. A holomorphic vector field Λ on Δ_X is said to be *complete* if, for each x in Δ_X, there exists a differentiable function $\varphi : \mathbb{R} \to \Delta_X$ satisfying

$$\varphi(0) = x \quad \text{and} \quad \frac{d}{dt}\varphi(t) = \Lambda(\varphi(t))$$

for every t in $\mathbb{R}$. The next lemma is due to L. Stacho [**25**] (see also [**1**, p. 139], [**26**], [**27**], and [**29**, Lecture 4]). The formulation that we give here is that of [**2**, Proposition 2.5].

LEMMA 2.8. *Let X be a complex Banach space, and let Λ be a holomorphic vector field on Δ_X. Then Λ is complete if and only if it has a holomorphic extension (say $\hat{\Lambda}$) to a neighborhood of B_X satisfying $\Re(\langle f, \hat{\Lambda}(x) \rangle) = 0$ for every $(x,f) \in \Pi(X)$.*

PROPOSITION 2.9. *Let X be a complex Banach space such that $(X^{**})_s \supseteq X$. Then X^{**} is a JBW*-triple.*

Proof. Since X is contained in the symmetric part of X^{**}, according to [**1**, Lemma 3.5 and Theorems 3.3 and 3.6; see also Definition 3.7], for each x in X there exists a unique continuous quadratic mapping $q_x : X^{**} \to X^{**}$ such that (the restriction to $\Delta_{X^{**}}$ of) the function $\Lambda_x : \alpha \to x - q_x(\alpha)$ from X^{**} to X^{**} becomes a complete holomorphic vector field on $\Delta_{X^{**}}$. Moreover, the mapping $x \to q_x$, from X to the Banach space of all X^{**}-valued continuous quadratic functions on X^{**}, is conjugate-linear and continuous. The continuity of the mapping $x \to q_x$ is not explicitly noted in [**1**], but follows easily from Lemma 2.8 and the closed graph theorem. For α in X^{**} consider the continuous conjugate-linear mapping $F_\alpha : x \to q_x(\alpha)$ from X to X^{**}. Denote by G_α the unique w^*-continuous conjugate-linear mapping from X^{**} to X^{**} which extends F_α (see, for instance, [**23**, Lemma 1.5]), and consider the function $\Lambda_\alpha : \beta \to \alpha - G_\beta(\alpha)$ from X^{**} to X^{**}. Note that the definition of Λ_α just given is consistent with the one previously introduced in the particular case that $\alpha = x \in X$. Now, let x, α and (f,β) be elements of X, X^{**} and $\Pi(X^*)$, respectively. Since Λ_x is a holomorphic vector field on $\Delta_{X^{**}}$ and (β, f) belongs to $\Pi(X^{**})$, Lemma 2.8 applies, giving

$$\Re(\langle x - F_\beta(x), f \rangle) = \Re(\langle \Lambda_x(\beta), f \rangle) = 0.$$

Since x is arbitrary in X, it follows from the w^*-density of X in X^{**} that

$$\Re(\langle \alpha - G_\beta(\alpha), f \rangle) = \Re(\langle \Lambda_\alpha(\beta), f \rangle) = 0.$$

Note now that Λ_α is a holomorphic mapping on X^{**} bounded on $B_{X^{**}}$ (indeed, it follows easily from the continuity of the mapping $x \to q_x$, and the way of defining $G_\diamond$, that $\gamma \to G_\gamma(\alpha)$ is a continuous quadratic mapping from X^{**} to X^{**}). Since (f,β) is arbitrary in $\Pi(X^*)$, it follows from Lemma 2.7 that $\Re(\langle \chi, \Lambda_\alpha(\gamma) \rangle) = 0$ for every $(\gamma, \chi) \in \Pi(X^{**})$. By Lemma 2.8, Λ_α is a complete holomorphic vector field on $\Delta_{X^{**}}$. By [**1**, Theorem 3.6], $\alpha = \Lambda_\alpha(0)$ belongs to $(X^{**})_s$. Finally, since α is arbitrary in X^{**}, we have $X^{**} = (X^{**})_s$. □

In relation to Proposition 2.9 above, it is worth mentioning that complex Banach spaces whose biduals are JBW*-triples need not be JB*-triples (see, for instance, [**6**, Example 3.10]).

THEOREM 2.10. *Let X be a nonzero complex Banach space. Then X is a Hilbert space if (and only if) $BL(X)$ is unitary and there exists a biholomorphic automorphism of $\Delta_{X^{**}}$ that does not lie in $\mathcal{G}_{X^{**}}$.*

Proof. Assume that $BL(X)$ is unitary, and that there exists a biholomorphic automorphism of $\Delta_{X^{**}}$ that is not in $\mathcal{G}_{X^{**}}$. By the second assumption and [**1**, Lemma 2.1], $(X^{**})_s$ is nonzero. Then, since $(X^{**})_s$ is invariant under $\mathcal{G}_{X^{**}}$, the first assumption, together with the implication $1 \Rightarrow 2$ in Lemma 2.1, gives $(X^{**})_s \supseteq X$. By Proposition 2.9, X^{**} is a JBW*-triple. Now the proof of the present theorem is finished by repeating verbatim the concluding paragraph of the proof of Theorem 2.2. □

Acknowledgements. The authors are grateful to W. Kaup and L. L. Stacho for fruitful comments about the subject matter of the paper.

References

1. J. ARAZY, 'An application of infinite dimensional holomorphy to the geometry of Banach spaces', *Geometrical aspects of functional analysis*, Lecture Notes in Math. 1267 (ed. J. Lindenstrauss and V. D. Milman, Springer, Berlin, 1980) 122–150.
2. J. ARAZY and B. SOLEL, 'Isometries of non-self-adjoint operator algebras', *J. Funct. Anal.* 90 (1990) 284–305.
3. T. J. BARTON and R. M. TIMONEY, 'Weak*-continuity of Jordan triple products and applications', *Math. Scand.* 59 (1986) 177–191.
4. J. BECERRA and A. RODRÍGUEZ, 'The geometry of convex transitive Banach spaces', *Bull. London Math. Soc.* 31 (1999) 323–331.
5. J. BECERRA and A. RODRÍGUEZ, 'Isometric reflections on Banach spaces after a paper of A. Skorik and M. Zaidenberg', *Rocky Mountain J. Math.* 30 (2000) 63–83.
6. J. BECERRA and A. RODRÍGUEZ, 'Transistivity of the norm on Banach spaces having a Jordan structure', *Manuscripta Math.* 102 (2000) 111–127.
7. J. BECERRA and A. RODRÍGUEZ, 'Characterizations of almost transitive superreflexive Banach spaces', *Comment. Math. Univ. Carolin.* 42 (2001) 629–636.
8. J. BECERRA and A. RODRÍGUEZ, 'Transitivity of the norm on Banach spaces', *Extracta Math.* 17 (2002) 1–58.

9. J. Becerra and A. Rodríguez, 'Convex-transitive Banach spaces, big points, and the duality mapping', *Quart. J. Math. Oxford* 53 (2002) 257–264.

10. F. F. Bonsall and J. Duncan, *Numerical ranges* II, London Math. Soc. Lecture Note Ser. 10 (Cambridge University Press, 1973).

11. F. Cabello, 'Regards sur le problème des rotations de Mazur', *Extracta Math.* 12 (1997) 97–116.

12. E. R. Cowie, 'Isometries in Banach algebras', Ph.D. Thesis, Swansea, 1981.

13. E. R. Cowie, 'A note on uniquely maximal Banach spaces', *Proc. Edinburgh Math. Soc.* 26 (1983) 85–87.

14. R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64 (Pitman, New York, 1993).

15. S. Dineen, 'The second dual of a JB*-triple system', *Complex analysis, functional analysis and approximation theory*, North-Holland Math. Stud. 125 (ed. J. Mújica, North-Holland, Amsterdam, 1986) 67–69.

16. C. Finet, 'Uniform convexity properties of norms on superreflexive Banach spaces', *Israel J. Math.* 53 (1986) 81–92.

17. G. Godefroy, 'Symetries isométriques. Applications à la dualité des espaces reticulées', *Israel J. Math.* 44 (1983) 61–74.

18. M. L. Hansen and R. V. Kadison, 'Banach algebras with unitary norms', *Pacific J. Math.* 175 (1996) 535–552.

19. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math. 1547 (Springer, Berlin, 1993).

20. N. J. Kalton and G. V. Wood, 'Orthonormal systems in Banach spaces and their applications', *Math. Proc. Camb. Phil. Soc.* 79 (1976) 493–510.

21. W. Kaup, 'Algebraic characterization of symmetric complex Banach manifolds', *Math. Ann.* 228 (1977) 39–64.

22. W. Kaup, 'A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces', *Math. Z.* 183 (1983) 503–529.

23. R. Payá, J. Pérez and A. Rodríguez, 'Type I factor representations of non-commutative JB*-algebras', *Proc. London Math. Soc.* (3) 48 (1984) 428–444.

24. S. Rolewicz, *Metric linear spaces* (Reidel, Dordrecht, 1985).

25. L. L. Stacho, 'A short proof of the fact that biholomorphic automorphisms of the unit ball in certain L^p spaces are linear', *Acta Sci. Math.* 41 (1979) 381–383.

26. L. L. Stacho, 'A projection principle concerning biholomorphic automorphisms', *Acta Sci. Math.* 44 (1982) 99–124.

27. L. L. Stacho, 'On nonlinear projections of vector fields', *Convex analysis and chaos*, Josai Math. Monogr. 1 (ed. K. Nishizava, Josai, 1999) 47–55.

28. S. K. Tarasov, 'Banach spaces with a homogeneous ball', *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 43 (1988) 62–64.

29. H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics*, CBMS Reg. Conf. Ser. Math. 67 (Amer. Math. Soc., Providence, RI, 1987).

30. G. V. Wood, 'Maximal symmetry in Banach spaces', *Proc. Royal Irish Acad.* 82A (1982) 177–186.

31. G. V. Wood, 'Maximal algebra norms', *Proc. Trends in Banach Spaces and Operator Theory*, Memphis 2000, Contemp. Math. (Amer. Math. Soc., Providence, RI, 2003).

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Bull. London Math. Soc. 35 (2003) 225–228
DOI: 10.1112/S0024609302001728

CONVERGENCE ALMOST EVERYWHERE OF CERTAIN PARTIAL SUMS OF FOURIER INTEGRALS

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Abstract

Suppose that R goes to infinity through a second-order lacunary set. Let S_R denote the Rth spherical partial inverse Fourier integral on $\mathbb{R}^d$. Then $S_R f$ converges almost everywhere to f, provided that f satisfies

$$\int \left|\widehat{f}(\xi)\log\log(8+|\xi|)\right|^2 d\xi < \infty.$$

1. *Introduction*

In this note we consider the problem of almost everywhere convergence of the partial sums of spherical Fourier integrals on $\mathbb{R}^d$. For $0 \leqslant a \leqslant b$, let

$$S_{[a,b]}f := \left[\chi_{[a,b]}(|\cdot|)\cdot\widehat{f}\right]^{\vee}.$$

Here, $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$, χ_E the characteristic function of a set E, and $\widehat{f}$ the Fourier transform of a function f on $\mathbb{R}^d$. We also let

$$S_R f := \left[\chi_{[0,R]}(|\cdot|)\cdot\widehat{f}\right]^{\vee}.$$

An important problem is to understand whether, for arbitrary f in L^2, $S_R f(x)$ converges almost everywhere to $f(x)$ as R tends to infinity.

When R goes to infinity through a *lacunary* sequence, this is true and is an easy consequence of the Littlewood–Paley theory and the Hardy–Littlewood maximal theorem.

When there is no restriction on the manner in which R tends to infinity, and $d = 1$, one can achieve almost everywhere convergence by the celebrated result of Carleson **[2]**. When $d > 1$, it seems that the best result to date is in **[1]**, where it is shown that $S_R f$ converges almost everywhere to f, provided that f satisfies

$$\int \left|\widehat{f}(\xi)\log(8+|\xi|)\right|^2 d\xi < \infty.$$

Notice that this is a result of Rademacher–Menshov type, and is formally weaker than even the Kolmogorov–Seliverstov–Plessner condition, which has a single power of the logarithm.

Here we address the situation when R goes to infinity through what one might term a *second-order lacunary set* such as

$$\{2^N(1-2^{-k}) : N \in \mathbb{Z},\ k \in \mathbb{Z}_+\}.$$

Received 18 February 2000; revised 20 March 2002.

2000 *Mathematics Subject Classification* 42B15, 42B25, 42B08.

The first and second authors were supported by the European Commission TMR Network 'Harmonic Analysis'. The third author was supported by EPSRC grant GR/J65594.