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Involutive and Peirce Gradings in JBW*-Triples

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ABSTRACT

A Peirce grading (J_0, J_1, J_2) of a Jordan*-triple A consists of subspaces J_0, J_1 and J_2 of A, with direct sum A, which satisfy the conditions that

$$\{J_0\ J_2\ A\} = \{J_2\ J_0\ A\} = \{0\},$$

and, for j, k, and l equal to 0, 1, or 2, if $j-k+l$ is equal to 0, 1 or 2 then

$$\{J_j\ J_k\ J_l\} \subseteq J_{j-k+l},$$

and, if not then

$$\{J_j\ J_k\ J_l\} = \{0\}.$$

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An involutive grading (B_+, B_-) of A consists of a pair of subtriples of A, with direct sum A, satisfying the conditions

$$\{B_+\ B_-\ B_+\} \subseteq B_-, \qquad \{B_-\ B_+\ B_-\} \subseteq B_+,$$
$$\{B_+\ B_+\ B_-\} \subseteq B_-, \qquad \{B_-\ B_-\ B_+\} \subseteq B_+.$$

Every Peirce grading (J_0, J_1, J_2) of A gives rise to an involutive grading $(J_0 \oplus J_2, J_1)$ of A. It is shown that, conversely, when A is a JBW*-triple factor and (B_+, B_-) is an involutive grading of A, either B_+ is also a JBW*-triple factor or, for each weak*-closed ideal J_0 of B_+, with complementary weak*-closed ideal J_2, writing J_1 for B_-, (J_0, J_1, J_2) is a Peirce grading of A.

Key Words: Jordan*-triple; JBW*-triple; Peirce grading; Involutive grading.

1. INTRODUCTION

A study of involutive and Peirce gradings of Jordan pairs, Jordan triple systems and Jordan algebras was carried out by Neher (1981) who showed, amongst other things, that, provided that the Jordan structure in question was simple, semi-simple, and satisfied both the ascending and descending chain conditions on principal inner ideals, the two concepts were essentially equivalent. One of the purposes of this note is to extend Neher's results to a large class of Jordan*-triples.

A complex Banach space A the open unit ball in which is a bounded symmetric domain has a natural triple product with respect to which it is a Jordan*-triple, known as a JB*-triple. The class of JB*-triples includes that of JB*-algebras, which itself includes the class of C*-algebras. When the complex Banach space A is also the dual of a Banach space then A is said to be a JBW*-triple. The class of JBW*-triples includes the class of JBW*-algebras, which itself includes the class of W*-algebras, or von Neumann algebras. A JBW*-triple A is said to be a JBW*-triple factor if it possesses no non-trivial weak*-closed ideals. A JBW*-triple factor need not be simple, nor need it satisfy either the ascending or descending chain conditions on principal inner ideals.

In this paper the properties of involutive and Peirce gradings of JB*-triples and JBW*-triples are investigated. One of the features of JB*-triples is that many of their algebraic properties automatically have topological and geometric consequences. For example, it is known that every structural projection on a JBW*-triple is automatically contractive and weak*-continuous (Edwards et al., 1996). Further examples of such phenomena occur in the study of involutive gradings of JB*-triples.

The subtriples B_+ and B_- occurring in an involutive grading (B_+, B_-) of a JB*-triple A are automatically norm-closed, and, if A is a JBW*-triple, the subtriples B_+ and B_- are automatically weak*-closed. Moreover, the set of bicontractive projections on a JB*-triple A can be characterised in the set of all linear projections on A purely in algebraic terms. Furthermore, bicontractive projections on a JBW*-triple are automatically weak*-continuous. Although this paper is mainly concerned with JBW*-triples, many of the results proved hold in part in much greater generality. For example, the results quoted or proved for Jordan*-triples hold for Jordan triple systems over arbitrary rings containing 1/2. Readers more interested in this approach are referred to Anquela and Cortes (to appear).

It is clear that every Peirce grading (J_0, J_1, J_2) of a Jordan*-triple A provides an involutive grading $(J_0 \oplus J_2, J_1)$ of A. The main result of the paper shows that, provided that A is a JBW*-triple factor, for every involutive grading (B_+, B_-) of A, either B_+ is itself a JBW*-triple factor or, for each non-trivial weak*-closed ideal J_0 in the JBW*-triple B_+, with complementary weak*-closed ideal J_2, when J_1 is written instead of B_-, (J_0, J_1, J_2) is a Peirce grading of A and J_0 and J_2 are themselves JBW*-triple factors. Of course, by symmetry, the same result applies when B_+ is replaced by B_-.

This provides another example of a phenomenon, which often appears in the theory of Jordan structures, in which a particular result that holds for a Jordan*-triple A only under strong and sometimes quite technical algebraic conditions, continues to hold in greater generality when these conditions are replaced by the geometrical requirement that A is a JB*-triple or a JBW*-triple of some kind. Such results depend upon the very intimate relationships that exist between the algebraic, geometric and holomorphic structures of JB*-triples and JBW*-triples.

The paper is organized as follows. In Sec. 2 basic definitions are given and notation is established. In Sec. 3 and in Sec. 4 a study of involutive and Peirce gradings of JB*-triples and JBW*-triples is undertaken, and the automatic topological and geometric consequences, referred to above, are described. The algebraic parts of most of the proofs are similar to those employed by Neher, and the topological parts of the proofs follow standard techniques which depend upon the work of Dineen, Kaup and Upmeier, Friedmann and Russo, Stachó and others. In Sec. 5 the main result connecting Peirce and involutive gradings for JBW*-triples is proved. The proof of this is less obvious, and requires the use of more recent techniques, developed by the authors in Edwards and Rüttimann (2003, to appear). In the final section the results are applied to the special case of in which A is a W*-algebra, for which the results do not appear to have been known previously.

2. PRELIMINARIES

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a b c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and, for elements a, b, c and d in A, satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\ b\ c\}, d) - D(c, \{d\ a\ b\}), \tag{2.1}$$

where $[.,.]$ denotes the commutator, and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a,b)c = \{a\ b\ c\},$$

is said to be a *Jordan*-triple*. A Jordan*-triple A for which the vanishing of $\{a\,a\,a\}$ implies that a itself vanishes is said to be *anisotropic*. For each element a in A, the conjugate linear mapping $Q(a)$ from A to itself is defined, for each element b in A, by

$$Q(a)b = \{a\ b\ a\}.$$

A subspace B of a Jordan*-triple A such that $\{B\,B\,B\}$ is contained in B is said to be a *subtriple* of A. A subtriple J of A for which $\{J\,A\,J\}$ is contained in J is said to be an *inner ideal* of A. An inner ideal I in A for which both $\{A\,I\,A\}$ and $\{A\,A\,I\}$ are contained in I is said to be an *ideal* in A.

An element u in a Jordan*-triple A is said to be a *tripotent* if $\{u\,u\,u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. For each tripotent u in A, the linear operators $P_0(u)$, $P_1(u)$ and $P_2(u)$, defined by

$$\begin{aligned} P_0(u) &= \mathrm{id}_A - 2D(u,u) + Q(u)^2, \\ P_1(u) &= 2(D(u,u) - Q(u)^2), \\ P_2(u) &= Q(u)^2, \end{aligned} \tag{2.2}$$

are mutually orthogonal projection operators on A with sum id_A. For j equal to 0, 1 or 2, the range of $P_j(u)$ is the eigenspace $A_j(u)$ of $D(u,u)$ corresponding to the eigenvalue $\frac{1}{2}j$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u) \tag{2.3}$$

is the *Peirce decomposition* of A relative to u. Moreover, $A_0(u)$ and $A_2(u)$ are inner ideals in A, $A_1(u)$ is a subtriple of A, and $A_j(u)$ is said to be the *Peirce j-space* corresponding to the tripotent u. Furthermore,

$$\{A\ A_2(u)\ A_0(u)\} = \{A\ A_0(u)\ A_2(u)\} = \{0\}, \tag{2.4}$$

and, for j, k and l equal to 0, 1 or 2,

$$\{A_j(u)\ A_k(u)\ A_l(u)\} \subseteq A_{j+l-k}(u) \tag{2.5}$$

when $j+l-k$ is equal to 0, 1 or 2, and

$$\{A_j(u)\ A_k(u)\ A_l(u)\} = \{0\} \tag{2.6}$$

otherwise. For details of the properties of Jordan*-triples the reader is referred to Meyberg (1972), Upmeier (1985) and Loos (1975).

A Jordan*-triple A which is also a Banach space such that D is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of Bonsall and Duncan (1971, Definition 5.1), with non-negative spectrum, and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

is said to be a *JB*-triple*. The final condition can be replaced by the apparently less restrictive condition that, for all elements a in A,

$$\|\{a\ a\ a\}\| = \|a\|^3.$$

A complex Banach space possesses a triple product with respect to which it forms a JB*-triple if and only if its open unit ball is a bounded symmetric domain (Kaup, 1983). Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. Every norm-closed subtriple of a JB*-triple A is a JB*-triple, and a norm-closed subspace J of A is an ideal if and only if $\{J\,J\,A\}$ is contained in J (Bunce and Chu, 1992). A JB*-triple A which is the dual of a Banach space A_* is said to be a *JBW* -triple*. In this case the *predual*A_* of A is unique and, for each element a in A, the operators $D(a, b)$ and $Q(a)$ are weak*-continuous (Barton and Timoney, 1986; Barton et al., 1987; Horn, 1987). It follows that a weak*-closed subtriple B of a JBW*-triple A is a JBW*-triple. The second dual A^{**} of a JB*-triple A is a JBW*-triple (Dineen, 1986a; Dineen, 1986b). For other important properties of JBW*-triples the reader is referred to Friedman and Russo (1985), Kaup (1984) and Stachó (1982). Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras, the properties of which may be found in Edwards (1980); Hanche-Olsen and Stormer (1984) and Wright (1977).

Let u be a tripotent in the JBW*-triple A. With respect to the multiplication $(a,b) \mapsto a.b$ defined by

$$a.b = \{a\,u\,b\},$$

and involution $a \mapsto a^\dagger$ defined by

$$a^\dagger = \{u\,a\,u\},$$

the Peirce two-space $A_2(u)$ forms a JBW*-algebra with unit u. Furthermore, for elements a, b and c in $A_2(u)$,

$$\{a\ b\ c\} = a.(b^\dagger.c) + c.(b^\dagger.a) - b^\dagger.(a.c). \tag{2.7}$$

3. INVOLUTIVE GRADINGS

Recall that a pair (B_+, B_-) of subtriples of a Jordan*-triple A is said to be an *involutive grading* of A if

$$A = B_+ \oplus B_-, \tag{3.1}$$

$$\{B_+\ B_-\ B_+\} \subseteq B_-, \qquad \{B_-\ B_+\ B_-\} \subseteq B_+, \tag{3.2}$$

$$\{B_+\ B_+\ B_-\} \subseteq B_-, \qquad \{B_-\ B_-\ B_+\} \subseteq B_+. \tag{3.3}$$

Observe that, by symmetry, if (B_+, B_-) is an involutive grading then so also is (B_-, B_+) which, in this case, is said to be the *opposite grading*. A linear mapping ϕ from A to itself, which is a triple automorphism of A and satisfies the condition that ϕ^2 coincides with the identity id_A, is said to be an *involutive automorphism* of A. Observe that, if ϕ is an involutive automorphism of A then so also is $-\phi$.

The first result, the proof of which is a routine calculation, describes the connection between involutive gradings and involutive automorphisms.

Lemma 3.1. *Let A be a Jordan*-triple, let ϕ be an involutive automorphism of A, and let*

$$B_+^\phi = \{a \in A : \phi a = a\} \qquad B_-^\phi = \{a \in A : \phi a = -a\}. \tag{3.4}$$

Then, (B_+^ϕ, B_-^ϕ) is an involutive grading and the mapping $\phi \mapsto (B_+^\phi, B_-^\phi)$ is a bijection from the set of involutive automorphisms of A onto the set of involutive gradings of A such that $(B_+^{-\phi}, B_-^{-\phi})$ coincides with (B_-^ϕ, B_+^ϕ).

It is clear that, for an involutive automorphism ϕ of the Jordan*-triple A, with corresponding involutive grading (B_+^ϕ, B_-^ϕ), the linear mapping T_ϕ, defined by

$$T_\phi = \tfrac{1}{2}(\mathrm{id}_A + \phi), \tag{3.5}$$

is the linear projection onto the subtriple B_+^ϕ and $T_{-\phi}$ is the linear projection onto the subtriple B_-^ϕ. Clearly, the projections T_ϕ and $T_{-\phi}$ are orthogonal. The next result describes their other properties.

Lemma 3.2. *Let A be a Jordan*-triple, let ϕ be an involutive automorphism of A, and let T_ϕ be the projection defined in (3.5). Then, for all elements a, b and c in A,*

(i) $T_\phi\{T_\phi a\, b\, T_\phi c\} = \{T_\phi a\, T_\phi b\, T_\phi c\}$.
(ii) $T_\phi\{T_\phi a\, T_\phi b\, c\} = \{T_\phi a\, T_\phi b\, T_\phi c\}$.

Proof. Let (B_+^ϕ, B_-^ϕ) be the involutive grading corresponding to ϕ. Then, since B_+^ϕ and B_-^ϕ are subtriples satisfying (3.2),

$$\{T_\phi a\ T_\phi b\ T_\phi c\} \subseteq \{B_+^\phi\ B_+^\phi\ B_+^\phi\} \subseteq B_+^\phi = T_\phi A,$$
$$\{T_\phi a\ T_{-\phi} b\ T_\phi c\} \subseteq \{B_+^\phi\ B_-^\phi\ B_+^\phi\} \subseteq B_-^\phi = T_{-\phi} A.$$

Hence, using the orthogonality of T_ϕ and $T_{-\phi}$,

$$\begin{aligned} T_\phi\{T_\phi a\ b\ T_\phi c\} &= T_\phi\{T_\phi a\ (T_\phi b + T_{-\phi} b)\ T_\phi c\} \\ &= T_\phi\{T_\phi a\ T_\phi b\ T_\phi c\} + T_\phi\{T_\phi a\ T_{-\phi} b\ T_\phi c\} \\ &= \{T_\phi a\ T_\phi b\ T_\phi c\} + T_\phi T_{-\phi}\{T_\phi a\ T_{-\phi} b\ T_\phi c\} \\ &= \{T_\phi a\ T_\phi b\ T_\phi c\}, \end{aligned}$$

and (i) holds. A similar proof, using (3.3), applies to (ii). □

For the case of a JB*-triple rather more can be said about involutive automorphisms.

Lemma 3.3. *Let A be a JB*-triple. A mapping ϕ from A to itself is an involutive automorphism if and only if ϕ is a linear isometry from A such that ϕ^2 and id_A coincide.*

Proof. By Kaup (1983, Proposition 5.5), ϕ is a linear triple automorphism of A if and only if ϕ is a linear isometry from A onto itself and the proof is complete. □

Recall that a linear projection T on a JB*-triple A is said to be *bicontractive* if both of the projections T and $\mathrm{id}_A - T$ are contractive. The next result relates involutive gradings on JB*-triples to bicontractive projections on A.

Lemma 3.4. *Let A be a JB*-triple, and, for each involutive automorphism ϕ on A, let (B_+^ϕ, B_-^ϕ) be the corresponding involutive grading of A, and let T_ϕ and $T_{-\phi}$ be the corresponding projections onto the subtriples B_+^ϕ and B_-^ϕ, respectively. Then the mapping $\phi \mapsto T_\phi$ is a bijection from the set of involutive automorphisms on A onto the set of bicontractive projections on A.*

Proof. By Lemma 3.3, ϕ is an isometry. It follows that

$$\|T_\phi\| = \|\tfrac{1}{2}(\mathrm{id}_A + \phi)\| \leq 1.$$

Similarly $T_{-\phi}$ is contractive, and it follows that T_ϕ is bicontractive. Conversely, from Friedman and Russo (1987, Theorem 4), if T is a bicontractive projection on A, there exists a linear isometry ϕ of order two such that

$$T = \tfrac{1}{2}(\mathrm{id}_A + \phi).$$

Using Lemma 3.3 it follows that ϕ is an involutive automorphism on A such that T and T_ϕ coincide. It is clear from above that the mapping $\phi \mapsto T_\phi$ is a bijection. □

This result allows a completely algebraic characterisation of bicontractive projections on JB*-triples to be given.

Corollary 3.5. *Let A be a JB*-triple, and let T be a linear projection on A. Then T is bicontractive if and only if, for all elements a, b and c in A,*

$$T\{Ta\ b\ Tc\} = \{Ta\ Tb\ Tc\}, \tag{3.6}$$

$$T\{Ta\ Tb\ c\} = \{Ta\ Tb\ Tc\}, \tag{3.7}$$

$$(\mathrm{id}_A - T)\{(\mathrm{id}_A - T)a\ b\ (\mathrm{id}_A - T)c\} = \{(\mathrm{id}_A - T)a\ (\mathrm{id}_A - T)b\ (\mathrm{id}_A - T)c\}, \tag{3.8}$$

$$(\mathrm{id}_A - T)\{(\mathrm{id}_A - T)a\ (\mathrm{id}_A - T)b\ c\} = \{(\mathrm{id}_A - T)a\ (\mathrm{id}_A - T)b\ (\mathrm{id}_A - T)c\}. \tag{3.9}$$

Proof. Let T be a bicontractive projection on A. Then, by Lemma 3.4, there exists an involutive automorphism ϕ such that T and T_ϕ coincide. Then,

$$\mathrm{id}_A - T = \mathrm{id}_A - T_\phi = T_{-\phi},$$

and, by Lemma 3.2, (3.6)–(3.9) hold.

Conversely, suppose that the linear projection T satisfies (3.6)–(3.9), and let B_+ and B_- be the ranges of the projections T and $\mathrm{id}_A - T$, respectively. From (3.6) and (3.8), B_+ and B_- are subtriples of A such that (3.1) holds. Let a_+ and c_+ be elements of B_+, and let b_- be an element of B_-. Then, using (3.6),

$$\begin{aligned} T\{a_+\ b_-\ c_+\} &= T\{Ta_+\ (\mathrm{id}_A - T)b_-\ Tc_+\} \\ &= \{Ta_+\ T(\mathrm{id}_A - T)b_-\ Tc_+\} = 0. \end{aligned}$$

Therefore, the element $\{a_+ b_- c_+\}$ is contained in the kernel of T which coincides with B_-, and it follows that

$$\{B_+\ B_-\ B_+\} \subseteq B_-.$$

Similarly, the other inclusions in (3.2) and (3.3) hold, and (B_+, B_-) is an involutive grading of A. By Lemma 3.1, there exists a unique involutive automorphism ϕ of A and corresponding projections T_ϕ and $T_{-\phi}$ such that

$$TA = B_+ = B_+^\phi = T_\phi A, \quad (\mathrm{id}_A - T)A = B_- = B_-^\phi = T_{-\phi}A.$$

It follows that T and T_ϕ coincide, and, by Lemma 3.4, that T is bicontractive. □

The following results show that involutive gradings have automatic topological properties for both JB*-triples and JBW*-triples.

Corollary 3.6. *Let A be a JB*-triple, and let (B_+, B_-) be an involutive grading of A. Then, the subtriples B_+ and B_- are JB*-triples.*

Proof. Since B_+ and B_- are the kernels of contractive projections, this follows from Lemma 3.4 and Corollary 3.5. □

Corollary 3.7. *Let A be a JBW*-triple. Then, the following results hold.*

(i) *For each involutive automorphism ϕ of A, with corresponding involutive grading (B_+^ϕ, B_-^ϕ), ϕ is weak*-continuous and the subtriples B_+^ϕ and B_-^ϕ are JBW*-triples.*
(ii) *Every bicontractive projection on A is weak*-continuous.*

Proof. Since A has a unique predual, every linear isometry from A onto itself is automatically weak*-continuous, and the result follows immediately from Lemma 3.3 and Lemma 3.4. □

In order to find examples of involutive gradings of Jordan*-triples it is sufficient to look at the Peirce decomposition

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

of A corresponding to a tripotent u in A. Writing

$$B_+ = A_0(u) \oplus A_2(u), \qquad B_- = A_1(u),$$

it is an easy consequence of the Peirce relations (2.4)–(2.6) that (B_+, B_-) is an involutive grading of A, with corresponding involutive automorphism ϕ given by

$$\phi = 2P_0(u) + 2P_2(u) - \mathrm{id}_A,$$

where $P_0(u)$, $P_1(u)$ and $P_2(u)$ are the Peirce projections corresponding to u. In the next section generalisations of this example will be considered.

4. PEIRCE GRADINGS

Let A be a Jordan*-triple. Using the terminology of Neher (1981), an ordered triple (J_0, J_1, J_2) of subspaces of a Jordan*-triple A is said to be a *Peirce grading* of A if

$$A = J_0 \oplus J_1 \oplus J_2, \tag{4.1}$$

$$\{J_0\ J_2\ A\} = \{J_2\ J_0\ A\} = \{0\}, \tag{4.2}$$

and, for j, k and l equal to 0, 1 or 2,

$$\{J_j\ J_k\ J_l\} \subseteq J_{j-k+l} \tag{4.3}$$

if $j-k+l$ is equal to 0, 1 or 2, and

$$\{J_j\ J_k\ J_l\} = \{0\} \tag{4.4}$$

otherwise. Observe that, if (J_0, J_1, J_2) is a Peirce grading then so also is (J_2, J_1, J_0). It is clear that when u is a tripotent in A and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is the corresponding Peirce decomposition of A, then $(A_0(u), A_1(u), A_2(u))$ is a Peirce grading of A.

The following result, the proof of which is a routine verification, relates Peirce gradings to involutive gradings.

Lemma 4.1. *Let A be a Jordan*-triple, let (J_0, J_1, J_2) be a Peirce grading of A, and let P_0, P_1, and P_2 be the linear projections onto the subspaces J_0, J_1 and J_2, respectively. Then, $(J_0 \oplus J_2, J_1)$ is an involutive grading of A, the corresponding involutive automorphism ϕ being given by*

$$\phi = 2P_0 + 2P_2 - \mathrm{id}_A = \mathrm{id}_A - 2P_1 = P_0 - P_1 + P_2,$$

and the corresponding projections T_ϕ and $T_{-\phi}$ being given by

$$T_\phi = P_0 + P_2, \quad T_{-\phi} = \mathrm{id}_A - T_\phi = P_1.$$

Recall that, for a subspace J of a Jordan*-triple A, the set of elements a in A for which $\{J\, a\, J\}$ is equal to $\{0\}$ is said to be the *kernel* of J and is denoted by Ker(J). The subspace J of A is said to be *complemented* if

$$A = J + \mathrm{Ker}(J).$$

It follows from Edwards and Rüttimann (1996a, Lemma 4.1), that a complemented subtriple of A is an inner ideal in A.

The next result describes some further properties of Peirce gradings.

Lemma 4.2. *Let A be an anisotropic Jordan*-triple and let (J_0, J_1, J_2) be a Peirce grading of A. Then, the following results hold.*

(i) *The subspaces J_0 and J_2 are inner ideals in A and the subspaces J_1 and $J_0 \oplus J_2$ are subtriples of A.*
(ii) *The subtriples J_0 and J_2 of the Jordan*-triple $J_0 \oplus J_2$ are ideals in $J_0 \oplus J_2$.*
(iii) *The inner ideals J_0 and J_2 are complemented in A and are such that*

$$\mathrm{Ker}(J_0) = J_1 \oplus J_2, \qquad \mathrm{Ker}(J_2) = J_1 \oplus J_0,$$

and

$$J_1 = \mathrm{Ker}(J_0) \cap \mathrm{Ker}(J_2).$$

Proof. The proofs of (i) and (ii) are immediate from (4.2)–(4.4). Observe that, by (4.4),

$$\{J_0\ (J_1 \oplus J_2)\ J_0\} = \{J_0\ J_1\ J_0\} + \{J_0\ J_2\ J_0\} = \{0\},$$

and it follows that $J_1 \oplus J_2$ is contained in Ker(J_0). Therefore, by (4.1),

$$A = J_0 \oplus J_1 \oplus J_2 \subseteq J_0 \oplus \mathrm{Ker}(J_0) \subseteq A,$$

and it can be seen that $J_1 \oplus J_2$ coincides with $\mathrm{Ker}(J_0)$. Similarly, $J_1 \oplus J_0$ coincides with $\mathrm{Ker}(J_2)$. Furthermore, the inner ideals J_0 and J_2 are complemented in A. From above, J_1 is contained in $\mathrm{Ker}(J_0) \cap \mathrm{Ker}(J_2)$. Let a be an element of $\mathrm{Ker}(J_0) \cap \mathrm{Ker}(J_2)$. Then, there exist elements b_0 in J_0, c_2 in J_2, and b_1 and c_1 in J_1 such that

$$a = b_0 + b_1 = c_1 + c_2,$$

from which it follows that

$$0 = b_0 + (b_1 - c_1) - c_2,$$

and, by (4.1),

$$b_0 = c_2 = 0, \qquad b_1 = c_1.$$

Therefore, a lies in J_1, and the proof of (iii) is complete. □

Recall that the *annihilator* $K^{\perp}$ of a subspace K of the Jordan*-triple A consists of elements a in A for which $\{K a A\}$ is equal to $\{0\}$. The annihilator $K^{\perp}$ is a subspace of the kernel $\mathrm{Ker}(K)$ of K and, if A is anisotropic, $K^{\perp}$ is an inner ideal in A consisting of those elements a in A for which $\{a K A\}$ is equal to $\{0\}$. In this case elements of $K^{\perp}$ are said to be *orthogonal* to those in K. For any complemented inner ideal K in the anisotropic Jordan*-triple A, the *Peirce spaces* K_0, K_1 and K_2 are defined by

$$K_0 = K^{\perp}, \quad K_1 = \mathrm{Ker}(K) \cap \mathrm{Ker}(K^{\perp}), \quad K_2 = K, \tag{4.5}$$

In the case in which K is self-compatible, or, equivalently, when the inner ideal $K^{\perp}$ is also complemented, A enjoys the *generalised Peirce decomposition*

$$A = K_0 \oplus K_1 \oplus K_2,$$

relative to K. For details, see Edwards and Rüttimann (1996b, Lemma 3.2), and Edwards et al. (1999, Lemma 3.1). In general (K_0, K_1, K_2) does not constitute a Peirce grading of A.

For a Peirce grading (J_0, J_1, J_2), Lemma 4.2 shows that both J_0 and J_2 are complemented inner ideals in A. The next result describes the relationship between their Peirce spaces and the subtriples occurring in the Peirce grading.

Lemma 4.3. *Let A be an anisotropic Jordan*-triple, let (J_0, J_1, J_2) be a Peirce grading of A, and let $(J_0)_0$, $(J_0)_1$, and $(J_0)_2$, and $(J_2)_0$, $(J_2)_1$, and $(J_2)_2$ be the Peirce spaces corresponding to the complemented inner ideals J_0 and J_2, respectively. Then:*

$$(J_0)_0 = J_2 \oplus (J_1 \cap (J_0)_0); \quad (J_0)_1 \subseteq J_1;$$
$$(J_2)_0 = J_0 \oplus (J_1 \cap (J_2)_0); \quad (J_2)_1 \subseteq J_1.$$

Proof. Let a be an element of $(J_0)_0$. Then, by (4.1), there exist elements a_0 in J_0, a_1 in J_1, and a_2 in J_2 such that

$$a = a_0 + a_1 + a_2.$$

Then, by (4.2)–(4.4),

$$0 = \{a_0\, a_0\, a_0\} + \{a_1\, a_0\, a_0\} + \{a_2\, a_0\, a_0\} \in J_0 \oplus J_1 \oplus \{0\},$$

and, therefore,

$$\{a_0\, a_0\, a_0\} = \{a_1\, a_0\, a_0\} = 0.$$

By anisotropy, a_0 is equal to zero. By (4.2), a_2 is contained in $(J_0)_0$, and it follows that a_1 is contained in $(J_0)_0$. Therefore,

$$(J_0)_0 \subseteq J_2 \oplus (J_1 \cap (J_0)_0),$$

and the opposite inclusion is trivially true. Similarly,

$$(J_2)_0 = J_0 \oplus (J_1 \cap (J_2)_0).$$

From (4.2) it can be seen that

$$(J_2)_2 = J_2 \subseteq (J_0)^{\perp} = (J_0)_0,$$

and, hence, that

$$\mathrm{Ker}((J_0)_0) \subseteq \mathrm{Ker}(J_2). \tag{4.6}$$

Furthermore, by (4.5), (4.6) and Lemma 4.2(iii),

$$\begin{aligned}(J_0)_1 &= \mathrm{Ker}((J_0)_2) \cap \mathrm{Ker}((J_0)_0) = \mathrm{Ker}(J_0) \cap \mathrm{Ker}((J_0)_0)\\ &\subseteq (J_1 \oplus J_2) \cap \mathrm{Ker}(J_2) = (J_1 \oplus J_2) \cap (J_1 \oplus J_0)\\ &= J_1,\end{aligned}$$

as required. Symmetrically, $(J_2)_1$ is also contained in J_1. □

Recall that, from Edwards and Rüttimann (1996b, Theorem 3.5), a self-compatible inner ideal K in an anisotropic Jordan*-triple A having

Peirce spaces K_0, K_1 and K_2 satisfies the *Peirce relations*, for j, k and l equal to 0, 1 or 2,

$$\{K_j\ K_k\ K_l\} \subseteq K_{j-k+l},$$

if $j-k+l$ is equal to 0, 1 or 2, and

$$\{K_j\ K_k\ K_l\} = \{0\},$$

otherwise, except for (j, k, l) equal to (0,1,1), (1, 1, 0), (1, 0, 1), (1, 2, 1), (1, 1, 2), (2, 1, 1), (1, 1, 1), (2, 1, 0), and (0, 1, 2). If K has the property that the Peirce relations hold for all these exceptional cases then K is said to be a *Peirce inner ideal* in A. It is clear that if K is a Peirce inner ideal then (K_0, K_1, K_2) is a Peirce grading of A. The next result describes conditions under which the converse assertion can be made.

Lemma 4.4. *Let A be an anisotropic Jordan*-triple, let (J_0, J_1, J_2) be a Peirce grading of A, and let $(J_2)_0$, $(J_2)_1$, and $(J_2)_2$ be the Peirce spaces corresponding to the complemented inner ideal J_2. Then, $(J_2)_1$ coincides with J_1 if and only if $(J_2)_0$ coincides with J_0 and, if this is the case, then J_2 is a Peirce inner ideal in A with Peirce spaces given by*

$$(J_2)_0 = J_0, \quad (J_2)_1 = J_1, \quad (J_2)_0 = J_0.$$

Proof. Observe that, by Lemma 4.3,

$$A = (J_2)_0 \oplus (J_2)_1 \oplus (J_2)_2 = J_0 \oplus (J_1 \cap (J_2)_0) \oplus (J_2)_1 \oplus J_2$$
$$\subseteq J_0 \oplus J_1 \oplus J_2 = A.$$

It follows that

$$(J_1 \cap (J_2)_0) + (J_2)_1 = J_1,$$

and, by Lemma 4.3, that J_1 and $(J_2)_1$ coincide if and only if $(J_2)_0$ coincides with J_0.

If the equivalent results hold then the Peirce spaces corresponding to the complemented inner ideal J_2 are given by

$$(J_2)_0 = J_0, \qquad (J_2)_1 = J_1, \qquad (J_2)_2 = J_2,$$

and, since (J_2, J_1, J_0) is a Peirce grading of A, J_2 is a Peirce inner ideal in A. □

Observe that, even under the favourable circumstances of the previous lemma, if K is a Peirce inner ideal in A with corresponding Peirce grading (K_0, K_1, K_2) then, since it is not in general true that $K^{\perp\perp}$ and K coincide, the inner ideal K_0 need not give rise to the opposite grading.

Peirce gradings of JBW*-triples have some automatic topological properties.

Lemma 4.5. *Let A be a JBW*-triple, and let (J_0, J_1, J_2) be a Peirce grading of A. Then J_0 and J_2 are weak*-closed inner ideals in A and J_1 is a weak*-closed subtriple of A.*

Proof. By Lemma 4.2, J_0 and J_2 are complemented inner ideals in A, and, by Edwards and Rüttimann (1996, Lemma 3.2), J_0 and J_2 are weak*-closed. Because of the separate weak*-continuity of the triple product, both Ker(J_0) and Ker(J_2) are weak*-closed. It follows from Lemma 4.2(iii) that J_1 is also weak*-closed. □

5. INVOLUTIVE AND PEIRCE GRADINGS

Recall that, for a JBW*-triple A, having a Peirce grading (J_0, J_1, J_2), J_0 and J_2 are weak*-closed inner ideals in A, J_1 and $J_0 \oplus J_2$ are weak*-closed subtriples of A, and J_0 and J_2 are complementary weak*-closed ideals in $J_0 \oplus J_2$. Furthermore, $(J_0 \oplus J_2, J_1)$ forms an involutive grading of A. The main result of the paper shows that, under certain circumstances, involutive gradings give rise to Peirce gradings. The next result is a major step towards that end.

Theorem 5.1. *Let A be a JBW*-triple, let (B_+, B_-) be an involutive grading of A, let J_0 be a weak*-closed ideal in the JBW*-triple B_+, and let*

$$J_1 = B_-, \qquad J_2 = (J_0)^{\perp} \cap B_+.$$

Then, the following are equivalent:

(i) *(J_0, J_1, J_2) is a Peirce grading of A.*
(ii) *J_0 and J_2 are inner ideals in A.*
(iii) *$\{J_0 J_1 J_0\} = \{J_2 J_1 J_2\} = \{0\}$.*

Proof. Observe that, by Edwards et al. (1999, Proposition 4.1 and Lemma 5.1), J_2 is also a weak*-closed ideal in B_+ and,

$$B_+ = J_0 \oplus J_2, \qquad J_0 = (J_2)^{\perp} \cap B_+.$$

It follows that the results hold for (J_0, J_1, J_2) if and only if they hold for (J_2, J_1, J_0). That (i) implies (ii) follows immediately from Lemma 4.2(iii).

If (ii) holds, then, since J_0 is an inner ideal in A and (B_+,B_-) is an involutive grading,

$$\{J_0\ J_1\ J_0\} \subseteq \{J_0\ A\ J_0\} \cap \{B_+\ B_-\ B_+\} \subseteq J_0 \cap B_- \subseteq B_+ \cap B_- = \{0\}.$$

Similarly,

$$\{J_2\ J_1\ J_2\} = \{0\},$$

and (iii) holds.

If (iii) holds then it is clear that

$$A = J_0 \oplus J_1 \oplus J_2,$$

and, by the orthogonality of J_0 and J_2,

$$\{J_0\ J_2\ A\} = \{J_2\ J_0\ A\} = \{0\}.$$

It remains to show that the Peirce relations (4.3)–(4.4) hold. Observe that, by the properties of involutive gradings, the orthogonality of J_0 and J_2, or by hypothesis, twenty-three of the twenty-seven relations hold immediately. It remains to show that

$$\{J_0\ J_1\ J_1\} \subseteq J_0, \quad \{J_2\ J_1\ J_1\} \subseteq J_2, \quad \{J_1\ J_0\ J_1\} \subseteq J_2, \quad \{J_1\ J_2\ J_1\} \subseteq J_0,$$

and, by symmetry, it is sufficient to prove the first and third. Let a_0 be an element of J_0, and let b_1 and c_1 be elements of J_1. Since J_0 is a weak*-closed subtriple of A, by spectral theory (Kaup, 1983), there exists an element d_0 in J_0 such that

$$a_0 = \{d_0\ d_0\ d_0\}.$$

Therefore, using (2.1) and (3.3),

$$\begin{aligned}\{a_0\ b_1\ c_1\} &= \{\{d_0\ d_0\ d_0\}\ b_1\ c_1\}\\ &= \{d_0\ d_0\ \{c_1\ b_1\ d_0\}\} - \{d_0\ \{d_0\ c_1\ b_1\}\ d_0\} + \{\{c_1\ b_1\ d_0\}\ d_0\ d_0\}\\ &\in \{J_0\ J_0\ \{B_-\ B_-\ B_+\}\} + \{J_0\ \{B_+\ B_-\ B_-\}\ J_0\}\\ &\quad + \{\{B_-\ B_-\ B_+\}\ J_0\ J_0\}\\ &\subseteq \{J_0\ J_0\ B_+\} + \{J_0\ B_+\ J_0\} + \{B_+\ J_0\ J_0\}\\ &\subseteq J_0, \end{aligned} \tag{5.1}$$

since J_0 is an ideal in B_+.

In order to prove the final inclusion, observe that, by (3.2),

$$\{J_1\ J_0\ J_1\} \subseteq \{B_-\ B_+\ B_-\} \subseteq B_+. \tag{5.2}$$

Let a_1 and b_1 be elements of J_1 and let c_0, d_0 and e_0 be elements of J_0. Then, using (2.1),

$$\begin{aligned}\{d_0\ \{a_1\ c_0\ b_1\}\ e_0\} &= \{\{c_0\ a_1\ d_0\}\ b_1\ e_0\} + \{d_0\ b_1\ \{c_0\ a_1\ e_0\}\}\\ &\quad - \{c_0\ a_1\ \{d_0\ b_1\ e_0\}\}\\ &\in \{\{J_0\ J_1\ J_0\}\ J_1\ J_0\} + \{J_0\ J_1\ \{J_0\ J_1\ J_0\}\}\\ &\quad + \{J_0\ J_1\ \{J_0\ J_1\ J_0\}\}\\ &= \{0\},\end{aligned}$$

by hypothesis. Using (5.2), it follows that

$$\{J_1\ J_0\ J_1\} \subseteq \mathrm{Ker}(J_0) \cap B_+. \tag{5.3}$$

Since J_0 is a weak*-closed ideal in B_+, it follows from Edwards et al. (1999, Proposition 4.1), that the kernel of J_0 in B_+ coincides with its annihilator in B_+ and, hence, from (5.3),

$$\{J_1\ J_0\ J_1\} \subseteq J_2,$$

as required. □

In order to prove the main result of the paper some further preliminary results are required. The first result concerns the relationship between involutive gradings and the Peirce spaces of tripotents.

Lemma 5.2. *Let A be a Jordan*-triple, let ϕ be an involutive automorphism of A, with corresponding involutive grading (B_+^ϕ, B_-^ϕ), and let u be a tripotent in B_+^ϕ with Peirce projections $P_0(u)$, $P_1(u)$, and $P_2(u)$. Then, for j equal to 0, 1 and 2,*

$$P_j(u)\phi = \phi P_j(u).$$

Proof. For each element a in A,

$$\phi D(u,u)a = \phi\{u\ u\ a\} = \{\phi u\ \phi u\ \phi a\} = \{u\ u\ \phi a\} = D(u,u)\phi a,$$

and

$$\phi Q(u)a = \phi\{u\ a\ u\} = \{\phi u\ \phi a\ \phi u\} = \{u\ \phi a\ u\} = Q(u)\phi a.$$

Since ϕ commutes with both $D(u,u)$ and $Q(u)$, from the definition of the Peirce projections (2.2), the result follows. □

The second result concerning pairs of orthogonal tripotents can be found in Meyberg (1972), Loos (1975), and McCrimmon (1979), but, since its proof relates to the proof of the main theorem, for completeness, it is reproduced here.

Lemma 5.3. *Let A be a JBW*-triple, and let u and v be orthogonal tripotents in A, having associated Peirce spaces $A_0(u)$, $A_1(u)$ and $A_2(u)$, and $A_0(v)$, $A_1(v)$ and $A_2(v)$, respectively. Then the following results hold.*

(i) *The element $u+v$ in A is a tripotent such that*

$$A_0(u+v) = A_0(u) \cap A_0(v),$$
$$A_1(u+v) = (A_0(u) \cap A_1(v)) \oplus (A_1(u) \cap A_0(v)),$$
$$A_2(u+v) = A_2(u) \oplus (A_1(u) \cap A_1(v)) \oplus A_2(v).$$

(ii) *In the weak*-closed inner ideal $A_2(u+v)$ of A, the Peirce spaces corresponding to the tripotent u are given by*

$$(A_2(u+v))_0(u) = A_2(v), \quad (A_2(u+v))_1(u) = A_1(u) \cap A_1(v),$$
$$(A_2(u+v))_2(u) = A_2(u).$$

(iii) *The following are equivalent:*

(a) $A_2(u+v) = A_2(u) + A_2(v)$;
(b) $A_1(u) \cap A_1(v) = \{0\}$;
(c) *$A_2(u)$ is an ideal in $A_2(u+v)$.*

Proof. Since u and v are orthogonal, it is clear that $u+v$ is a tripotent such that

$$D(u+v, u+v) = D(u,u) + D(v,v). \tag{5.4}$$

By McCrimmon (1979a, Corollary 1.8), the tripotents u and v are compatible and their Peirce projections commute. Furthermore, for j equal to 0, 1 or 2, the Peirce spaces $A_j(u+v)$ are the eigenspaces of $D(u+v,\ u+v)$ corresponding to the eigenvalues $\frac{j}{2}$. The first statement now follows immediately from (5.4). When the linear operator $D(u,\ u)$ is restricted to the JBW*-triple $A_2(u+v)$ it is clear that its eigenspaces relative to the eigenvalues 0, $\frac{1}{2}$ and 1 are given by $A_2(v)$, $A_1(u) \cap A_1(v)$, and $A_2(u)$, respectively, and the proof of (ii) is complete. The proof of (iii) is immediate from (ii) and Edwards et al. (1999, Proposition 4.1). □

Let A be a JBW*-triple. A linear projection S on A is said to be an *M-projection* if, for each element a in A,

$$\|a\| = \max\{\|Sa\|, \|a - Sa\|\}.$$

A closed subspace which is the range of an M-projection is said to be an *M-summand* of A, and A is said to be equal to the *M-sum*

$$A = SA \oplus_M (\mathrm{id}_A - S)A$$

of the M-summands SA and $(\mathrm{id}_A - S)A$. The results of Horn (1987) show that the set M-summands in A coincides with the set of its weak*-closed ideals. Furthermore, for each weak*-closed ideal I in A the complementary M-summand is the annihilator $I^\perp$ of I.

Let J be a weak*-closed inner ideal in a weak*-closed subtriple B of a JBW*-triple A. Then the *central kernel* $k_B(J)$ of J in B is the largest weak*-closed ideal of B that is contained in J. For the properties of central kernels the reader is referred to Edwards and Rüttimann (2003, to appear). A purely algebraic proof of a result closely related to the following one can be produced by applying the results of McCrimmon (1979b).

Lemma 5.4. *Under the conditions of Lemma 5.3, the central kernel $k_{A_2(u+v)}(A_2(u))$ of the weak*-closed inner ideal $A_2(u)$ in the weak*-closed inner ideal $A_2(u+v)$ of A is the set of elements a in $A_2(u)$ for which*

$$\{a\ u\ (A_1(u) \cap A_1(v))\} = \{0\}.$$

Proof. Let I be the set of elements a in $A_2(u)$ for which

$$\{a\ u\ (A_1(u) \cap A_1(v))\} = \{0\}. \tag{5.5}$$

By Lemma 5.3(ii), I is the set of elements a in $A_2(u)$ for which

$$\{a\ u\ (A_2(u+v))_1(u)\} = \{0\}, \tag{5.6}$$

which, by Edwards and Rüttimann (2002, Corollary 3.8), contains the central kernel $k_{A_2(u+v)}(A_2(u))$ of the weak*-closed inner ideal $A_2(u)$ in the JBW*-triple $A_2(u+v)$. In order to complete the proof it remains to show that I is an ideal in the JBW*-triple $A_2(u+v)$. Since the set of weak*-closed ideals in a JBW*-triple and the set of weak*-closed ideals in a JBW*-algebra both coincide with the set of M-summands, using

(2.7), it suffices to show that I is an ideal in the JBW*-algebra $A_2(u+v)$. Therefore, it is required to show that

$$\{I\ u+v\ A_2(u+v)\} \subseteq I.$$

Since u and v are orthogonal, v is contained in $A_0(u)$, and, hence, using (2.2),

$$\{I\ v\ A_2(u+v)\} \subseteq \{A_2(u)\ A_0(u)\ A\} = \{0\}.$$

Therefore, it remains to show that

$$\{I\ u\ A_2(u+v)\} \subseteq I. \tag{5.7}$$

However, by Lemma 5.3(i), (5.5), and the orthogonality of u and v,

$$\begin{aligned}\{I\ u\ A_2(u+v)\} &= \{I\ u\ (A_2(u) \oplus (A_1(u)\cap A_1(v)) \oplus A_2(v))\}\\ &= \{I\ u\ A_2(u)\} + \{I\ u\ (A_1(u)\cap A_1(v))\} + \{I\ u\ A_2(v)\}\\ &= \{I\ u\ A_2(u)\} + \{0\} + \{0\}.\end{aligned} \tag{5.8}$$

From (5.7) and (5.8), it remains to show that

$$\{I\ u\ A_2(u)\} \subseteq I. \tag{5.9}$$

To this end, let a lie in I, b lie in $A_2(u)$ and let c lie in $(A_2(u+v))_1(u)$. Then, using (2.1), (2.3), (2.4), Lemma 5.3(ii), and (5.6),

$$\begin{aligned}&\{\{a\ u\ b\}\ u\ c\}\\ &\quad= \{\{c\ u\ a\}\ u\ b\} - \{a\ \{u\ c\ u\}\ b\} + \{a\ u\ \{c\ u\ b\}\}\\ &\quad\in \{\{I\ u\ (A_2(u+v))_1(u)\}\ u\ b\}\\ &\qquad + \{a\ \{(A_2(u+v))_2(u)\ (A_2(u+v))_1(u)\ (A_2(u+v))_2(u)\}\ c\}\\ &\qquad + \{a\ u\ \{(A_2(u+v))_1(u)\ (A_2(u+v))_2(u)\ (A_2(u+v))_2(u)\}\}\\ &\quad\subseteq \{0\} + \{0\} + \{a\ u\ (A_2(u+v))_1(u)\} = \{0\},\end{aligned}$$

and it follows that $\{a\,u\,b\}$ lies in I. Therefore, (5.9) holds, and the proof is complete. □

Recall that a JBW*-triple A is said to be a *JBW*-triple factor* if the only weak*-closed ideals in A are $\{0\}$ and A. Let (B_+, B_-) be an involutive grading of A and suppose that J_0 is a weak*-closed ideal in B_+. Then, writing

$$J_1 = B_-, \qquad J_2 = (J_0)^\perp \cap B_+$$

the main question to be answered is when (J_0, J_1, J_2) is a Peirce grading of A. Observe that, if J_0 or J_2 is equal to $\{0\}$ then this is always the case, and if J_1 is equal to $\{0\}$ then, since A is a JBW*-triple factor, J_0 or J_2 is also equal to $\{0\}$. It is now possible to give the proof of the main result.

Theorem 5.5. *Let A be a JBW*-triple factor and let (B_+, B_-) be an involutive grading of A with both B_+ and B_- non-zero. Then, either, the weak*-closed subtriple B_+ of A is a JBW*-triple factor, or there exists a non-zero proper weak*-closed ideal J_0 in B_+ such that, if*

$$J_1 = B_-, \qquad J_2 = (J_0)^\perp \cap B_+$$

then (J_0, J_1, J_2) is a Peirce grading of A, and J_0 and J_2 are weak-closed JBW*-triple factors.*

Proof. Suppose that B_+ is not a JBW*-triple factor, and let J_0 be a non-zero proper weak*-closed ideal in B_+. Defining J_1 and J_2 as in the statement of the theorem, by Theorem 5.1, in order to prove that (J_0, J_1, J_2) is a Peirce grading, it is sufficient to show that J_0 and J_2 are inner ideals in A. If not, let u be a non-zero tripotent in J_0, and let v be a non-zero tripotent in J_2. Then u and v form an orthogonal pair. Using the results of Sec. 3, let ϕ be the involutive automorphism corresponding to the involutive grading (B_+, B_-) and let T_ϕ be the corresponding weak*-continuous bicontractive projection onto $J_0 \oplus J_2$. It follows from McCrimmon (1979a, Corollary 1.8), (3.5), and Lemma 5.2 that T_ϕ, $T_{-\phi}$, $P_0(u)$, $P_1(u)$, $P_2(u)$, $P_0(v)$, $P_1(v)$, and $P_2(v)$ form a commutative family of weak*-continuous projections.

Since u is contained in J_0 and v is contained in J_2, it follows that

$$(J_0)^\perp \subseteq \{u\}^\perp = A_0(u), \quad (J_2)^\perp \subseteq \{v\}^\perp = A_0(v),$$

and, hence, that

$$J_2 = (J_0)^\perp \cap B_+ \subseteq A_0(u), \quad J_0 = (J_2)^\perp \cap B_+ \subseteq A_0(v). \tag{5.10}$$

Therefore, using (5.10), the compatibility of u and v, and Lemma 5.3,

$$\begin{aligned}A_1(u)\cap A_1(v)\cap B_+ \subseteq B_+ = J_0 \oplus J_2 &\subseteq A_0(u) + A_0(v)\\ &= A_2(v) \oplus (A_0(u)\cap A_1(v)) \oplus (A_0(u)\cap A_0(v))\\ &\quad \oplus (A_1(u)\cap A_0(v)) \oplus A_2(u).\end{aligned} \tag{5.11}$$

Let a be an element of $A_1(u)\cap A_1(v)\cap B_+$. Then, from (5.11), for j and k equal to 0, 1 and 2, there exist elements a_{jk} in $A_j(u)\cap A_k(v)$ such that

$$a = a_{02} + a_{01} + a_{00} + a_{10} + a_{20}.$$

It follows that,

$$a = P_1(u)P_1(v)T_\phi a = T_\phi P_1(u)P_1(v)a = 0.$$

Therefore, $A_1(u)\cap A_1(v)\cap B_+$ is equal to $\{0\}$, and, again using the commutation properties of T_ϕ and the corresponding Peirce projections, and Lemma 5.3,

$$A_1(u)\cap A_1(v) = (A_2(u+v))_1(u) \subseteq B_-.$$

Therefore, using (2.1) and (3.3),

$$\begin{aligned}&\{(A_2(u)\cap B_-)\ u\ (A_1(u)\cap A_1(v))\}\\ &\quad\subseteq \{B_-\ B_+\ B_-\}\cap\{(A_2(u+v))_2(u)\ (A_2(u+v))_2(u)\ (A_2(u+v))_1(u)\}\\ &\quad\subseteq B_+\cap (A_2(u+v))_1(u) = A_1(u)\cap A_1(v)\cap B_+ = \{0\}.\end{aligned}$$

It therefore follows from Lemma 5.4 that

$$A_2(u)\cap B_- \subseteq k_{A_2(u+v)}(A_2(u)), \tag{5.12}$$

which is a weak*-closed ideal in the weak*-closed inner ideal $A_2(u+v)$ of A. By Horn and Neher (1988, Lemma 3.2), or Edwards and Rüttimann (2001, Corollary 3.6), there exists a weak*-closed ideal K in A such that

$$k_{A_2(u+v)}(A_2(u)) = K\cap A_2(u+v).$$

Since A is a factor, K is equal either to A or to $\{0\}$. If the former holds then

$$A_2(u+v) = K\cap A_2(u+v) = k_{A_2(u+v)}(A_2(u)) \subseteq A_2(u) \subseteq A_2(u+v),$$

and it follows from Lemma 5.3 that $A_2(v)$ is equal to $\{0\}$. In this case v is equal to zero, giving a contradiction. It follows that $k_{A_2(u+v)}(A_2(u))$ is equal to zero, and, hence that $A_2(u)\cap B_-$ is equal to $\{0\}$. Again using the commutativity of T_ϕ and the Peirce projections of u it can be seen that $A_2(u)$ is contained in B_+. Then, since J_0 is a weak*-closed ideal in B_+, using Bunce and Chu (1992, Lemma 3.1), for every element a in $A_2(u)$,

$$a = \{u\ u\ a\} \in \{J_0\ J_0\ B_+\} \subseteq J_0.$$

It follows that, for each non-zero tripotent u in J_0, $A_2(u)$ is contained in J_0, which, by Edwards et al. (1996, Lemma 2.1(ii)), shows that J_0 is an inner ideal in A. By symmetry, J_2 is also an inner ideal in A, and, by Theorem 5.1, the proof that (J_0, J_1, J_2) is a Peirce grading is complete.

Suppose that I is a weak*-closed ideal in J_0. Since J_0 is a weak*-closed inner ideal in A, it follows from Edwards and Rüttimann (2001, Corollary 3.6), that there exists a weak*-closed ideal in $\tilde{I}$ in A such that $\tilde{I}\cap J_0$ coincides with I. However, since A is a factor, $\tilde{I}$ is equal to $\{0\}$ or A, which implies that I is equal to $\{0\}$ or J_0. Hence, J_0, and similarly J_2 are JBW*-triple factors, as required. □

The authors are grateful to the referee for suggesting that, in the result above, J_0 and J_2 might themselves be JBW*-triple factors. By symmetry, the result above also holds with the rôles of B_+ and B_- interchanged.

Before proceeding to study applications of the results, it should be remarked that special cases of both involutive and Peirce gradings occur. For example, for each weak*-closed ideal I in the JBW*-triple A, by choosing B_+ equal to I and B_- equal to the annihilator $I^\perp$ of I, (B_+, B_-) forms an involutive grading, the corresponding involutive automorphism ϕ being given by

$$\phi = 2P_2(I) - \mathrm{id}_A,$$

where $P_2(I)$ is the structural M-projection on A with range I. For a Peirce grading (J_0, J_1, J_2) of A, two obvious special cases arise. The first occurs when J_1 is equal to $\{0\}$ in which case, writing B_+ for J_0 and B_- for J_2, (B_+, B_-) is the involutive grading described above. The second occurs when J_2, or, symmetrically, J_0, is equal to $\{0\}$. In this case, writing B_+ for J_0 and B_- for J_1, (B_+, B_-) is an involutive grading of A, the corresponding involutive automorphism ϕ of A being given by

$$\phi = 2P_2(J_0) - \mathrm{id}_A,$$

where $P_2(J_0)$ is the structural projection onto the weak*-closed inner ideal J_0 in A, which will not, in general, be an M-projection.

6. W*-ALGEBRAS

An example of a JBW*-triple is a W*-algebra A endowed with the triple product defined, for elements a, b, and c of A, by

$$\{a\ b\ c\} = \tfrac{1}{2}(ab^*c + cb^*a).$$

For the properties of W*-algebras, the reader is referred to Pedersen (1979) and Alfsen and Shultz (2001). Recall that the set $\mathscr{P}(A)$ of self-adjoint idempotents, or *projections*, in the W*-algebra A forms a complete orthomodular lattice, which is order isomorphic to the set $\mathscr{S}(A)$ of self-adjoint elements s of A such s^2 is equal to the unit 1 in A, or *symmetries*, the order isomorphism being given by $e \mapsto 2e-1$. The family $\mathscr{Z}\mathscr{P}(A)$ of projections in the centre of A forms a complete Boolean lattice which is order isomorphic to the complete Boolean lattice of weak*-closed ideals in A, the order isomorphism being given by $z \mapsto zA$. Observe that this complete Boolean lattice coincides with $\mathscr{Z}\mathscr{I}(A)$ the family of weak*-closed triple ideals in A. For each element e in $\mathscr{P}(A)$ the *central support* $c(e)$ of e is the smallest element of $\mathscr{Z}\mathscr{P}(A)$ majorizing e. Before proceeding to apply the results of Sec. 5 to this example the following lemma is required.

Lemma 6.1. *Let A be a W*-algebra and let $e_1, e_2, \ldots, e_n$ be projections in A, with central supports $c(e_1), c(e_2), \ldots, c(e_n)$, respectively, such that*

$$e_1Ae_2A\cdots e_{n-1}Ae_n = \{0\}.$$

Then,

$$c(e_1)c(e_2)\cdots c(e_n) = 0.$$

Proof. Let the weak*-closed subspace I_1 of A be defined by

$$I_1 = \{a \in A : e_1Ae_2A\cdots Ae_{n-1}Aa = \{0\}\}.$$

Clearly, for each element a in I_1 and each element b in A, the elements ab and ba lie in I_1, which is, therefore, a weak*-closed ideal in A containing e_n. It therefore contains the smallest weak*-closed ideal $c(e_n)A$ in A containing e_n, and, consequently,

$$c(e_n)e_1Ae_2A\cdots Ae_{n-1}A = \{0\}.$$

In particular,

$$c(e_n)e_1Ae_2A\cdots Ae_{n-1} = \{0\}.$$

Proceeding inductively, it can be seen that

$$c(e_2)c(e_3)\cdots c(e_{n-1})e_1A = \{0\}.$$

Let I be the weak*-closed ideal given by

$$I = \{a \in A : c(e_2)c(e_3)\cdots c(e_n)aA = \{0\}\}.$$

Then, as before, e_1 lies in I, and, hence, I contains the weak*-closed ideal $c(e_1)A$, thereby completing the proof. □

Observe that, in this example, the results of the previous section reduce to the following lemma.

Lemma 6.2. *Let A be a W*-factor and let (B_+, B_-) be an involutive grading of A. Then, one of the following occurs.*

(i) *B_+ and B_- are both JBW*-triple factors.*
(ii) *If B_+ or B_- is not a JBW*-triple factor then there exist non-zero JBW*-triple factors J_0 and J_2 which are weak*-closed inner ideals in A and complementary weak*-closed ideals in the JBW*-triple B_+ or B_-, respectively, such that, writing J_1 for B_- or B_+, respectively, (J_0, J_1, J_2) is a Peirce grading of A.*

Since, in this paper, interest is centered upon the situations in which involutive gradings give rise to non-trivial Peirce gradings, the cases to be considered are those which fall under (ii) of Lemma 6.2. The result below shows that very much more can be said in this special case. For each projection e in the W*-algebra A, the complementary projection $1-e$ will be written e'.

Theorem 6.3. *Let A be a W*-factor, let (B_+, B_-) be an involutive grading of A such that B_+ is not a JBW*-triple factor, and let ϕ be the corresponding involutive automorphism of A. Then, the following results hold.*

(i) *There exist unique symmetries s and t in A such, that for all elements a in A,*

$$\phi(a) = sat.$$

(ii) *There exists a Peirce grading (J_0, J_1, J_2) of A, unique up to the interchange of J_0 and J_2, given by*

$$J_0 = eAf, \quad J_1 = eAf' + e'Af, \quad J_2 = e'Af',$$

where the projections e and f in A are defined by

$$e = \tfrac{1}{2}(1+s), \qquad f = \tfrac{1}{2}(1+t),$$

such that

$$B_+ = J_0 \oplus J_2, \qquad B_- = J_1.$$

(iii) *The JBW*-triple B_- is not a JBW*-triple factor, and there exists a Peirce grading (H_0, H_1, H_2) of A, unique up to the interchange of H_0 and H_2, given by*

$$H_0 = eAf', \quad H_1 = eAf + e'Af', \quad H_2 = e'Af,$$

such that

$$B_+ = H_1, \qquad B_- = H_0 \oplus H_2,$$

the corresponding involutive automorphism being that opposite to ϕ.

Proof. Observe that if B_- is zero then A possesses non-trivial weak*-closed M-summands and is not a W*-factor. Hence B_- is non-zero. By Lemma 6.2(ii) and Lemma 4.5, there exist weak*-closed inner ideals J_0 and J_2 in A such that

$$B_+ = J_0 \oplus J_2, \qquad B_- = J_1.$$

Therefore, by Edwards and Rüttimann (1989, Theorem 3.16), there exist unique projections e, f, g and h in A, none of which is equal to 0 or 1, such that

$$J_0 = eAf, \qquad J_2 = gAh.$$

Since J_2 is contained in $(J_0)^{\perp}$, by Edwards and Rüttimann (1998, Lemma 4.3),

$$e \le g', \qquad f \le h', \tag{6.1}$$

and, by Edwards and Rüttimann (1998, Lemma 3.2), the weak*-closed inner ideals J_0 and J_2 are compatible. Observe that, by Lemma 4.3 and Edwards and Rüttimann (1996b, Lemma 5.1),

$$eAh \subseteq eAf' \subseteq (J_0)_1 \subseteq J_1, \tag{6.2}$$

$$eAf'h' \subseteq eAf' \subseteq (J_0)_1 \subseteq J_1, \tag{6.3}$$

and, since (J_0, J_1, J_2) is a Peirce grading,

$$\{J_2\ J_1\ J_1\} \subseteq J_2. \tag{6.4}$$

It follows from (6.2), (6.3) and (6.4) that

$$\begin{aligned}(gAh)(hAe)(eAf'h') &= (gAh)(hAe)(eAf'h') + (eAf'h')(hAe)(gAh) \\ &\subseteq \{J_2\ J_1\ J_1\} \subseteq J_2.\end{aligned} \tag{6.5}$$

However, again using Lemma 4.3,

$$(gAh)(hAe)(eAf'h') \subseteq gAh' \subseteq (J_2)_1 \subseteq J_1, \tag{6.6}$$

and it follows from (6.5) and (6.6) that

$$gAhAeAf'h' = (gAh)(hAe)(eAf'h') = \{0\}.$$

Applying Lemma 6.1,

$$c(g)c(h)c(e)c(f'h') = 0.$$

However, since A is a factor and g, h and e are non-zero, it can be seen that

$$c(g) = c(h) = c(e) = 1,$$

which implies that $c(f'h')$ is equal to zero, and, hence, that $f'h'$ is equal to zero. Combining this result with (6.1) it follows that h and f' coincide. Similarly g and e' coincide, and, therefore,

$$J_0 = eAf, \qquad J_2 = e'Af'.$$

Furthermore, again using Lemma 4.3 and Edwards and Rüttimann (1996b, Lemma 5.1),

$$\begin{aligned}A &= eAf + e'Af' + eAf' + e'Af \\ &= J_0 + J_2 + (J_0)_1 \subseteq J_0 + J_2 + J_1 \subseteq A,\end{aligned}$$

and

$$J_1 = eAf' + e'Af,$$

as required. Hence,

$$B_+ = eAf + e'Af', \qquad B_- = eAf' + e'Af,$$

and, using Lemma 3.3, the projection T_ϕ onto B_+ is defined, for all elements a in A, by

$$T_\phi a = eaf + e'af'.$$

Therefore, using (3.5), for all elements a in A,

$$\begin{aligned}\phi(a) &= 2T_\phi a - a = 2eaf + 2e'af' - a \\ &= (2e-1)a(2f-1) = sat,\end{aligned}$$

where s and t are the symmetries given by

$$s = 2e - 1, \qquad t = 2f - 1.$$

This completes the proof of (i) and (ii). Defining H_0, H_1 and H_2 as in the statement of the theorem, it is clear that (H_0, H_1, H_2) is a Peirce grading satisfying the conditions required to complete the proof of (iii). □

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