Monatsh. Math. 152, 135–150 (2007)
DOI 10.1007/s00605-007-0464-6
Printed in The Netherlands

Monatshefte für
Mathematik

On some equations in prime rings

By

Maja Fošner and **Joso Vukman**

University of Maribor, Slovenia

Communicated by D. Segal

Received April 13, 2006; accepted in final form February 6, 2007
Published online August 13, 2007

Abstract. The main purpose of this paper is to prove the following result. Let R be a prime ring of characteristic different from two and let $T: R \to R$ be an additive mapping satisfying the relation $T(x^3) = T(x)x^2 - xT(x)x + x^2T(x)$ for all $x \in R$. In this case T is of the form $4T(x) = qx + xq$, where q is some fixed element from the symmetric Martindale ring of quotients. This result makes it possible to solve some functional equations in prime rings with involution which are related to bicircular projections.

2000 Mathematics Subject Classification: 16N60, 16W10
Key words: Prime ring, semiprime ring, functional identity, derivation, Jordan derivation, involution, bicircular projection

1. Introduction

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geqslant 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that R is prime if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D: R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [13] asserts that any Jordan derivation on a prime ring of a characteristic different from two is a derivation. Cusack [11] generalized Herstein's result to 2-torsion free semiprime rings (see also [8] for an alternative proof). We denote by Q_{mr}, Q_s, and C the maximal Martindale right ring of quotients, symmetric Martindale ring of quotients and extended centroid of a semiprime ring R, respectively. For the explanation of Q_{mr}, Q_s, and C we refer the reader to [5, Chapter 2].

Brešar [9] proved the following result.

Theorem 1.1 [9, Theorem 4.3]. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping satisfying the relation*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

for all pairs $x, y \in R$. In this case D is a derivation.

One can easily prove that any Jordan derivation on arbitrary 2-torsion free ring satisfies the relation (1), which means that Theorem 1.1 generalizes Cusack's generalization of Herstein's result we have just mentioned above. Motivated by Theorem 1.1 the second named author, Kosi-Ulbl, and Eremita [18] have recently proved the following result.

Theorem 1.2 [18, Theorem 1]. *Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x), \tag{2}$$

for all pairs $x, y \in R$. In this case T is of the form $2T(x) = qx + xq$ where q is a fixed element from Q_s.

Putting in (1) and (2) $y = x$ one obtains

$$D(x^3) = D(x)x^2 + xD(x)x + x^2D(x), \quad \text{for all } x \in R \tag{3}$$

and

$$T(x^3) = T(x)x^2 - xT(x)x + x^2T(x), \quad \text{for all } x \in R. \tag{4}$$

Beidar, Brešar, Chebotar and Martindale have proved [2, Theorem 4.4] that in case an additive mapping $D : R \to R$, where R is a prime ring of characteristic different from two, satisfies the relation (3), then D is a derivation (actually they proved more general result). It is our aim in this paper to prove that in case an additive mapping $T : R \to R$, where R is a prime ring of characteristic different from two, satisfies the relation (4), then T is of the form $4T(x) = qx + xq$, where q is a fixed element from Q_s.

Let X be a complex Banach space and let $L(X)$ be the algebra of all bounded linear operators on X. A projection $P \in L(X)$ is called bicircular in case all mappings of the form $e^{i\alpha}P + e^{i\beta}(I - P)$, where I denotes the identity operator, are isometric for all pairs of real numbers α, β. Stachó and Zalar [15, 16] investigated bicircular projections on the C^*-algebra $L(H)$, the algebra of all bounded linear operators on a Hilbert space H. According to [16, Proposition 3.4] every bicircular projection $P : L(H) \to L(H)$ satisfies the relation

$$P(xyx) = P(x)yx - xP(y^*)^*x + xyP(x), \tag{5}$$

for all pairs $x, y \in L(H)$. The first named author and Ilišević [12] investigated the above functional equation in 2-torsion free semiprime *-rings. They expressed the solution of Eq. (5) in terms of derivations and so-called double centralizers. The second named author showed that applying Theorem 1.1 and Theorem 1.2 a more direct approach makes it possible to prove a more general result [17, Theorem 1]. In this paper we prove a result concerning bicircular projections on a prime ring with involution which is related to the conjecture in [17].

2. Preliminaries

Let R be a ring and let X be a subset of R. By $C(X)$ we denote the set $\{r\in R|[r,X]=0\}$. Let $m\in\mathbb{N}$ and let $E: X^{m-1}\to R$, $p: X^{m-2}\to R$ be arbitrary mappings. In the case when $m=1$ this should be understood as that E is an element in R and $p=0$. Let $1\leqslant i<j\leqslant m$ and define $E^i, p^{ij}, p^{ji}: X^m\to R$ by

$$E^i(\overline{x}_m)=E(x_1,\dots,x_{i-1},x_{i+1},\dots,x_m),$$
$$p^{ij}(\overline{x}_m)=p^{ji}(\overline{x}_m)=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_m),$$

where $\overline{x}_m=(x_1,\dots,x_m)\in X^m$.

Let $I,J\subseteq\{1,\dots,m\}$, and for each $i\in I, j\in J$ let $E_i, F_j: X^{m-1}\to R$ be arbitrary mappings. Consider functional identities

$$\sum_{i\in I}E_i^i(\overline{x}_m)x_i+\sum_{j\in J}x_jF_j^j(\overline{x}_m)=0 \qquad (\overline{x}_m\in X^m), \tag{6}$$

$$\sum_{i\in I}E_i^i(\overline{x}_m)x_i+\sum_{j\in J}x_jF_j^j(\overline{x}_m)\in C(X) \qquad (\overline{x}_m\in X^m). \tag{7}$$

A natural possibility when (6) and (7) are fulfilled is when there exist mappings $p_{ij}: X^{m-2}\to R$, $i\in I$, $j\in J$, $i\neq j$, and $\lambda_k: X^{m-1}\to C(X)$, $k\in I\cup J$, such that

$$\begin{aligned} E_i^i(\overline{x}_m)&=\sum_{j\in J, j\neq i}x_jp_{ij}^{ij}(\overline{x}_m)+\lambda_i^i(\overline{x}_m),\\ F_j^j(\overline{x}_m)&=-\sum_{i\in I, j\neq i}p_{ij}^{ij}(\overline{x}_m)x_i-\lambda_j^j(\overline{x}_m),\\ \lambda_k&=0 \quad \text{if} \quad k\notin I\cap J \end{aligned} \tag{8}$$

for all $\overline{x}_m\in X^m$, $i\in I$, $j\in J$. We shall say that every solution of the form (8) is a standard solution of (6) and (7).

The case when one of the sets I or J is empty is not excluded. The sum over the empty set of indexes should be simply read as zero. So, when $J=0$ (resp. $I=0$) (6) and (7) reduce to

$$\sum_{i\in I}E_i^i(\overline{x}_m)x_i=0 \quad \left(\text{resp.} \sum_{j\in J}x_jF_j^j(\overline{x}_m)=0\right) \quad (\overline{x}_m\in X^m), \tag{9}$$

$$\sum_{i\in I}E_i^i(\overline{x}_m)x_i\in C(X) \quad \left(\text{resp.} \sum_{j\in J}x_jF_j^j(\overline{x}_m)\in C(X)\right) \quad (\overline{x}_m\in X^m). \tag{10}$$

In that case the (only) standard solution is

$$E_i=0, \quad i\in I \quad (\text{resp. } F_j=0, \quad j\in J). \tag{11}$$

The d-freeness of X will play an important role in this paper. For a definition of d-freeness we refer the reader to [6]. Under some natural assumptions one can establish that various subsets (such as ideals, Lie ideals, the sets of symmetric or skew symmetric elements in a ring with involution) of certain types of rings are d-free. We refer the reader to [1] and [3] for results of this kind. Let us mention that a prime ring R is a d-free subset of its maximal right ring of quotients, unless R satisfies the standard polynomial identity of degree less than $2d$ (see [3, Theorem 2.4]).

Let R be a ring and let

$$p(x_1,x_2,x_3)=\sum_{\pi\in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$$

be a fixed multilinear polynomial in noncommutative indeterminates x_1, x_2 and x_3. Further, let L be a subset of R closed under p, i.e. $p(\overline{x}_3)\in L$ for all $x_1,x_2,x_3\in L$, where $\overline{x}_3=(x_1,x_2,x_3)$. We shall consider a mapping $T:L\to R$ satisfying

$$T\big(p(\overline{x}_3)\big)=\sum_{\pi\in S_3}\Big(T\big(x_{\pi(1)}\big)x_{\pi(2)}x_{\pi(3)}-x_{\pi(1)}T\big(x_{\pi(2)}\big)x_{\pi(3)}+x_{\pi(1)}x_{\pi(2)}T\big(x_{\pi(3)}\big)\Big), \quad (12)$$

for all $x_1,x_2,x_3\in L$. In the first step of the proof of the following theorem we derive a functional identity from (12). Let us mention that the idea of considering the expression $[p(\overline{x}_3),p(\overline{y}_3)]$ in its proof is taken from [4].

For the proof of our main result (Theorem 3.2) we need the result below which is of independent interest.

Theorem 2.1. *Let L be a 6-free Lie subring of R closed under p. If $T:L\to R$ is an additive mapping satisfying (4), then there exists $q\in R$ such that $4T(x)=xq+qx$ for all $x\in L$.*

Proof. Note that for any $a\in R$ and $\overline{x}_3\in L^3$ we have

$$\big[p(\overline{x}_3),a\big]=p\big([x_1,a],x_2,x_3\big)+p\big(x_1,[x_2,a],x_3\big)+p\big(x_1,x_2,[x_3,a]\big).$$

Thus

$$T\big[p(\overline{x}_3),a\big]=T\big(p([x_1,a],x_2,x_3)\big)+T\big(p(x_1,[x_2,a],x_3)\big)+T\big(p(x_1,x_2,[x_3,a])\big).$$

Using (12) it follows that

$$\begin{aligned}
T\big[p(\overline{x}_3),a\big] &= \sum_{\pi\in S_3} T\big[x_{\pi(1)},a\big]x_{\pi(2)}x_{\pi(3)} - \sum_{\pi\in S_3}\big[x_{\pi(1)},a\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}\\
&\quad+\sum_{\pi\in S_3}\big[x_{\pi(1)},a\big]x_{\pi(2)}T\big(x_{\pi(3)}\big)+\sum_{\pi\in S_3}T\big(x_{\pi(1)}\big)\big[x_{\pi(2)},a\big]x_{\pi(3)}\\
&\quad-\sum_{\pi\in S_3}x_{\pi(1)}T\big[x_{\pi(2)},a\big]x_{\pi(3)}+\sum_{\pi\in S_3}x_{\pi(1)}\big[x_{\pi(2)},a\big]T\big(x_{\pi(3)}\big)\\
&\quad+\sum_{\pi\in S_3}T\big(x_{\pi(1)}\big)x_{\pi(2)}\big[x_{\pi(3)},a\big]-\sum_{\pi\in S_3}x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},a\big]\\
&\quad+\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},a\big]\\
&=\sum_{\pi\in S_3} T\big[x_{\pi(1)},a\big]x_{\pi(2)}x_{\pi(3)} - \sum_{\pi\in S_3}\big[x_{\pi(1)},a\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}\\
&\quad+\sum_{\pi\in S_3}\big[x_{\pi(1)}x_{\pi(2)},a\big]T\big(x_{\pi(3)}\big)+\sum_{\pi\in S_3}T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},a\big]\\
&\quad-\sum_{\pi\in S_3}x_{\pi(1)}T\big[x_{\pi(2)},a\big]x_{\pi(3)}-\sum_{\pi\in S_3}x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},a\big]\\
&\quad+\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},a\big].
\end{aligned}$$

In particular

$$
\begin{aligned}
&T\left[p(\overline{x}_3),p(\overline{y}_3)\right]\\
&\quad=\sum_{\pi\in S_3}T\left[x_{\pi(1)},p(\overline{y}_3)\right]x_{\pi(2)}x_{\pi(3)}-\sum_{\pi\in S_3}\left[x_{\pi(1)},p(\overline{y}_3)\right]T\left(x_{\pi(2)}\right)x_{\pi(3)}\\
&\qquad+\sum_{\pi\in S_3}\left[x_{\pi(1)}x_{\pi(2)},p(\overline{y}_3)\right]T\left(x_{\pi(3)}\right)+\sum_{\pi\in S_3}T\left(x_{\pi(1)}\right)\left[x_{\pi(2)}x_{\pi(3)},p(\overline{y}_3)\right]\\
&\qquad-\sum_{\pi\in S_3}x_{\pi(1)}T\left[x_{\pi(2)},p(\overline{y}_3)\right]x_{\pi(3)}-\sum_{\pi\in S_3}x_{\pi(1)}T\left(x_{\pi(2)}\right)\left[x_{\pi(3)},p(\overline{y}_3)\right]\\
&\qquad+\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}T\left[x_{\pi(3)},p(\overline{y}_3)\right]
\end{aligned}
\tag{13}
$$

for all $\overline{x}_3,\overline{y}_3\in L^3$. For $i=1,2,3$ we also have

$$
\begin{aligned}
T\left[x_{\pi(i)},p(\overline{y}_3)\right]&=-T\left[p(\overline{y}_3),x_{\pi(i)}\right]\\
&=\sum_{\sigma\in S_3}T\left[x_{\pi(i)},y_{\sigma(1)}\right]y_{\sigma(2)}y_{\sigma(3)}-\sum_{\sigma\in S_3}\left[x_{\pi(i)},y_{\sigma(1)}\right]T\left(y_{\sigma(2)}\right)y_{\sigma(3)}\\
&\quad+\sum_{\sigma\in S_3}\left[x_{\pi(i)},y_{\sigma(1)}y_{\sigma(2)}\right]T\left(y_{\sigma(3)}\right)+\sum_{\sigma\in S_3}T\left(y_{\sigma(1)}\right)\left[x_{\pi(i)},y_{\sigma(2)}y_{\sigma(3)}\right]\\
&\quad-\sum_{\sigma\in S_3}y_{\sigma(1)}T\left[x_{\pi(i)},y_{\sigma(2)}\right]y_{\sigma(3)}-\sum_{\sigma\in S_3}y_{\sigma(1)}T\left(y_{\sigma(2)}\right)\left[x_{\pi(i)},y_{\sigma(3)}\right]\\
&\quad+\sum_{\sigma\in S_3}y_{\sigma(1)}y_{\sigma(2)}T\left[x_{\pi(i)},y_{\sigma(3)}\right].
\end{aligned}
$$

for all $\overline{y}_3\in L^3$. Therefore (13) can be written as

$$
\begin{aligned}
&T\left[p(\overline{x}_3),p(\overline{y}_3)\right]\\
&\quad=\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(T\left[x_{\pi(1)},y_{\sigma(1)}\right]y_{\sigma(2)}y_{\sigma(3)}-\left[x_{\pi(1)},y_{\sigma(1)}\right]T\left(y_{\sigma(2)}\right)y_{\sigma(3)}\\
&\qquad+\left[x_{\pi(1)},y_{\sigma(1)}y_{\sigma(2)}\right]T\left(y_{\sigma(3)}\right)+T\left(y_{\sigma(1)}\right)\left[x_{\pi(1)},y_{\sigma(2)}y_{\sigma(3)}\right]-y_{\sigma(1)}T\left[x_{\pi(1)},y_{\sigma(2)}\right]y_{\sigma(3)}\\
&\qquad-y_{\sigma(1)}T\left(y_{\sigma(2)}\right)\left[x_{\pi(1)},y_{\sigma(3)}\right]+y_{\sigma(1)}y_{\sigma(2)}T\left[x_{\pi(1)},y_{\sigma(3)}\right]\Big)x_{\pi(2)}x_{\pi(3)}\\
&\qquad-\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\left[x_{\pi(1)},y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}\right]T\left(x_{\pi(2)}\right)x_{\pi(3)}\\
&\qquad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\left[x_{\pi(1)}x_{\pi(2)},y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}\right]T\left(x_{\pi(3)}\right)\\
&\qquad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}T\left(x_{\pi(1)}\right)\left[x_{\pi(2)}x_{\pi(3)},y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}\right]\\
&\qquad-\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\pi(1)}\Big(T\left[x_{\pi(2)},y_{\sigma(1)}\right]y_{\sigma(2)}y_{\sigma(3)}-\left[x_{\pi(2)},y_{\sigma(1)}\right]T\left(y_{\sigma(2)}\right)y_{\sigma(3)}\\
&\qquad+\left[x_{\pi(2)},y_{\sigma(1)}y_{\sigma(2)}\right]T\left(y_{\sigma(3)}\right)+T\left(y_{\sigma(1)}\right)\left[x_{\pi(2)},y_{\sigma(2)}y_{\sigma(3)}\right]-y_{\sigma(1)}T\left[x_{\pi(2)},y_{\sigma(2)}\right]y_{\sigma(3)}\\
&\qquad-y_{\sigma(1)}T\left(y_{\sigma(2)}\right)\left[x_{\pi(2)},y_{\sigma(3)}\right]+y_{\sigma(1)}y_{\sigma(2)}T\left[x_{\pi(2)},y_{\sigma(3)}\right]\Big)x_{\pi(3)}\\
&\qquad-\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\pi(1)}T\left(x_{\pi(2)}\right)\left[x_{\pi(3)},y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}\right]
\end{aligned}
$$

$$+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} x_{\pi(1)}x_{\pi(2)}\Big(T\big[x_{\pi(3)},y_{\sigma(1)}\big]y_{\sigma(2)}y_{\sigma(3)}-\big[x_{\pi(3)},y_{\sigma(1)}\big]T\big(y_{\sigma(2)}\big)y_{\sigma(3)}$$
$$+\big[x_{\pi(3)},y_{\sigma(1)}y_{\sigma(2)}\big]T\big(y_{\sigma(3)}\big)+T\big(y_{\sigma(1)}\big)\big[x_{\pi(3)},y_{\sigma(2)}y_{\sigma(3)}\big]-y_{\sigma(1)}T\big[x_{\pi(3)},y_{\sigma(2)}\big]y_{\sigma(3)}$$
$$-y_{\sigma(1)}T\big(y_{\sigma(2)}\big)\big[x_{\pi(3)},y_{\sigma(3)}\big]+y_{\sigma(1)}y_{\sigma(2)}T\big[x_{\pi(3)},y_{\sigma(3)}\big]\Big)$$

for all $\bar{x}_3,\bar{y}_3\in L^3$.

On the other hand, using $\big[p(\bar{x}_3),p(\bar{y}_3)\big]=-\big[p(\bar{y}_3),p(\bar{x}_3)\big]$, we get from the above identity

$$T\big[p(\bar{x}_3),p(\bar{y}_3)\big]$$
$$=\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(T\big[x_{\pi(1)},y_{\sigma(1)}\big]x_{\pi(2)}x_{\pi(3)}-\big[x_{\pi(1)},y_{\sigma(1)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}$$
$$+\big[x_{\pi(1)}x_{\pi(2)},y_{\sigma(1)}\big]T\big(x_{\pi(3)}\big)+T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},y_{\sigma(1)}\big]-x_{\pi(1)}T\big[x_{\pi(2)},y_{\sigma(1)}\big]x_{\pi(3)}$$
$$-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(1)}\big]+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(1)}\big]\Big)y_{\sigma(2)}y_{\sigma(3)}$$
$$-\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},y_{\sigma(1)}\big]T\big(y_{\sigma(2)}\big)y_{\sigma(3)}$$
$$+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},y_{\sigma(1)}y_{\sigma(2)}\big]T\big(y_{\sigma(3)}\big)$$
$$+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}T\big(y_{\sigma(1)}\big)\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},y_{\sigma(2)}y_{\sigma(3)}\big]$$
$$-\sum_{\pi\in S_3}\sum_{\sigma\in S_3}y_{\sigma(1)}\Big(T\big[x_{\pi(1)},y_{\sigma(2)}\big]x_{\pi(2)}x_{\pi(3)}-\big[x_{\pi(1)},y_{\sigma(2)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}$$
$$+\big[x_{\pi(1)}x_{\pi(2)},y_{\sigma(2)}\big]T\big(x_{\pi(3)}\big)+T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},y_{\sigma(2)}\big]-x_{\pi(1)}T\big[x_{\pi(2)},y_{\sigma(2)}\big]x_{\pi(3)}$$
$$-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(2)}\big]+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(2)}\big]\Big)y_{\sigma(3)}$$
$$-\sum_{\pi\in S_3}\sum_{\sigma\in S_3}y_{\sigma(1)}T\big(y_{\sigma(2)}\big)\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},y_{\sigma(3)}\big]$$
$$+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}y_{\sigma(1)}y_{\sigma(2)}\Big(T\big[x_{\pi(1)},y_{\sigma(3)}\big]x_{\pi(2)}x_{\pi(3)}-\big[x_{\pi(1)},y_{\sigma(3)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}$$
$$+\big[x_{\pi(1)}x_{\pi(2)},y_{\sigma(3)}\big]T\big(x_{\pi(3)}\big)+T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},y_{\sigma(3)}\big]-x_{\pi(1)}T\big[x_{\pi(2)},y_{\sigma(3)}\big]x_{\pi(3)}$$
$$-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(3)}\big]+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(3)}\big]\Big)$$

for all $\bar{x}_3,\bar{y}_3\in L^3$.

Comparing the identities so obtained we arrive at

$$0=\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(T\big[x_{\pi(1)},y_{\sigma(1)}\big]x_{\pi(2)}x_{\pi(3)}y_{\sigma(2)}-T\big(x_{\pi(1)}\big)y_{\sigma(1)}x_{\pi(2)}x_{\pi(3)}y_{\sigma(2)}$$
$$+T\big(y_{\sigma(1)}\big)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}y_{\sigma(2)}-\big[x_{\pi(1)},y_{\sigma(1)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}y_{\sigma(2)}$$
$$+\big[x_{\pi(1)}x_{\pi(2)},y_{\sigma(1)}\big]T\big(x_{\pi(3)}\big)y_{\sigma(2)}-x_{\pi(1)}T\big[x_{\pi(2)},y_{\sigma(1)}\big]x_{\pi(3)}y_{\sigma(2)}$$
$$-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(1)}\big]y_{\sigma(2)}+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(1)}\big]y_{\sigma(2)}$$
$$-\big[x_{\pi(1)}x_{\pi(2)},y_{\sigma(1)}\big]x_{\pi(3)}T\big(y_{\sigma(2)}\big)\Big)y_{\sigma(3)}$$

$$
\begin{aligned}
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(-T\big[x_{\pi(1)},y_{\sigma(1)}\big]y_{\sigma(2)}y_{\sigma(3)}x_{\pi(2)}-T\big(y_{\sigma(1)}\big)x_{\pi(1)}y_{\sigma(2)}y_{\sigma(3)}x_{\pi(2)}\\
&+T\big(x_{\pi(1)}\big)y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}x_{\pi(2)}+\big[x_{\pi(1)},y_{\sigma(1)}\big]T\big(y_{\sigma(2)}\big)y_{\sigma(3)}x_{\pi(2)}\\
&-\big[x_{\pi(1)},y_{\sigma(1)}y_{\sigma(2)}\big]T\big(y_{\sigma(3)}\big)x_{\pi(2)}+y_{\sigma(1)}T\big[x_{\pi(1)},y_{\sigma(2)}\big]y_{\sigma(3)}x_{\pi(2)}\\
&+y_{\sigma(1)}T\big(y_{\sigma(2)}\big)\big[x_{\pi(1)},y_{\sigma(3)}\big]x_{\pi(2)}-y_{\sigma(1)}y_{\sigma(2)}T\big[x_{\pi(1)},y_{\sigma(3)}\big]x_{\pi(2)}\\
&+\big[x_{\pi(1)},y_{\sigma(1)}y_{\sigma(2)}\big]y_{\sigma(3)}T\big(x_{\pi(2)}\big)\Big)x_{\pi(3)}\\
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}y_{\sigma(1)}\Big(y_{\sigma(2)}x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(3)}\big]-y_{\sigma(2)}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}T\big(y_{\sigma(3)}\big)\\
&+y_{\sigma(2)}x_{\pi(1)}x_{\pi(2)}y_{\sigma(3)}T\big(x_{\pi(3)}\big)+y_{\sigma(2)}T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},y_{\sigma(3)}\big]\\
&-y_{\sigma(2)}x_{\pi(1)}T\big[x_{\pi(2)},y_{\sigma(3)}\big]x_{\pi(3)}-y_{\sigma(2)}x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(3)}\big]\\
&+y_{\sigma(2)}T\big[x_{\pi(1)},y_{\sigma(3)}\big]x_{\pi(2)}x_{\pi(3)}-T\big(y_{\sigma(2)}\big)\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},y_{\sigma(3)}\big]\\
&-\Big(T\big[x_{\pi(1)},y_{\sigma(2)}\big]x_{\pi(2)}x_{\pi(3)}-\big[x_{\pi(1)},y_{\sigma(2)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}\\
&+\big[x_{\pi(1)}x_{\pi(2)},y_{\sigma(2)}\big]T\big(x_{\pi(3)}\big)+T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},y_{\sigma(2)}\big]-x_{\pi(1)}T\big[x_{\pi(2)},y_{\sigma(2)}\big]x_{\pi(3)}\\
&-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(2)}\big]+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(2)}\big]\Big)y_{\sigma(3)}\Big)\\
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\pi(1)}\Big(-x_{\pi(2)}y_{\sigma(1)}y_{\sigma(2)}T\big[x_{\pi(3)},y_{\sigma(3)}\big]+x_{\pi(2)}y_{\sigma(1)}y_{\sigma(2)}x_{\pi(3)}T\big(y_{\sigma(3)}\big)\\
&-x_{\pi(2)}y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}T\big(x_{\pi(3)}\big)-x_{\pi(2)}T\big[x_{\pi(3)},y_{\sigma(1)}\big]y_{\sigma(2)}y_{\sigma(3)}\\
&-x_{\pi(2)}T\big(y_{\sigma(1)}\big)\big[x_{\pi(3)},y_{\sigma(2)}y_{\sigma(3)}\big]+x_{\pi(2)}y_{\sigma(1)}T\big[x_{\pi(3)},y_{\sigma(2)}\big]y_{\sigma(3)}\\
&+x_{\pi(2)}y_{\sigma(1)}T\big(y_{\sigma(2)}\big)\big[x_{\pi(3)},y_{\sigma(3)}\big]+T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}\big]\\
&+\Big(T\big[x_{\pi(2)},y_{\sigma(1)}\big]y_{\sigma(2)}y_{\sigma(3)}-\big[x_{\pi(2)},y_{\sigma(1)}\big]T\big(y_{\sigma(2)}\big)y_{\sigma(3)}\\
&+\big[x_{\pi(2)},y_{\sigma(1)}y_{\sigma(2)}\big]T\big(y_{\sigma(3)}\big)+T\big(y_{\sigma(1)}\big)\big[x_{\pi(2)},y_{\sigma(2)}y_{\sigma(3)}\big]-y_{\sigma(1)}T\big[x_{\pi(2)},y_{\sigma(2)}\big]y_{\sigma(3)}\\
&-y_{\sigma(1)}T\big(y_{\sigma(2)}\big)\big[x_{\pi(2)},y_{\sigma(3)}\big]+y_{\sigma(1)}y_{\sigma(2)}T\big[x_{\pi(2)},y_{\sigma(3)}\big]\Big)x_{\pi(3)}\Big) \qquad (14)
\end{aligned}
$$

for all $x_1,x_2,x_3,y_1,y_2,y_3\in L$. Let $s:\mathbb{Z}\to\mathbb{Z}$ be the mapping defined by $s(i)=i-3$. For each $\sigma\in S_3$ the mapping $s^{-1}\sigma s:\{4,5,6\}\to\{4,5,6\}$ will be denoted by $\overline{\sigma}$. Writing x_{3+i} instead of y_i, $i=1,2,3$, in the above identity we can express this relation as

$$\sum_{i=1}^{6}E_i^i(\overline{x}_6)x_i+\sum_{j=1}^{6}x_jF_j^j(\overline{x}_6)=0.$$

For example

$$
\begin{aligned}
E_6^6(\overline{x}_6)=&\sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \overline{\sigma}(6)=6}}\Big(T\big[x_{\pi(1)},x_{\overline{\sigma}(4)}\big]x_{\pi(2)}x_{\pi(3)}x_{\overline{\sigma}(5)}-T\big(x_{\pi(1)}\big)x_{\overline{\sigma}(4)}x_{\pi(2)}x_{\pi(3)}x_{\overline{\sigma}(5)}\\
&+T\big(x_{\overline{\sigma}(4)}\big)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\overline{\sigma}(5)}-\big[x_{\pi(1)},x_{\overline{\sigma}(4)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}x_{\overline{\sigma}(5)}\\
&+\big[x_{\pi(1)}x_{\pi(2)},x_{\overline{\sigma}(4)}\big]T\big(x_{\pi(3)}\big)x_{\overline{\sigma}(5)}-x_{\pi(1)}T\big[x_{\pi(2)},x_{\overline{\sigma}(4)}\big]x_{\pi(3)}x_{\overline{\sigma}(5)}\\
&-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}\\
&-\big[x_{\pi(1)}x_{\pi(2)},x_{\overline{\sigma}(4)}\big]x_{\pi(3)}T\big(x_{\overline{\sigma}(5)}\big)\Big), \qquad (15)
\end{aligned}
$$

and

$$F_4^4(\overline{x}_6) = \sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \overline{\sigma}(4)=4}} \Big(x_{\overline{\sigma}(5)}x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(6)}\big] - x_{\overline{\sigma}(5)}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}T\big(x_{\overline{\sigma}(6)}\big)$$

$$+ x_{\overline{\sigma}(5)}x_{\pi(1)}x_{\pi(2)}x_{\overline{\sigma}(6)}T\big(x_{\pi(3)}\big) + x_{\overline{\sigma}(5)}T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},x_{\overline{\sigma}(6)}\big]$$

$$- x_{\overline{\sigma}(5)}x_{\pi(1)}T\big[x_{\pi(2)},x_{\overline{\sigma}(6)}\big]x_{\pi(3)} - x_{\overline{\sigma}(5)}x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(6)}\big]$$

$$+ x_{\overline{\sigma}(5)}T\big[x_{\pi(1)},x_{\overline{\sigma}(6)}\big]x_{\pi(2)}x_{\pi(3)} - T\big(x_{\overline{\sigma}(5)}\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},x_{\overline{\sigma}(6)}\big]$$

$$- \Big(T\big[x_{\pi(1)},x_{\overline{\sigma}(5)}\big]x_{\pi(2)}x_{\pi(3)} - \big[x_{\pi(1)},x_{\overline{\sigma}(5)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}$$

$$+ \big[x_{\pi(1)}x_{\pi(2)},x_{\overline{\sigma}(5)}\big]T\big(x_{\pi(3)}\big) + T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},x_{\overline{\sigma}(5)}\big]$$

$$- x_{\pi(1)}T\big[x_{\pi(2)},x_{\overline{\sigma}(5)}\big]x_{\pi(3)} - x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(5)}\big]$$

$$+ x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(5)}\big]\Big)x_{\overline{\sigma}(6)}\Big) \tag{16}$$

for all $\overline{x}_6 \in L^6$. Since L is 6-free, it follows that the functional identity (15) has only a standard solution. In particular, there exist mappings $p_{6j} : L^4 \to R$, $j = 1,2,3,4,5$ and $\lambda_6 : L^5 \to C(L)$ such that

$$E_6^6(\overline{x}_6) = \sum_{j=1}^{5} x_j p_{6j}^{6j}(\overline{x}_6) + \lambda_6^6(\overline{x}_6)$$

for all $\overline{x}_6 \in L^6$. Note that this is also a functional identity which can be rewritten as

$$\sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \overline{\sigma}(6)=6}} \Big(T\big[x_{\pi(1)},x_{\overline{\sigma}(4)}\big]x_{\pi(2)}x_{\pi(3)}$$

$$- T\big(x_{\pi(1)}\big)x_{\overline{\sigma}(4)}x_{\pi(2)}x_{\pi(3)} + T\big(x_{\overline{\sigma}(4)}\big)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}\Big)x_{\overline{\sigma}(5)}$$

$$+ \sum_{\pi\in S_3}\sum_{\sigma\in S_3,\sigma(6)=6} x_{\overline{\sigma}(4)}\Big(x_{\pi(1)}T\big(x_{\pi(2)}\big)x_{\pi(3)}x_{\overline{\sigma}(5)}$$

$$- x_{\pi(1)}x_{\pi(2)}T\big(x_{\pi(3)}\big)x_{\overline{\sigma}(5)} + x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}T\big(x_{\overline{\sigma}(5)}\big)\Big)$$

$$+ \sum_{\pi\in S_3}\sum_{\sigma\in S_3,\sigma(6)=6} x_{\pi(1)}\Big(- x_{\overline{\sigma}(4)}T\big(x_{\pi(2)}\big)x_{\pi(3)}x_{\overline{\sigma}(5)} + x_{\pi(2)}x_{\overline{\sigma}(4)}T\big(x_{\pi(3)}\big)x_{\overline{\sigma}(5)}$$

$$- T\big[x_{\pi(2)},x_{\overline{\sigma}(4)}\big]x_{\pi(3)}x_{\overline{\sigma}(5)} - T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}$$

$$+ x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)} - x_{\pi(2)}x_{\overline{\sigma}(4)}x_{\pi(3)}T\big(x_{\overline{\sigma}(5)}\big)\Big) - \sum_{j=1}^{5} x_j p_{6j}^{6j}(\overline{x}_6) \in C(L)$$

for all $\overline{x}_6 \in L^6$. Thus

$$E_4^4(\overline{x}_5)x_4 + E_5^5(\overline{x}_5)x_5 + x_1F_1^1(\overline{x}_5) + x_2F_2^2(\overline{x}_5)$$
$$+ x_3F_3^3(\overline{x}_5) + x_4F_4^4(\overline{x}_5) + x_5F_5^5(\overline{x}_5) \in C(L),$$

where in particular

$$E_5^5(\overline{x}_5) = \sum_{\pi\in S_3} \Big(T\big[x_{\pi(1)},x_4\big]x_{\pi(2)}x_{\pi(3)} - T\big(x_{\pi(1)}\big)x_4x_{\pi(2)}x_{\pi(3)} + T(x_4)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}\Big).$$

Again using that L is 6-free, this identity has only a standard solution. Hence there are mappings $p_{5j} : L^3 \to R$, $j = 1,2,3,4$, such that

$$E_5^5(\overline{x}_5) - \sum_{j=1}^{4} x_j p_{5j}^{5j}(\overline{x}_5) \in C(L).$$

We continue with the same procedure as above. Finally there exist mappings $p, q : L \to R$ and $\lambda : L^2 \to C(L)$ such that

$$T[x,y] - T(x)y + T(y)x = xp(y) + yq(x) + \lambda(x,y) \tag{17}$$

for all $x, y \in L$. Similarly, using (16) and the same method as above, we can show that there exist mappings $p', q' : L \to R$ and $\lambda' : L^2 \to C(L)$ such that

$$T[x,y] - xT(y) + yT(x) = p'(x)y + q'(y)x + \lambda'(x,y) \tag{18}$$

for all $x, y \in L$. Using (17) we obtain

$$\begin{aligned} 0 &= T[x,y] - T(x)y + T(y)x + T[y,x] - T(y)x + T(x)y \\ &= xp(y) + yq(x) + \lambda(x,y) + yp(x) + xq(y) + \lambda(y,x) \end{aligned}$$

for all $x, y \in L$. Thus

$$x(p(y) + q(y)) + y(q(x) + p(x)) \in C(L)$$

for all $x, y \in L$. It follows that $p(x) + q(x) = 0$ for all $x \in L$, so $p = -q$. Analogously we can prove that $p' = -q'$. Therefore by (17) and (18) we have

$$\begin{aligned} &T(x)y - T(y)x + xp(y) - yp(x) + \lambda(x,y) \\ &\qquad = xT(y) - yT(x) + p'(x)y - p'(y)x + \lambda'(x,y) \end{aligned}$$

for all $x, y \in L$. Note that this functional identity can be written as

$$\begin{aligned} &\big(T(x) - p'(x)\big)y + \big(-T(y) + p'(y)\big)x \\ &\qquad + x\big(-T(y) + p(y)\big) + y\big(T(x) - p(x)\big) \in C(L). \end{aligned}$$

Hence there exist $r \in R$ and $\mu, \mu' : L \to C(L)$ such that

$$\begin{aligned} T(x) - p'(x) &= xr + \mu(x) \\ T(x) - p(x) &= rx + \mu'(x). \end{aligned} \tag{19}$$

Note that (14) can be rewritten as

$$\begin{aligned} 0 = \sum_{\pi \in S_3} \sum_{\sigma \in S_3} \Big(&\big(x_{\pi(1)} p\big(x_{\overline{\sigma}(4)}\big) - x_{\overline{\sigma}(4)} p\big(x_{\pi(1)}\big) + \lambda\big(x_{\pi(1)}, x_{\overline{\sigma}(4)}\big)\big) x_{\pi(2)} x_{\pi(3)} x_{\overline{\sigma}(5)} \\ &- \big[x_{\pi(1)}, x_{\overline{\sigma}(4)}\big] T\big(x_{\pi(2)}\big) x_{\pi(3)} x_{\overline{\sigma}(5)} + \big[x_{\pi(1)} x_{\pi(2)}, x_{\overline{\sigma}(4)}\big] T\big(x_{\pi(3)}\big) x_{\overline{\sigma}(5)} \\ &- x_{\pi(1)} T\big[x_{\pi(2)}, x_{\overline{\sigma}(4)}\big] x_{\pi(3)} x_{\overline{\sigma}(5)} - x_{\pi(1)} T\big(x_{\pi(2)}\big) \big[x_{\pi(3)}, x_{\overline{\sigma}(4)}\big] x_{\overline{\sigma}(5)} \\ &+ x_{\pi(1)} x_{\pi(2)} T\big[x_{\pi(3)}, x_{\overline{\sigma}(4)}\big] x_{\overline{\sigma}(5)} - \big[x_{\pi(1)} x_{\pi(2)}, x_{\overline{\sigma}(4)}\big] x_{\pi(3)} T\big(x_{\overline{\sigma}(5)}\big) \Big) x_{\overline{\sigma}(6)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(\big(-x_{\pi(1)}p\big(x_{\overline{\sigma}(4)}\big)+x_{\pi(4)}p\big(x_{\pi(1)}\big)-\lambda\big(x_{\pi(1)},x_{\overline{\sigma}(4)}\big)\big)x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}x_{\pi(2)}\\
&+\big[x_{\pi(1)},x_{\overline{\sigma}(4)}\big]T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}x_{\pi(2)}-\big[x_{\pi(1)},x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big]T\big(x_{\overline{\sigma}(6)}\big)x_{\pi(2)}\\
&+x_{\overline{\sigma}(4)}T\big[x_{\pi(1)},x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)}x_{\pi(2)}+x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)\big[x_{\pi(1)},x_{\overline{\sigma}(6)}\big]x_{\pi(2)}\\
&-x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}T\big[x_{\pi(1)},x_{\overline{\sigma}(6)}\big]x_{\pi(2)}+\big[x_{\pi(1)},x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)}T\big(x_{\pi(2)}\big)\Big)x_{\pi(3)}\\
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\overline{\sigma}(4)}\Big(x_{\overline{\sigma}(5)}x_{\pi(1)}x_{\pi(2)}\big(p'\big(x_{\pi(3)}\big)x_{\overline{\sigma}(6)}-p'\big(x_{\overline{\sigma}(6)}\big)x_{\pi(3)}+\lambda'\big(x_{\pi(3)},x_{\overline{\sigma}(6)}\big)\big)\\
&+x_{\overline{\sigma}(5)}T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},x_{\overline{\sigma}(6)}\big]-x_{\overline{\sigma}(5)}x_{\pi(1)}T\big[x_{\pi(2)},x_{\overline{\sigma}(6)}\big]x_{\pi(3)}\\
&-x_{\overline{\sigma}(5)}x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(6)}\big]+x_{\overline{\sigma}(5)}T\big[x_{\pi(1)},x_{\overline{\sigma}(6)}\big]x_{\pi(2)}x_{\pi(3)}\\
&-T\big(x_{\overline{\sigma}(5)}\big)\big[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},x_{\overline{\sigma}(6)}\big]-\Big(T\big[x_{\pi(1)},x_{\overline{\sigma}(5)}\big]x_{\pi(2)}x_{\pi(3)}\\
&-\big[x_{\pi(1)},x_{\overline{\sigma}(5)}\big]T\big(x_{\pi(2)}\big)x_{\pi(3)}+\big[x_{\pi(1)}x_{\pi(2)},x_{\overline{\sigma}(5)}\big]T\big(x_{\pi(3)}\big)\\
&+T\big(x_{\pi(1)}\big)\big[x_{\pi(2)}x_{\pi(3)},x_{\overline{\sigma}(5)}\big]-x_{\pi(1)}T\big[x_{\pi(2)},x_{\overline{\sigma}(5)}\big]x_{\pi(3)}\\
&-x_{\pi(1)}T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(5)}\big]+x_{\pi(1)}x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(5)}\big]\Big)x_{\overline{\sigma}(6)}\Big)\\
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\pi(1)}\Big(x_{\pi(2)}x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big(-p'\big(x_{\pi(3)}\big)x_{\overline{\sigma}(6)}+p'\big(x_{\overline{\sigma}(6)}\big)x_{\pi(3)}\\
&-\lambda'\big(x_{\pi(3)},x_{\overline{\sigma}(6)}\big)\big)-x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}-x_{\pi(2)}T\big(x_{\overline{\sigma}(4)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big]\\
&+x_{\pi(2)}x_{\overline{\sigma}(4)}T\big[x_{\pi(3)},x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)}+x_{\pi(2)}x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(6)}\big]\\
&+T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big]+\Big(T\big[x_{\pi(2)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\\
&-\big[x_{\pi(2)},x_{\overline{\sigma}(4)}\big]T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}+\big[x_{\pi(2)},x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big]T\big(x_{\overline{\sigma}(6)}\big)\\
&+T\big(x_{\overline{\sigma}(4)}\big)\big[x_{\pi(2)},x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big]-x_{\overline{\sigma}(4)}T\big[x_{\pi(2)},x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)}\\
&-x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)\big[x_{\pi(2)},x_{\overline{\sigma}(6)}\big]+x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}T\big[x_{\pi(2)},x_{\overline{\sigma}(6)}\big]\Big)x_{\pi(3)}\Big).
\end{aligned}
$$

It is easy to see that this functional identity is of the form $\sigma_{j=1}^{6}x_jF_j^j(\overline{x}_6)=0$, where in particular

$$
\begin{aligned}
F_1^1(\overline{x}_6)=&\sum_{\substack{\pi\in S_3\\ \pi(1)=1}}\sum_{\sigma\in S_3}\Big(p\big(x_{\overline{\sigma}(4)}\big)x_{\pi(2)}x_{\pi(3)}x_{\overline{\sigma}(5)}+\lambda\big(x_{\pi(2)},x_{\overline{\sigma}(4)}\big)x_{\pi(3)}x_{\overline{\sigma}(5)}\\
&-x_{\overline{\sigma}(4)}T\big(x_{\pi(2)}\big)x_{\pi(3)}x_{\overline{\sigma}(5)}+x_{\pi(2)}x_{\overline{\sigma}(4)}T\big(x_{\pi(3)}\big)x_{\overline{\sigma}(5)}\\
&-T\big[x_{\pi(2)},x_{\overline{\sigma}(4)}\big]x_{\pi(3)}x_{\overline{\sigma}(5)}-T\big(x_{\pi(2)}\big)\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}\\
&+x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}-x_{\pi(2)}x_{\overline{\sigma}(4)}x_{\pi(3)}T\big(x_{\overline{\sigma}(5)}\big)\Big)x_{\overline{\sigma}(6)}\\
&+\sum_{\substack{\pi\in S_3\\ \pi(1)=1}}\sum_{\sigma\in S_3}\Big(-p\big(x_{\overline{\sigma}(4)}\big)x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}x_{\pi(2)}+x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}x_{\pi(2)}\\
&-x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}T\big(x_{\overline{\sigma}(6)}\big)x_{\pi(2)}+x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}T\big(x_{\pi(2)}\big)\Big)x_{\pi(3)}\\
&+\sum_{\substack{\pi\in S_3\\ \pi(1)=1}}\sum_{\sigma\in S_3}\Big(x_{\pi(2)}x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big(-p'\big(x_{\pi(3)}\big)x_{\overline{\sigma}(6)}+p'\big(x_{\overline{\sigma}(6)}\big)x_{\pi(3)}\\
&-\lambda'\big(x_{\pi(3)},x_{\overline{\sigma}(6)}\big)\big)-x_{\pi(2)}T\big[x_{\pi(3)},x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}
\end{aligned}
$$

$$
\begin{aligned}
&- x_{\pi(2)}T\big(x_{\overline{\sigma}(4)}\big)\big[x_{\pi(3)}, x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big] + x_{\pi(2)}x_{\overline{\sigma}(4)}T\big[x_{\pi(3)}, x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)}\\
&+ x_{\pi(2)}x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)\big[x_{\pi(3)}, x_{\overline{\sigma}(6)}\big] + T\big(x_{\pi(2)}\big)\big[x_{\pi(3)}, x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big]\\
&+ \Big(T\big[x_{\pi(2)}, x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)} - \big[x_{\pi(2)}, x_{\overline{\sigma}(4)}\big]T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}\\
&+ \big[x_{\pi(2)}, x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big]T\big(x_{\overline{\sigma}(6)}\big) + T\big(x_{\overline{\sigma}(4)}\big)\big[x_{\pi(2)}, x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big]\\
&- x_{\overline{\sigma}(4)}T\big[x_{\pi(2)}, x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)} - x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)\big[x_{\pi(2)}, x_{\overline{\sigma}(6)}\big]\\
&+ x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}T\big[x_{\pi(2)}, x_{\overline{\sigma}(6)}\big]\Big)x_{\pi(3)}\Big).
\end{aligned}
$$

Thus $F_1^1(\overline{x}_6) = 0$ for all $\overline{x}_6 \in L^6$. Further, this identity can be written as $\sum_{i=2}^{6} E_i^i(x_2, x_3, x_4, x_5, x_6)x_i = 0$, which in turn implies $E_i = 0$ for all $i = 2, 3, 4, 5, 6$. In particular

$$
\begin{aligned}
0 = {}& E_3^3(x_2, x_3, x_4, x_5, x_6)\\
&\sum_{\sigma \in S_3}\Big(-p\big(x_{\overline{\sigma}(4)}\big)x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}x_2 + x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}x_2 - x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}T\big(x_{\overline{\sigma}(6)}\big)x_2\\
&+ x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}T(x_2)\Big) + \sum_{\sigma \in S_3}\Big(x_2x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}p'\big(x_{\overline{\sigma}(6)}\big)\\
&+ x_2T\big(x_{\overline{\sigma}(4)}\big)x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)} - x_2x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}\\
&- T(x_2)x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)} + T\big[x_2, x_{\overline{\sigma}(4)}\big]x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)} - \big[x_2, x_{\overline{\sigma}(4)}\big]T\big(x_{\overline{\sigma}(5)}\big)x_{\overline{\sigma}(6)}\\
&+ \big[x_2, x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}\big]T\big(x_{\overline{\sigma}(6)}\big) + T\big(x_{\overline{\sigma}(4)}\big)\big[x_2, x_{\overline{\sigma}(5)}x_{\overline{\sigma}(6)}\big] - x_{\overline{\sigma}(4)}T\big[x_2, x_{\overline{\sigma}(5)}\big]x_{\overline{\sigma}(6)}\\
&- x_{\overline{\sigma}(4)}T\big(x_{\overline{\sigma}(5)}\big)\big[x_2, x_{\overline{\sigma}(6)}\big] + x_{\overline{\sigma}(4)}x_{\overline{\sigma}(5)}T\big[x_2, x_{\overline{\sigma}(6)}\big]\Big).
\end{aligned}
$$

This is also a functional identity,

$$
\begin{aligned}
&E_2^2(x_2, x_4, x_5, x_6)x_2 + E_4^4(x_2, x_4, x_5, x_6)x_4 + E_5^5(x_2, x_4, x_5, x_6)x_5\\
&\quad + E_6^6(x_2, x_4, x_5, x_6)x_6 + x_2F_2(x_2, x_4, x_5, x_6) + x_4F_4^4(x_2, x_4, x_5, x_6)\\
&\quad + x_5F_5^5(x_2, x_4, x_5, x_6) + x_6F_6^6(x_2, x_4, x_5, x_6) = 0.
\end{aligned}
$$

In particular

$$
\begin{aligned}
E_2^2(x_2, x_4, x_5, x_6) = \sum_{\sigma \in S_3}&\Big(-p\big(x_{\sigma(4)}\big)x_{\sigma(5)}x_{\sigma(6)} - x_{\sigma(4)}x_{\sigma(5)}T\big(x_{\sigma(6)}\big)\\
&- T\big(x_{\sigma(4)}\big)x_{\sigma(5)}x_{\sigma(6)} + x_{\sigma(4)}T\big(x_{\sigma(5)}\big)x_{\sigma(6)}\Big),
\end{aligned}
$$

which in turn implies

$$E_2^2(x_4, x_5, x_6) = x_4h_4(x_5, x_6) + x_5h_5(x_4, x_6) + x_6h_6(x_4, x_5) + \lambda_1(x_4, x_5, x_6)$$

for all $x_4, x_5, x_6 \in L$, where $h_4, h_5, h_6 : L^2 \to R$ and $\lambda_1 : L^3 \to C(L)$. Consequently (after two more steps), there exist $r' \in R$ and a mapping $\mu'' : L \to C(L)$ such that

$$T(x) + p(x) = xr' + \mu''(x)$$

for all $x \in L$. On the other hand (by (19)) we have

$$T(x) - p(x) = rx + \mu'(x)$$

for all $x \in L$. Thus

$$2T(x) = xr' + rx + \mu'(x) + \mu''(x).$$

By (12) we arrive at

$$\begin{aligned}
&\sum_{\pi \in S_3} \big(x_{\pi(1)} r' x_{\pi(2)} x_{\pi(3)} + (\mu' + \mu'')\big(x_{\pi(1)}\big) x_{\pi(2)} x_{\pi(3)} - x_{\pi(1)} r x_{\pi(2)} x_{\pi(3)} \\
&\qquad - x_{\pi(1)} x_{\pi(2)} r' x_{\pi(3)} - x_{\pi(1)} (\mu' + \mu'')\big(x_{\pi(2)}\big) x_{\pi(3)} + x_{\pi(1)} x_{\pi(2)} r x_{\pi(3)} \\
&\qquad + x_{\pi(1)} x_{\pi(2)} (\mu' + \mu'')\big(x_{\pi(3)}\big)\big) \\
&\quad = \sum_{\pi \in S_3} \big(x_{\pi(1)} r' x_{\pi(2)} x_{\pi(3)} + (\mu' + \mu'')\big(x_{\pi(1)}\big) x_{\pi(2)} x_{\pi(3)} - x_{\pi(1)} r x_{\pi(2)} x_{\pi(3)} \\
&\qquad - x_{\pi(1)} x_{\pi(2)} r' x_{\pi(3)} + x_{\pi(1)} x_{\pi(2)} r x_{\pi(3)}\big) \in C(L).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{\pi \in S_3, \pi(3)=3} &\big(x_{\pi(1)} r' x_{\pi(2)} + (\mu' + \mu'')\big(x_{\pi(1)}\big) x_{\pi(2)} \\
&- x_{\pi(1)} r x_{\pi(2)} - x_{\pi(1)} x_{\pi(2)} r' + x_{\pi(1)} x_{\pi(2)} r\big) = 0
\end{aligned}$$

Note that this implies

$$(r' - r)x + x(r - r') = -(\mu' + \mu'')(x)$$

for all $x \in L$. It follows that $\mu' + \mu'' = 0$ and $r' - r \in C(L)$. Thus $[r, x] = [r', x]$ for all $x \in L$, which implies

$$2T(x) = xr + r'x, \ 2T(x) = rx + xr'.$$

Consequently $4T(x) = (r + r')x + x(r + r')$ for all $x \in L$. Thereby the proof is completed. □

3. Results

The main purpose of this paper is to prove Theorem 3.2. Note that Theorem 2.1 and Theorem 3.2 are almost trivially true for any ring with a unit 1. Indeed, setting $x = 1$ in a partial linearization of (4)

$$\begin{aligned}
T(x^2y + xyx + yx^2) &= T(x)xy + T(x)yx + T(y)x^2 \\
&\quad - xT(x)y - xT(y)x - yT(x)x + x^2T(y) + xyT(x) + yxT(x),
\end{aligned}$$

$x, y \in R$, one sees that $2T(y) = T(1)y + yT(1)$ for all $y \in R$.

Theorem 3.1. *Let R be a 2-torsion free ring and let $T : R \to R$ be an additive mapping satisfying the relation (4). If the center of R is nonzero and contains no nonzero divisors of R, then T is of the form $4T(x) = qx + xq$ for some fixed element $q \in Q_s$.*

Proof. Let c be a nonzero central element. Pick any $x \in R$ and set $x_1 = x_2 = cx$ and $x_3 = x$ in (12). We arrive at

$$\begin{aligned}
T(6c^2x^3) &= 2c\big(2T(cx)x^2 + T(x)x^2c - 2xT(cx)x - xT(x)xc + x^2T(x)c + 2x^2T(cx)\big) \\
&= 2c\big(2T(cx)x^2 + T(x^3)c - 2xT(cx)x + 2x^2T(cx)\big).
\end{aligned}$$

On the other hand, setting $x_1 = x_2 = c$ and $x_3 = x^3$ in (12) we obtain

$$T(6c^2x^3) = 2c\big(T(c)x^3 + T(x^3)c + x^3T(c)\big).$$

Comparing so obtained relations we get

$$\big(2T(cx) - T(c)x\big)x^2 + x^2\big(2T(cx) - xT(c)\big) - 2xT(cx)x = 0. \tag{20}$$

In particular, when $x = c$, we have

$$2c^2\big(T(c^2) - cT(c)\big) = 0.$$

By the primeness of R this implies

$$T(c^2) = cT(c). \tag{21}$$

Replace x_1 by cx in (12). On the other hand replace x_2 by cy in (12), where $y \in R$. In both cases let $x_3 = c$. Comparing so obtained results we arrive at

$$\begin{aligned}&\big(T(cx) - cT(x)\big)y + y\big(T(cx) - cT(x)\big)\\ &\quad = \big(T(cy) - cT(y)\big)x + x\big(T(cy) - cT(y)\big)\end{aligned} \tag{22}$$

for all $x, y \in R$. Setting $y = c$ in (22) and using (21) one concludes that

$$T(cx) = cT(x) \tag{23}$$

for all $x \in R$.

Now let F be the field of fractions of R. Enlarge R to the ring RF, noting that any element of RF can be written in the form rc^{-1}. Then T can be extended to RF by defining

$$T\big(xc^{-1}\big) = c^{-1}T(x)$$

for all $x \in R$. This is well-defined. Namely, if $xc^{-1} = yd^{-1}$ then $cy = dx$, whence by (23)

$$cT(y) = T(cy) = T(dx) = dT(x),$$

which implies $c^{-1}T(x) = d^{-1}T(y)$. It is easily seen that this extended T satisfies (4). Since RF has the identity element $1 = cc^{-1}$, the conclusion of the theorem holds, as noted above. Thereby the proof is completed. □

Theorem 3.2. *Let R be a 2-torsion free prime ring and let $T : R \to R$ be an additive mapping satisfying the relation* (4). *In this case T is of the form $4T(x) = qx + xq$ for some fixed element $q \in Q_s$.*

Proof. Note that the complete linearization of (4) gives us (12).

First suppose that R is not a PI ring (satisfying the standard polynomial identity of degree less than 12). According to Theorem 2.1 there exist $q \in Q_{mr}$ such that $4T(x) = xq + qx$ for all $x \in R$. Since $qx + xq \in R$ for all $x \in R$ it follows from the end of the proof of [18, Theorem 2.1] that $q \in Q_s$.

Assume now that R is a PI ring. It is well-known that in this case R has a nonzero center [14]. Since the center of a prime ring R contains no nonzero divisors of R the proof is completed by Theorem 3.1. □

Our last result is related to bicircular projections on prime ring with involution which is related to the conjecture in [17].

Theorem 3.3. *Let R be a prime *-ring of characteristic different from two. Suppose that $D, G : R \to R$ are additive mappings satisfying the relations*

$$D(x^3) = D(x)x^2 + xG(x^*)^*x + x^2D(x),$$
$$G(x^3) = G(x)x^2 + xD(x^*)^*x + x^2G(x)$$

for all $x \in R$. In this case D and G are of the form

$$16D(x) = 4\big(d(x) + g(x)\big) + (p+q)x + x(p+q),$$
$$16G(x) = 4\big(d(x) - g(x)\big) + (p-q)x + x(p-q)$$

for all $x \in R$ where d and g are a derivations and p and q are some fixed elements from Q_s. Besides, $d(x) = -d(x^*)^*$, $g(x) = g(x^*)^*$ for all $x \in R$ and $p^* = p$, $q^* = q$.

Proof. The proof goes through in several steps. Let us first assume that $D = G$. In this case we have the relation

$$F\big(x^3\big) = F(x)x^2 + xF(x^*)^*x + x^2F(x) \tag{24}$$

for all $x \in R$. It is our aim to prove that F is of the form

$$8F(x) = 4d(x) + qx + xq, \tag{25}$$

for all $x \in R$, where d is a derivation of R and q is a fixed element from Q_s. Besides, $d(x) = d(x^*)^*$, for all $x \in R$ and $q^* = -q$. Let us introduce mappings $d : R \to R$ and $f : R \to R$ by

$$\begin{aligned} d(x) &= F(x) + F(x^*)^* \\ f(x) &= F(x) - F(x^*)^* \end{aligned} \tag{26}$$

for all $x \in R$. Now we have

$$d(x^*)^* = (F(x^*) + F(x)^*)^* = F(x) + F(x^*)^* = d(x), \quad x \in R.$$

From the relation (24) one obtains easily that

$$d\big(x^3\big) = d(x)x^2 + xd(x)x + x^2d(x) \tag{27}$$

and

$$f\big(x^3\big) = f(x)x^2 - xf(x)x + x^2f(x), \tag{28}$$

is fulfilled for all $x \in R$. Now it follows from the relation (27) and [2, Theorem 4.4] that d is a derivation. On the other hand one can conclude from the relation (28) applying Theorem 3.2 that f is of the form $4f(x) = qx + xq$, for all $x \in R$ and some fixed element $q \in Q_s$. We have therefore

$$4F(x) - 4F(x^*)^* = qx + xq \tag{29}$$

for all $x\in R$. Putting in the above relation x^* for x we obtain $4F(x^*) - 4F(x)^* = qx^* + x^*q$, $x\in R$, which gives

$$4F(x^*)^* - 4F(x) = q^*x + xq^*$$

for all $x\in R$. Combining the above relation with the relation (29) we obtain $(q+q^*)x + x(q+q^*) = 0$, for all $x\in R$, whence it follows after some calculation because of the primeness of R that $q^* = -q$. Combining the relation (26) with the relation (29) we obtain $8F(x) = 4d(x) + qx + xq$, $x\in R$, which completes the proof of the first step.

Let us now assume that $D = -G$. Thus we have the relation

$$H(x^3) = H(x)x^2 - xH(x^*)^*x + x^2H(x),$$

for all $x\in R$. In this case H is of the form

$$8H(x) = 4g(x) + px + xp, \tag{30}$$

for all $x\in R$ where g is a derivation of R and $p\in Q_s$ is some fixed element. Besides, $g(x) = -g(x^*)^*$, for all $x\in R$ and $p^* = p$. The proof of the second step will be omitted since it goes through using the same arguments as in the proof of the first step.

We are ready for the proof of general case. We have therefore relations

$$D(x^3) = D(x)x^2 + xG(x^*)^*x + x^2D(x), \quad x\in R \tag{31}$$

and

$$G(x^3) = G(x)x^2 + xD(x^*)^*x + x^2G(x), \quad x\in R. \tag{32}$$

Combining (31) with (32) we obtain

$$F(x^3) = F(x)x^2 + xF(x^*)^*x + x^2F(x), \quad x\in R$$

where $F(x)$ stands for $D(x) + G(x)$. On the other hand subtracting the relation (32) from the relation (31) we arrive at

$$H(x^3) = H(x)x^2 - xH(x^*)^*x + x^2H(x), \quad x\in R$$

where $H(x)$ denotes $D(x) - G(x)$. Now according to (25) and (30) we have

$$8D(x) + 8G(x) = 4d(x) + qx + xq, \quad x\in R \tag{33}$$

and

$$8D(x) - 8G(x) = 4g(x) + px + xp, \quad x\in R. \tag{34}$$

From (33) and (34) one obtains

$$16D(x) = 4(d(x) + g(x)) + (p+q)x + x(p+q), \quad x\in R$$

and

$$16G(x) = 4(d(x) - g(x)) + (q-p)x + x(q-p), \quad x\in R$$

which completes the proof of the theorem. □

Acknowledgment. The authors would like to thank the referee for careful reading of the paper and some useful suggestions.

References

[1] Beidar KI, Brešar M, Chebotar MA (2002) Functional identities revisited: the functional and the strong degree. Comm Algebra **30**: 935–969
[2] Beidar KI, Brešar M, Chebotar MA, Martindale WS 3rd (2001) On Herstein's Lie map Conjectures II. J Algebra **238**: 239–264
[3] Beidar KI, Chebotar MA (2000) On functional identities and d-free subsets of rings I. Comm Algebra **28**: 3925–3952
[4] Beidar KI, Fong Y (1999) On additive isomorphisms of prime rings preserving polynomials. J Algebra **217**: 650–667
[5] Beidar KI, Martindale WS III, Mikhalev AV (1996) Rings with Generalized Identities. New York: Marcel Dekker
[6] Beidar KI, Mikhalev AV, Chebotar MA (2004) Functional identities in rings and their applications. Russian Math Surveys **59**: 403–428
[7] Brešar M (2000) Functional identities: A survey. Contemporary Math **259**: 93–109
[8] Brešar M (1988) Jordan derivations on semiprime rings. Proc Amer Math Soc **104**: 1003–1004
[9] Brešar M (1989) Jordan mappings of semiprime rings. J Algebra **127**: 218–228
[10] Brešar M, Vukman J (1988) Jordan derivations on prime rings. Bull Austral Math Soc **37**: 321–322
[11] Cusak J (1975) Jordan derivations on rings. Proc Amer Math Soc **53**: 321–324
[12] Fošner M, Ilišelić D (2005) On a class of projections on *-rings. Commun Algebra **33**: 3293–3310
[13] Herstein IN (1957) Jordan derivations of prime rings. Proc Amer Math Soc **8**: 1104–1119
[14] Rowen LH (1973) Some results on the center of a ring with polynomial identity. Bull Amer Math Soc **79**: 219–223
[15] Stachó LL, Zalar B (2004) Bicircular projections on some matrix and operator spaces. Linear Algebra Appl **384**: 9–20
[16] Stachó LL, Zalar B (2004) Bicircular projections and characterization of Hilbert spaces. Proc Amer Math Soc **132**: 3019–3025
[17] Vukman J (2007) On functional equations related to bicircular projections. Glasnik Mat (to appear)
[18] Vukman J, Kosi-Ulbl I, Eremita D (2005) On certain equations in rings. Bull Austral Math Soc **71**: 53–60

Authors' addresses: Maja Fošner, Faculty of logistics, University of Maribor, Mariborska cesta 2, 3000 Celje, Slovenia, e-mail: maja.fosner@uni-mb.si; Joso Vukman, Department of Mathematics and Computer Science, Faculty of Sciences and Mathematics, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia, e-mail: joso.vukman@uni-mb.si