Differential and Integral Equations Volume 14, Number 5, May 2001, Pages 559–576

UNBOUNDEDNESS OF SOLUTIONS AND INSTABILITY OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH DELAYED ARGUMENT

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(Submitted by: Roger Nussbaum)

Abstract. It is well known that, for $\varepsilon = 0$, all solutions of the equations

$$\begin{cases} x''(t) + p(t)x(t-\varepsilon) = 0, & t \in [1,+\infty), \\ x''(t) + p(t)x(t) + \frac{1}{t^\alpha}x\left(t - \frac{\varepsilon}{t^\beta}\right) = 0, & t \in [1,+\infty), \end{cases} \qquad p(t) \geq c > 0,$$

are bounded on $[1,+\infty)$ and even tend to zero, as $p(t) \underset{t\to+\infty}{\longrightarrow} +\infty$. Here we obtain the following results: 1) for each positive ε there exist unbounded solutions of the first equation; 2) for each positive ε there exist unbounded solutions of the second equation in the case when $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \leq 1$ and $p(t)$ is bounded; 3) all solutions of the equation

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [1,+\infty),$$

with positive nondecreasing and bounded on $[1,+\infty)$ coefficient $p(t)$ are bounded if and only if $\int_1^{+\infty} \tau(t)\,dt < \infty$.

1. Preliminaries

An ordinary second-order equation

$$x''(t) + p(t)x(t) = 0, \quad t \in [0,+\infty), \tag{1.1}$$

with a positive coefficient $p(t)$, is one of the classical objects in a qualitative theory of linear differential equations. In spite of the quite simple shape of this equation it appears to provide a variety of different oscillatory and asymptotic properties to its solutions. Asymptotic properties of the solutions have been studied in classical monographs by R. Bellman [1], G. Sansone [2] and P. Hartman [3]. A number of new results on asymptotic properties of solutions to ordinary differential equations have been obtained in the

[1]This research was supported in part by the Ministry of Science and the Ministry of Absorption of the State of Israel.
Accepted for publication December 1999.
AMS Subject Classifications: 34K15.

recent monograph by I.T. Kiguradze and T.A. Chanturia [4]. It should be mentioned that this monograph states the current situation in the subject and at the same time encourages further investigation. One of the most important trends is generalization of results for equations with a deviation argument. The equation of second order with delayed argument,

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \tag{1.2}$$
$$x(\xi) = \varphi(\xi) \quad for\ \xi < 0,$$

has its own history. Oscillation and asymptotic properties of this equation were considered in the well-known monographs by A.D. Myshkis [5]; S.B. Norkin [6]; G.S. Ladde; V. Lakshmikantham and B.G. Zhang [7]; I.Gyori and G. Ladas [8]; and L.N. Erbe, Q. Kong and B.G. Zhang [9].

Our approach to investigation of asymptotic behavior is based on analysis of oscillation properties of solutions. Note the paper by N.V. Azbelev [10] in which the space of solutions of the equation

$$x''(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \quad p_i(t) \geqslant 0, \quad t \in [0, +\infty), \tag{1.3}$$
$$x(\xi) = 0 \text{ for } \xi < 0 \tag{1.4}$$

is shown to be two-dimensional and a Wronskian $W(t)$ of a certain fundamental system is considered. Nonvanishing of $W(t)$ on the semiaxis $[0, +\infty)$ is equivalent to validity of Sturm's separation theorem (between two adjoint zeros of every solution there is one and only one zero of each other solution). Nonvanishing of the Wronskian was obtained in [10] due to the "smallness" of delays (see the H-condition, following below). In the paper by S.M. Labovskii [11] it is proved that $W(t) \neq 0$ for $t \in [0, +\infty)$ if the following conditions hold: $m = 1$, and $h_1(t) \equiv t - \tau_1(t)$ does not decrease. In the paper by A. Domoshnitsky [12] of 1993 nonvanishing of the Wronskian was obtained through several other conditions, basic of them being the "smallness" of the difference of delays $\tau_i - \tau_j$, where $i, j = 1, \ldots, m$. In the present paper tests of the Wronskian increase are obtained and a correlation between growth of the Wronskian and existence of unbounded solutions is established.

In our approach estimates of the distance between adjoint zeros of oscillating solutions are essential. This distance for solutions of delay equations was estimated in works by N.V. Azbelev [10], Yu. Domshlak [34], S.V.Eliason [41], A.D. Myshkis [5] and S.B. Norkin [6]. Assertions on solution comparison with some test functions (comparison theorems) were obtained in the

following works: [5, 6, 10]. Note also a recent paper by L. Berezansky and E. Braverman [13] in which several oscillation results known for ordinary second order equations were extended to delay equations.

The delay equation is generally known to inherit oscillation properties of the corresponding ordinary equation. For example, it was proved by J.J.A.M. Brands [14] that for each bounded delay $\tau(t)$ equation (1.2) is oscillatory if and only if the corresponding ordinary differential equation (1.1) is oscillatory. Thus, the following question arises: are the asymptotic properties of an ordinary differential equation inherited by its delay equation? The answer is negative. A.D. Myshkis [5] proved that there exists an unbounded solution of the equation

$$x''(t) + px(t - \varepsilon) = 0, \quad t \in [0, +\infty),$$

for each couple of positive constants p and ε. The problem of solutions unboundedness in case of nonconstant coefficients was formulated in [5] as one to be solved. The first results in this subject were obtained by A. Domoshnitsky in [15]: if there exists a positive constant ε such that $\tau_i(t) > \varepsilon$, then there exists an unbounded solution to equation (1.3), (1.4). In the recent paper by Yu. Dolgii and S.G. Nikolaev [16] the following system of delay equations on the whole axis t was considered: $y''(t) + P(t)y(t - \omega) = 0$, $t \in (-\infty, +\infty)$, where $y : R \to R^n$, $\omega > 0$ and $P(t)$ is an ω-periodic symmetric matrix function.Using the monodromy operator (see the monograph by J. Hale and S. Lunel [17]), to be a fundamental in the theory of periodic systems, the authors obtained instability of this system in the following case: $\det P_\omega \neq 0$, where $P_\omega = \frac{1}{\omega}\int_{-\omega}^{0} P(t)\,dt$.

In the monograph by S.B. Norkin [6] the following boundary value problem on the semiaxis is considered:

$$x''(t) + \lambda x(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \quad x(0)\cos\alpha + x'(0)\sin\alpha = 0,$$
$$x(t - \tau(t)) = x(0)\varphi(t - \tau(t)) \quad \text{for} \quad t - \tau(t) < 0, \quad \sup_{t \in [0, +\infty)} |x(t)| < \infty,$$

where $\varphi(t)$ is a continuous, bounded function on the initial set $(-\infty, 0)$ such that $\varphi(0) = 1$, λ is a real parameter, and α is a real number.

If $|p(t)|$ is a summable function on the semiaxis, then every positive parameter λ is an eigenvalue of this problem [6]. We can interpret this result as the one concerning solutions boundedness of delay equations.

Results on the boundedness of a delay equations solution in which the "smallness" of the coefficient $p(t)$ is combined with the "smallness" of the delay $\tau(t)$ were obtained by D.V. Izumova [18]. The asymptotic formula of solutions of a second-order equation with a summable delay $\tau(t)$ was obtained by M. Pinto [19].

Note also that investigation of the equation $x''(t)+p(t)x(t-\tau(t))=0$, with nonpositive coefficient $p(t)$, was started by G.A. Kamenskii [20, 21]. Assertions on existence of bounded solutions, their uniqueness and oscillation were obtained in the monograph by G.S. Ladde, V. Lakshmikantham and B. Zhang [7, pages 130–139]. Several possible types of solutions' behavior of this equation in case $p(t)$ and $\tau(t)$ are bounded functions on the semiaxis and $\int_0^\infty |p(t)|\,dt=\infty$, can be only as following [20, 21]:

a) $|x(t)|\to\infty$ for $t\to\infty$; b) $x(t)$ oscillates; c) $x(t)\to 0$, $x'(t)\to 0$ for $t\to\infty$.

Existence and uniqueness of solutions of each of these types were discussed by R.G. Koplatadze [22], A.L. Skubachevskii [23] and M.G. Shmul'yan [39]. S.M. Labovskii [24] proved that nonvanishing of the Wronskian $W(t)$ on the semiaxis was necessary and sufficient for existence of a positive decreasing solution to equation (1.3), (1.4) with nonpositive coefficients p_i $(i=1,\dots,m)$ and obtained several coefficient tests of $W(t)\neq 0$ for $t\in[0,+\infty)$. Solutions tending to zero were considered in the paper by T.A. Burton and J.R. Haddock [40].

Note that the approach for studying asymptotic properties of equations with linear transformations of arguments

$$x''(t)=\sum_{j=-l,j\neq 0}^{l} a_j x(q^j t)+\lambda x(t),\quad t\in(-\infty,+\infty),$$

where a_j, q and λ are constants, was proposed by E. Yu. Romanenko and A.N. Sharkovskii [25], and G.A. Derfel and S.A. Molchanov [26]. In [26] equations with a combination of delayed and advanced arguments are considered. Systematic study of advanced equations ($\tau(t)\le 0$) can be found in the recent paper by Z. Dosla and I. Kiguradze [27] in which results on boundedness, stability and asymptotic representations of solutions are obtained.

Let us consider the following equation,

$$x''(t)+\sum_{i=1}^{n} p_i(t)x(t-\tau_i(t))=f(t),\quad t\in[0,+\infty), \tag{1.5}$$

$$x(\xi)=\varphi(\xi)\quad \text{for } \xi<0, \tag{1.6}$$

where $f\colon[0,+\infty)\mapsto(-\infty,+\infty)$ and $\varphi\colon(-\infty,0)\mapsto(-\infty,+\infty)$ are measurable functions essentially bounded, and p_i and $\tau_i:[0,+\infty)\to[0,+\infty)$ are locally summable functions.

It is known [10] that a general solution of equation (1.5), (1.6) has the following representation:

$$x(t)=\int_0^t C(t,s)\bar f(s)\,ds+x_1(t)x(0)+x_2(t)x'(0). \tag{1.7}$$

Here $C(t,s)$ is the Cauchy function of equation (1.3), (1.4). Note that for every fixed $s\in[0,+\infty)$ the function $C(\cdot,s)$ is a solution of the "s-truncated" equation

$$(\mathcal{L}_s x)(t)\equiv x''(t)+\sum_{i=1}^{n} p_i(t)x(t-\tau_i(t))=0,\quad t\in[s,+\infty), \tag{1.8}$$

$$x(\xi)=0\quad \text{for } \xi<s. \tag{1.9}$$

The function $\bar f$ is determined by the equality

$$\bar f(t)=f(t)-\sum_{i=1}^{n} p_i(t)\varphi(t-\tau_i(t))\big(1-\sigma(t-\tau_i(t),0)\big),$$

where $\sigma(t,s)=\begin{cases}1 & \text{for } t\ge s,\\ 0 & \text{for } t<s.\end{cases}$ Functions x_1 and x_2 are solutions of equation (1.3), (1.4), satisfying the conditions $x_1(0)=1$, $x_1'(0)=0$, $x_2(0)=0$, and $x_2'(0)=1$.

Equation (1.5), (1.6) is said to be unstable if for each positive ε there exist two solutions x and $\bar x$ so that

$$|x(0)-\bar x(0)|<\varepsilon \quad\text{and}\quad |x'(0)-\bar x'(0)|<\varepsilon,$$

but their difference $x(t)-\bar x(t)$ is unbounded on $[0,+\infty)$. It is clear from representation (1.7) that existence of an unbounded solution of equation (1.3), (1.4) is equivalent to instability of equation (1.5), (1.6).

In this paper several criteria for existence of unbounded solutions to equation (1.3), (1.4) are obtained. The following examples show some of them. If $\varepsilon=0$, then all solutions of the equations

$$x''(t)+e^t x(t-\varepsilon)=0, \tag{1.10}$$

$$x''(t)+t^2x(t)+t^{3/2}x\Big(t-\frac{\varepsilon}{t}\Big)=0, \tag{1.11}$$

$$x''(t) + t^{\alpha}x(t - \frac{\varepsilon}{t^{\beta}}) = 0, \quad \alpha + 2 > 2\beta, \tag{1.12}$$

$$x''(t) + x(t) + \frac{1}{\sqrt{t}}x(t - \frac{\varepsilon}{\sqrt{t}}) = 0, \tag{1.13}$$

$$x(\xi) = 0 \text{ for } \xi < 0,$$

are bounded on $(1, +\infty)$, and for equations (1.10)–(1.12) they even tend to zero when $t \longrightarrow +\infty$ (see the monograph by V.N. Shevelo [28, p. 24]). If $\varepsilon > 0$, then there exist unbounded solutions to equations (1.10) and (1.12). If in addition ε is small enough, then there exist unbounded solutions to equations (1.11) and (1.13).

We will obtain the following criteria of boundedness of all solutions of the equation

$$x''(t) + p(t)x(t - \tau(t)) = 0, \quad t \in [0, +\infty), \qquad x(\xi) = 0 \text{ for } \xi < 0. \tag{1.14}$$

Theorem 1.1. *All solutions of equation* (1.14) *with positive, nondecreasing and bounded coefficient* $p(t)$ *and nondecreasing* $h(t) \equiv t - \tau(t)$ *are bounded if and only if*

$$\int_0^{\infty} \tau(t)\, dt < \infty. \tag{1.15}$$

The following result shows that solutions of delay equation (1.14) only in case of summable delay τ are getting closer and closer to solutions of the corresponding ordinary equation (1.1).

Theorem 1.2. *Assume that* $p(t) = c^2 > 0$, *and* $h(t) \equiv t - \tau(t)$ *does not decrease. Then any solution* $x(t)$ *of equation* (1.14) *satisfies the formulas*

$x(t) = (\alpha + o(1)) \sin ct + (\beta + o(1)) \cos ct,$

$x'(t) = c(\alpha + o(1)) \cos ct - c(\beta + o(1)) \sin ct$, *for* $t \to \infty$ *where* α *and* β *are constants, if and only if* $\tau(t)$ *is a summable function on* $[0, +\infty)$.

Note that asymptotic properties of solutions of the same delay equation (1.2) can be very distinct. The problem of similar asymptotic behavior of all solutions to the same equation has not been solved even with ordinary equations. For example, H. Milloux [29] discovered that if $p(t) \to \infty$ for $t \to \infty$, then there exists a solution of equation (1.1) tending to zero when $t \to \infty$. There are also several examples of other solutions without tending to zero. The problem of finding conditions under which all solutions tend to zero remains one highlighted in the qualitative theory of differential equations (see recent papers by A. Elbert [30], and L. Hatvani and L. Stacho [31]). If the coefficient $p(t) \to 0$ for $t \to +\infty$, then there exists an unbounded solution of the ordinary equation (1.1) (see the monograph by I.T. Kiguradze and T.A. Chanturia [4]). The equation $x''(t) + \frac{2}{t^2(t-1)}x(t) = 0$, $t \in [2, +\infty)$, gives an example, when the second solution $x(t) = \frac{t-1}{t}$ does not tend to zero.

In almost all statements of this paper something is said about existence of a certain unbounded solution. Can we say anything about unboundedness of **all solutions**? No, according to the following example: the function $x = \sin t$ is one of the solutions of the equation

$$x''(t) + x(t - \tau(t)) = 0, \quad t \in [0, +\infty),$$

where $\tau(t) = \begin{cases} 0, & 0 \le t \le \frac{\pi}{2}, \\ 2t - \pi, & \frac{\pi}{2} < t < \pi, \end{cases}$ $\tau(t + \pi) = \tau(t)$. Other solutions are unbounded by Theorem 1.1. Note that in this example the distance between adjoint zeros (π) is equal to the period of the coefficients (π). This has some logical ground. In a paper by A. Domoshnitsky [32] conditions of unboundedness of all solutions to delay equations with ω-periodic coefficients, such that $t - \tau(t) \geqslant 0$, are obtained. For example, if distances between every two zeros are not equal to 2ω, then all solutions are unbounded [32]. Note the classical Lyapunov's result for the ordinary differential equation $x''(t) + p(t)x(t) = 0$, $t \in [0, +\infty)$ with ω-periodic coefficient $p(t)$. If a period ω is less than the distance between adjoint zeros, then all solutions are bounded on the semiaxis (see the classical monograph by N.E. Zhukovskii [33]). The inequality $\int_0^{\omega} p(t)\, dt \leqslant \frac{4}{\omega}$ implies that all solutions are bounded. A similar inequality (1 instead of 4) provides unboundedness of all solutions in case there is a delay equation ($\tau(t) = \tau(t + \omega) \not\equiv 0$) [32]. Using estimates of distance between zeros [5, 6, 10, 34] one can obtain tests of unboundedness of all solutions even in cases when ω is greater than a distance between adjoint zeros [32]. Note that our approach also includes research of behavior of oscillating solutions' amplitudes. The amplitude behavior of solutions of ordinary differential equations was investigated by C.T. Taam [35] and L. Lasota [36]. Results on solutions unboundedness of partial differential equations with delayed argument are formulated in [38].

Let us introduce the operator $K_{\nu\mu}\colon \mathcal{C}_{[\nu,\mu]} \mapsto \mathcal{C}_{[\nu,\mu]}$ by the following equality:

$$(K_{\nu\mu}x)(t) = -\int_{\nu}^{\mu} G_{\nu\mu}(t, s) \sum_{j=1}^{m} p_j(s)x(s - \tau_j(s))\, ds, \tag{1.16}$$

where $x(\xi) = 0$ for $\xi < \nu$ and $G_{\nu\mu}(t, s)$ is the Green's function of the boundary value problem

$$x''(t) = f(t), \quad t \in [\nu, \mu], \quad x(\nu) = 0, \quad x(\mu) = 0. \tag{1.17}$$

Denote the minimal positive characteristic number of the operator $K_{\nu\mu}$ by $\lambda_{\nu,\mu}$. Let us denote $h_i(t) = t - \tau_i(t)$ and $h(t) = \min_{1\le i\le m} h_i(t)$. All the results submitted in this paper are obtained under the condition

$$\lambda_{h(t),t} > 1 \quad \text{for} \quad t \in (0, +\infty), \tag{1.18}$$

known as the H-condition [10]. Each of the following conditions a), b) and c) allows inequality (1.18) [10, 11, 12]:

a) $(t - h(t)) \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s)\, ds \le 4$ for $t \in (0, +\infty)$,

b) $(t - h(t))^2 \quad \text{ess sup}_{s\in[h(t),t]} \sum_{i=1}^{m} p_i(s) \le 8$ for $t \in (0, +\infty)$,

c) $m = 1$ and h_1 is a nondecreasing function.

Note: The H-condition ensures that Sturm's separation theorem is valid, $|W(t)|$ does not decrease, and there is no more than one zero of a nontrivial solution x to equation (1.3), (1.4) on $[h(t), t]$ for every $t \in (0, +\infty)$.

2. Main results

Let us formulate results on the unboundedness of solutions of equation (1.3), (1.4).

Theorem 2.1. *Let it be that*

$$M \equiv ess\ sup_{t\in[0,+\infty)} \sum_{j=1}^{m} p_j(t) < +\infty,$$

and there exists $i \in \{1, \dots, m\}$ so that

$$\int_0^{\infty} p_i(t)\tau_i(t)(2\sqrt{2}/\sqrt{M} - \tau_i(t))\, dt = +\infty.$$

Then there exists an unbounded solution of equation (1.3), (1.4).

For equation

$$\begin{aligned} &x''(t) + p_1(t)x(t) + p_2(t)x(t - \tau_2(t)) = 0, \quad t \in [0, +\infty), \\ &x(\xi) = 0 \text{ for } \xi < 0, \end{aligned} \tag{2.1}$$

the following result is obtained:

Theorem 2.2. *Let p_1 and p_2 be bounded on $[0, +\infty)$, $\tau_2(t) \underset{t\to+\infty}{\longrightarrow} 0$ and*

$$\int_0^{\infty} p_2(t)\tau_2(t)\, dt = +\infty. \tag{2.2}$$

Then there exists an unbounded solution of equation (2.1).

Example 2.3. The equation

$$x''(t) + p(t)x(t) + \frac{1}{t^\alpha} x(t - \frac{\varepsilon}{t^\beta}) = 0, \quad t \in [1, +\infty), \tag{2.3}$$

has an unbounded solution if $\alpha + \beta \le 1$, $\alpha \ge 0$, $\beta \ge 0$.

Unboundedness of the solution of equation (1.13) follows from the above assertion in case $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $p = 1$.

Denote $\tau(t) = \min_{1\le i\le m} \tau_i(t)$.

Theorem 2.4. *Let there be an index i so that*

$$\int_0^{\infty} p_i(t)\tau(t)\, dt = \infty. \tag{2.4}$$

Assume that at least one of the following two conditions a) *or* b) *is fulfilled:*

a) *there exists $\varepsilon > 0$ so that $\tau(t) \ge \varepsilon$ for $t \ge \nu \ge 0$;*

b) *ess $\sup_{t\in[\nu,+\infty)} \sum_{i=1}^{m} p_i(t) < \infty$.*

Then there exists an unbounded solution of equation (1.3), (1.4).

Existence of unbounded solutions of equation (1.10) results from Theorem 2.4 if condition a) is provided.

Example 2.5. The equation

$$x''(t) + t^\alpha |\sin t| x(t - \varepsilon) = 0, \quad t \in (0, +\infty)$$

has an unbounded solution if $\alpha \ge -1$.

3. Growth of the Wronskian and existence of unbounded solutions

Denote the Wronskian of the fundamental system of equation (1.3), (1.4) by

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}.$$

To simplify it let us assume that $W(0) > 0$.

Theorem 3.1. *If*

$$\lim_{t\to+\infty} W(t) = +\infty \tag{3.1}$$

and there exists positive ε so that $\tau_i(t) \ge \varepsilon$ for $i = 1, \dots, m$ and almost all $t \ge \nu$, then there exist unbounded solutions of equation (1.3), (1.4).

Introduce the function $\theta\colon [0, +\infty) \mapsto [0, +\infty)$ so that the minimal positive characteristic number $\lambda_{\nu,\theta(\nu)}$ of the operator $K_{\nu,\theta(\nu)}$ satisfies the inequality $\lambda_{\nu,\theta(\nu)} \le 1$ for each $\nu \in [0, +\infty)$. Denote

$$R(t) = \text{ess sup}_{s\in[t,\theta(t)]} \sum_{i=1}^{m} p_i(s).$$

Theorem 3.2. *Let it be that*

$$\operatorname*{esslim}_{t\to+\infty} \frac{W(t)}{\sqrt{R(t)}} = \infty. \tag{3.2}$$

Then there exists an unbounded solution of equation (1.3), (1.4).

Corollary 3.3. *If*

$$\operatorname*{esssup}_{t\in[0,+\infty)} \sum_{i=1}^{m} p_i(t) < \infty \text{ and } \lim_{t\to+\infty} W(t) = \infty,$$

then there exist unbounded solutions of equation (1.3), (1.4).

Remark 3.4. In many cases it is possible to replace (3.2) with the following condition:

$$\operatorname*{esslim}_{t\to+\infty} \frac{W(t)}{\sqrt{\sum_{i=1}^{m} p_i(s)}} = \infty. \tag{3.3}$$

This replacement is interesting only in case the conditions of Theorem 3.1 and Corollary 3.3 are not provided; i.e., there exists an index i so that

$$\operatorname*{ess\,sup}_{t\in[0,+\infty)} p_i(t) = \infty$$

and an index j so that

$$\operatorname*{ess\,inf}_{t\in[0,+\infty)} \tau_j(t) = 0.$$

A more typical example is the following: functions $p(t) = \sum_{i=1}^{m} p_i(t)$ and $h_i(t) = t - \tau_i(t)$ are nondecreasing on $[\nu, +\infty)$. In this case the function θ can be assigned, for example, by the following formula:

$$\theta(t) = t + \frac{\pi}{2\sqrt{p(t)}} + \bar{g}(t), \tag{3.4}$$

where $p(t) = \sum_{i=1}^{m} p_i(t)$, $\bar{g}(t) \geq \max_{1\leq i\leq m} \tau_i(t)$, $\bar{g}(t)\sum_{i=1}^{m} p_i(t) \geq 1$ (see Lemma 3.6 below). It is clear that the replacement is possible if

$$\overline{\operatorname*{esslim}_{t\to+\infty}} \frac{\sum_{i=1}^{m} p_i(t + \pi/(2\sqrt{p(t)}) + \bar{g}(t))}{p(t)} = K < \infty, \tag{3.5}$$

where $p(t) = \sum_{i=1}^{m} p_i(t)$. For example, this condition is provided for polynomial coefficients p_i, $i = 1, \ldots, m$.

Proof of Theorem 3.2. Let x_1 and x_2 be a fundamental system of equation (1.3), (1.4). Let us assume that x_1 is bounded. Let us suppose without loss of generality that $\max_{t\in[0,+\infty]} |x_1(t)| \leq 1$ and

$$W(0) \equiv \begin{vmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{vmatrix} = A > 0.$$

Let us start with the option of an oscillating solution x_2; i.e., there exists a sequence $\{t_j\}$ such that $x_2(t_j) = 0$ for $j = 1, 2, 3, \ldots$, $0 \leq t_1 < t_2 < \cdots < t_j < t_{j+1} < \cdots$. At these points $W(t_j) = x_1(t_j)x_2'(t_j)$. We assume that $x_2'(t_j) > 0$. It is clear that $x_2'(t_j) \geq W(t_j)$. The solution x_2 satisfies the following equality

$$x_2''(t) + \sum_{i=1}^{m} p_i(t)x_2(h_i(t))\sigma(h_i(t), t_j) = -\sum_{i=1}^{m} p_i(t)x_2(h_i(t))[1 - \sigma(h_i(t), t_j)]$$

on the segment $[t_j, t_{j+1}]$. The H-condition [10] implies that $h_i(t) \geq t_{j-1}$ for $i = 1, \ldots, m$ and almost all $t \geq t_j$. Now it is evident that

$$\varphi(t) \equiv -\sum_{i=1}^{m} p_i(t)x_2(h_i(t))[1 - \sigma(h_i(t), t_j)] \geq 0$$

for almost all $t \in [t_j, t_{j+1}]$. Let us estimate x_2 from below on the segment $[t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}]$. Set

$$v(t) = \frac{W(t_j)}{\sqrt{R(t_j)}} \sin[\sqrt{R(t_j)}(t - t_j)]. \tag{3.6}$$

It is clear that $v(t_j) = 0$, $v'(t_j) = W(t_j)$ and

$$\psi(t) \equiv v''(t) + \sum_{i=1}^{m} p_i(t)v(h_i(t))\sigma(h_i(t), t_j) \leq 0$$

for almost all $t \in [t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}]$. The last inequality, according to a generalization of the de la Vallee-Poussin theorem, obtained by N.V. Azbelev in [10], provided that the Cauchy function $C(t, s)$ is positive in the triangle $s, t \in (t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}})$, $s < t$.

Using representation (1.7) and inequality $\psi(t) \leq 0 \leq \varphi(t)$ for almost all $t \in [t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}]$, we obtain that $v(t) \leq x_2(t)$ for $t \in [t_j, t_j + \frac{\pi}{2\sqrt{R(t_j)}}]$. If $j \to \infty$, then $t_j \to \infty$ (see [10]). It is clear that

$$\operatorname*{esslim}_{t\to+\infty} \frac{W(t)}{\sqrt{R(t)}} \leq \lim_{t\to+\infty} \max_{s\in[0,t]} |x_2(s)|.$$

Now let us consider another option; i.e., nonoscillating solutions. Let t_0 be the last zero of $x_2(t)$ on $[0,+\infty)$. We assume without loss of generality that $x_2'(t_0) > 0$. Let $\bar{t}_0$ be the last zero of the solution $x_1(t)$ on $[0,+\infty)$. We assume without loss of generality that $x_1'(\bar{t}_0) > 0$. The H-condition provides that the Wronskian $W(t)$ does not decrease:

$$W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t) \geq W(0) > 0.$$

Since $x_2(t)x_1'(t) \geq 0$ for sufficiently large t, then $W(t) \leq x_1(t)x_2'(t)$ and $W(0) \leq W(t) \leq x_2'(t)$. This implies that x_2 is not bounded on $(0,+\infty)$. Theorem 3.2 is completely proved. □

The fact that $W(t)$ is nondecreasing implies the following:

Corollary 3.5. *If*

$$\operatorname*{esslim}_{t\to+\infty} \sum_{i=1}^{m} p_i(t) = 0,$$

then there exists an unbounded solution of equation (1.3), (1.4).

Proof of Theorem 3.1. Let x_1 and x_2 be a fundamental system of (1.3), (1.4), such that

$$\max_{t\in[0,+\infty)} |x_1(t)| \leq 1 \text{ and } W(0) > 0.$$

We will prove that x_2 is an unbounded solution. In the nonoscillating option the unboundedness of x_2 is obvious from the proof of Theorem 3.2. Let us consider a sequence $\{t_j\}$ in the oscillating option so that $x_2(t_j) = 0$, $j = 1,2,3,\cdots$, $t_1 < t_2 \cdots < t_j < t_{j+1} < \ldots$. At these points $W(t_j) = x_1(t_j)x_2'(t_j)$. Upon the assumption that $\max_{t\in[0,+\infty)} |x_1(t)| \leq 1$ it is obtained that $W(t_j) \leq x_2'(t_j)$. Let us consider the proof of Theorem 3.2 for $v(t) = W(t_j)(t-t_j)$.

It is evident that $v(t_j) = 0$, $v(t_j+\varepsilon) = W(t_j)\varepsilon$ and

$$\psi(t) \equiv v''(t) + \sum_{i=1}^{m} p_i(t)v(h_i(t))\sigma(h_i(t),t_j) \leq 0$$

for almost all $t \in [t_j, t_j+\varepsilon]$.

Since $v(t) \leq x_2(t)$ for $t \in [t_j, t_j+\varepsilon]$ (see the proof of Theorem 3.2), $x_2(t_j+\varepsilon) \geq W(t_j)\varepsilon$, and we conclude that x_2 is unbounded.

Lemma 3.6. *Let p and h be nondecreasing functions and*

$$\mu = \nu + \frac{\pi}{2\sqrt{p(\nu)}} + \bar{g}(\nu) \quad for \quad \nu \in [0,+\infty),$$

where $p(\nu) = \sum_{i=1}^{m} p_i(\nu)$. Then $\lambda_{\nu,\mu} \leq 1$.

Proof. Let us set

$$v(t) = \begin{cases} \sin\left[\sqrt{\sum_{i=1}^{m} p_i(\nu)}(t-\nu-\bar{g}(\nu))\right] + \sqrt{\sum_{i=1}^{m} p_i(\nu)}\bar{g}(\nu), & t \geq \bar{g}(\nu)+\nu, \\ \sqrt{\sum_{i=1}^{m} p_i(\nu)}(t-\nu), & t < \bar{g}(\nu)+\nu. \end{cases}$$

for each fixed ν. It is evident that

$$v''(t) + \sum_{i=1}^{m} p_i(t)v(h_i(t))\sigma(h_i(t),t_j) \geq 0$$

for almost all $t \in [\nu,\mu]$. It implies the inequality $v(t) \leq (K_{\nu\mu}v)(t)$ for $t \in [\nu,\mu]$. According to the known result [37, p. 81] that results in $\lambda_{\nu,\mu} \leq 1$.

4. Estimates of the Wronskian

In order to use the results of Part 3 we have to obtain estimates of the Wronskian.

Theorem 4.1. *The Wronskian $W(t)$ of the fundamental system satisfies the following differential inequality:*

$$W'(t) \geq \sum_{i=1}^{m} p_i(t)C(t,h_i(t))W(h_i(t)), \quad t \in [0,+\infty), \tag{4.1}$$

where $W(s) = 0$ for $s < 0$, $C(\cdot,s) = 0$ for $s < 0$.

Proof. Let us introduce the following function of two variables:

$$Q(t,s) = \begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t) \end{vmatrix}.$$

For a fixed s the function $q_s(t) \equiv Q(t,s)$ (as a function of the argument t only) is a solution of equation (1.3), (1.4), and moreover $q_s(s) = 0$, $q_s'(s) = W(s)$.

It is clear that

$$W'(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1''(t) & x_2''(t) \end{vmatrix} = -\begin{vmatrix} x_1(t) & x_2(t) \\ \sum_{i=1}^{m} p_i(t)x_1(h_i(t)) & \sum_{i=1}^{m} p_i(t)x_2(h_i(t)) \end{vmatrix}$$

$$= -\sum_{i=1}^{m} p_i(t)[x_1(t)x_2(h_i(t)) - x_2(t)x_1(h_i(t))]$$

$$= -\sum_{i=1}^{m} p_i(t)Q(h_i(t),t) = \sum_{i=1}^{m} p_i(t)Q(t,h_i(t)).$$

In order to continue the proof let us obtain the following:

Lemma 4.2. *If $\lambda_{s,t} > 1$ $(0 \leq s < t < \infty)$, then $Q(t,s) \geq W(s)C(t,s)$.*

Proof. Solution q_s of equation (1.3), (1.4) satisfies the equality

$$(\mathcal{L}_s q_s)(\xi) = -\sum_{i=1}^{m} p_i(\xi) q_s(h_i(\xi))[1-\sigma(h_i(\xi),s)] \text{ for } \xi \in [s,t].$$

Denote the zero of q_s nearest to s from below by ν. It is apparent that

$$\varphi(\xi) \equiv -\sum_{i=1}^{m} p_i(\xi) q_s(h_i(\xi))[1-\sigma(h_i(\xi),s)] \geq 0, \quad \xi \in [s,t].$$

Under the condition $\lambda_{s,t} > 1$ positivity of the Cauchy function $C(\xi,\eta)$ for $\xi,\eta \in (s,t)$, $\xi > \eta$, is provided. Inequality $\varphi \geq 0$ implies that $Q(\xi,s) \geq W(s)C(\xi,s)$ for $s \leq \xi \leq t$. Lemma 4.2 is completely proved. □

Continue the proof of Theorem 4.1.

From Lemma 4.2 it follows that

$$\sum_{i=1}^{m} p_i(t)Q(t,h_i(t)) \geq \sum_{i=1}^{m} p_i(t)W(h_i(t))C(t,h_i(t)).$$

The equality $W'(t) = \sum_{i=1}^{m} p_i(t)Q(t,h_i(t))$ completes the proof of Theorem 4.1. □

Theorem 4.1 makes it possible to obtain estimates of the Wronskian $W(t)$. Let us use the following estimate,

$$W(t) \geq W(0)\Big(1+\int_0^t \sum_{i=1}^{m} p_i(s)C(s,h_i(s))\,ds\Big), \tag{4.2}$$

where $C(t,h_i(t)) = 0$ if $h_i(t) < 0$, in order to obtain the following result.

Theorem 4.3. *If there exists a function $v(\cdot,\cdot)\colon [\nu,+\infty) \mapsto [0,+\infty)$ so that*

1) *$v(\cdot,s)$ for each fixed s has an absolutely continuous derivative on each segment $[s,b]$;*

2) *$v(\cdot,h_i(\cdot))\colon [\nu,+\infty) \mapsto [0,+\infty)$ is measurable for $i=1,\dots,m$;*

3)

$$v(t,h_i(s)) \begin{cases} >0 & t\in(h_i(s),s], \quad h_i(s)\in[\nu,s),\\ =0 & t=h_i(s),\\ =0 & t\in[\nu,+\infty), \quad h_i(s)\notin[\nu,s),\end{cases}$$

$$v'(h_i(s),h_i(s)) = \begin{cases} 1, & h_i(s)\in[\nu,s),\\ 0, & h_i(s)\notin[\nu,s),\end{cases}$$

and

$$\psi(t) \equiv v''(t,h_i(s)) + \sum_{j=1}^{m} p_j(t)v(h_j(t),h_i(s)) \leq 0$$

for $i=1,\dots,m$ and almost all $t\in[h_i(s),s]$. Then

$$W(t) \geq W(\nu)\Big(1+\int_\nu^t \sum_{i=1}^{m} p_i(s)v(s,h_i(s))\,ds\Big), \quad t\in[\nu,+\infty).$$

Proof. In order to prove Theorem 4.3 let us show that $v(s,h_i(s)) \leq C(s,h_i(s))$ for $i=1,\dots,m$ and almost all $s\in[\nu,+\infty)$. De la Vallée-Poussin's theorem [10] implies that $C(t,\xi)\geq 0$ for $t,\xi\in[h_i(s),s]$, $t\geq\xi$. The function $C(\cdot,h_i(s))$ for almost all fixed $h_i(s)\in[\nu,+\infty)$ is a solution of the equation

$$(\mathcal{L}_{h_i(s)}x)(t) = 0, \quad t\in[h_i(s),s],$$

and the function $v(\cdot,h_i(s))$ satisfies the equation

$$(\mathcal{L}_{h_i(s)}x)(t) = \psi(t), \quad t\in[h_i(s),s],$$

where $\psi\leq 0$. The positivity of the Cauchy function $C(t,\xi)$ implies the inequality $v(s,h_i(s)) \leq C(s,h_i(s))$, $s\in[\nu,+\infty)$.

5. Proofs

In order to prove Theorem 2.1, let us replace $v(t,s) = (t-s)(s+2\sqrt{2}/\sqrt{M}-t)$ in the conditions of Theorem 4.3. Theorem 2.2 is a corollary of Theorem 2.1.

Proof of Theorem 2.4. Let us replace $v(t,s)=t-s$ in the conditions of Theorem 4.3. Condition (2.4) implies that $\lim_{t\to+\infty} W(t)=+\infty$. Theorem 3.1 implies the sufficiency of condition a), and Theorem 3.2 implies the sufficiency of condition b) for existence of an unbounded solution.

Corollary 5.1. *If there exists an index $i\in\{1,\dots,m\}$ so that*

$$\int_0^\infty \frac{p_i(t)}{\sqrt{R(t-\tau_i(t))}} \sin\Big(\sqrt{R(t-\tau_i(t))}\,\tau_i(t)\Big)\,dt = \infty,$$

then $\lim_{t\to+\infty} W(t)=+\infty$. If also

$$\operatorname*{esslim}_{t\to+\infty} \frac{\int_0^\infty \frac{p_i(s)}{\sqrt{R(s-\tau_i(s))}} \sin(\sqrt{R(s-\tau_i(s))}\,\tau_i(s))\,ds}{\sqrt{R(t)}} = \infty,$$

then there exists an unbounded solution of equation (1.3), (1.4).

In order to prove Corollary 5.1, let us set

$$v(t,s) = \frac{1}{\sqrt{R(s)}}\sin[\sqrt{R(s)}(t-s)].$$

Note that existence of an unbounded solution of equation (1.11) results from Corollary 5.1.

If conditions a) and b) of Theorem 2.4 are not provided, the following assertion is proposed.

Corollary 5.2. *If there exists an index $i \in \{1, \dots, m\}$ so that at least one of the following conditions is provided,*

$$\operatorname*{esslim}_{t\to+\infty} \tau(t) \int_0^t p_i(s)\tau(s)\,ds = \infty, \tag{5.1}$$

or

$$\operatorname*{esslim}_{t\to+\infty} \frac{\int_0^t p_i(s)\tau(s)\,ds}{\sqrt{R(t)}} = \infty, \tag{5.2}$$

then there exists an unbounded solution of equation (1.3), (1.4).

In order to prove Corollary 5.2 let us set $v(t,s) = t - s$. The sufficiency of condition (5.1) results from the proof of Theorem 3.1. The sufficiency of condition (5.2) results from Theorem 3.2. Existence of an unbounded solution of equation (1.12) results from Corollary 5.2.

Proof of Theorem 1.1. Whereas we have proved that necessity follows from Theorem 2.4, sufficiency was proved by D.V. Izumova in her paper [18].

Proof of Theorem 1.2. Sufficiency was proved by M. Pinto [19]. Necessity results from Theorem 1.1.

Remark 5.3. The fact that the coefficient $p(t)$ is nondecreasing in Theorem 1.1 is essential; for example, the equation

$$x''(t) + \frac{1}{t^2}x\left(t - \frac{1}{t^2}\right) = 0, \quad t \in [1, +\infty)$$

has an unbounded solution by Corollary 3.5, but

$$\int^{\infty} \tau(t)\,dt = \int^{\infty} \frac{1}{t^2} < \infty.$$

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Differential and Integral Equations Volume 14, Number 5, May 2001, Pages 577–587

LOCAL EXACT CONTROLLABILITY OF A REACTION–DIFFUSION SYSTEM

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Abstract. The local exact controllability of a reaction–diffusion system with homogeneous Neumann boundary conditions is studied. The methods we use combine the Schauder–Kakutani fixed point theorem, Carleman estimates for the backward adjoint linearized system and some estimates in the theory of parabolic boundary value problems in L^k.

1. INTRODUCTION

This paper concerns the internal controllability of the following reaction–diffusion system:

$$\begin{cases} y_t - k_1\Delta y + \alpha a(x)y(x,t)z(x,t) = f(x) + u(x,t), & (x,t) \in Q \\ z_t - k_2\Delta z + \beta a(x)y(x,t)z(x,t) = g(x) + v(x,t), & (x,t) \in Q \\ \dfrac{\partial y}{\partial \nu}(x,t) = \dfrac{\partial z}{\partial \nu}(x,t) = 0, & (x,t) \in \Sigma \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \in \{1,2,3\}$, with a smooth boundary $\partial\Omega$, $Q = \Omega\times(0,T)$, $\Sigma = \partial\Omega\times(0,T)$ and u and v are control functions acting in $\overline{\omega}\times[0,T]$. Here $\omega \subset \Omega$ is a nonempty open subset, $T, k_1, k_2 \in (0,+\infty)$ and $\alpha, \beta \in \mathbb{N}^*$ are constants.

The first two equations in (1.1) describe a reaction–diffusion process, where $y(x,t)$ and $z(x,t)$ denote the densities of two substances A and B, respectively, in the position $x \in \overline{\Omega}$ and at the moment $t \in [0,T]$. The two substances diffuse in the domain Ω with the diffusion constants k_1 and k_2, respectively and react ($a(x)$ is a reaction coefficient; α molecules of A and β molecules of B react to produce some other substances). The functions f and g represent two infusions of substances A and B, respectively.

Accepted for publication March 2000.
AMS Subject Classifications: 93B05, 35K57, 46E35.