Strong Approximations in Probability and Statistics

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5.3. The law of iterated logarithm for the quantile process

In Section 4.5 we called attention to the fact that the analogue of the Glivenko–Cantelli theorem for the quantile process does not hold true without further restrictions. In this section we intend to point out that most of the strong laws proved for the empirical process are also true for the quantile process if we assume the conditions (4.5.10) and (4.5.12) of Theorem 4.5.6. We can do this in three ways: (i) apply Theorem 4.5.7 saying that the quantile process is near to a Kiefer process and deduce that theorems proved for a Kiefer process are also true for the quantile process; (ii) apply Theorem 5.2.2 saying that the quantile process is near to an empirical process and deduce that theorems proved for empirical processes are also true for quantile processes; (iii) apply Theorem 4.5.6 saying that a quantile process is near to a suitable uniform quantile process, hence theorems for the uniform quantile process extend to more general quantile processes. We followed the latter approach in the proofs of Theorems 5.2.2 and 5.2.5.

Applying any of the methods (i) and (ii) one gets (5.3.1)–(5.3.5) immediately, while method (iii) together with Theorem 4.5.6 implies (5.3.6). Thus we have:

Theorem 5.3.1. *Under the conditions of Theorem 5.2.2 on F we have*

(5.3.1) $$\overline{\lim_{n\to\infty}} (\log\log n)^{-1/2} \sup_{0<y<1} f(\operatorname{inv} F(y))|q_n(y)| \overset{\text{a.s.}}{=} 2^{-1/2},$$

(5.3.2) $$\overline{\lim_{n\to\infty}} (2\log\log n)^{-1} \int_0^1 f^2(\operatorname{inv} F(y))\, q_n^2(y)\, dy \overset{\text{a.s.}}{=} \pi^{-2}, \text{ (cf. Theorem 5.1.3)},$$

(5.3.3) $$\underline{\lim_{n\to\infty}} (\log\log n)^{1/2} \sup_{0<y<1} f(\operatorname{inv} F(y))|q_n(y)| \overset{\text{a.s.}}{=} 8^{-1/2}\pi, \quad \text{(cf. (5.1.8))}$$

(5.3.4) $$\underline{\lim_{n\to\infty}} (\log\log n)^{1/2} \Big(\int_0^1 f^2(\operatorname{inv} F(y))\, q_n^2(y)\, dy\Big)^{1/2} \overset{\text{a.s.}}{=} 8^{-1/2}, \quad \text{(cf. (5.1.9))},$$

(5.3.5) $$\overline{\lim_{n\to\infty}} (2\log\log n)^{-1/2} \sup_{\varepsilon<y<1-\varepsilon} (y(1-y))^{-1/2} f(\operatorname{inv} F(y))|q_n(y)| \overset{\text{a.s.}}{=} 1.$$

Assume that conditions (4.5.10), (4.5.12) *with* $\gamma<1$ *hold true. Then there exists a* $C>0$ *such that*

(5.3.6) $$\overline{\lim_{n\to\infty}} \sup_{\delta_n<y<1-\delta_n} (y(1-y)\log\log n)^{-1/2} |f(\operatorname{inv} F(y))\, q_n(y)| \le C \quad a.s.$$

where $\delta_n = 25n^{-1}\log\log n$.

The version of Theorem 5.1.2 in terms of $f(\operatorname{inv} F(y))q_n(y)$ is also straightforward. However the generalization of Theorem 5.1.5 resp. 5.1.6 to the quantile process is not immediate at all. The only such version we have at present is that of (5.3.6).

5.4. Asymptotic distribution results for some classical functionals of the empirical process

We have already seen that Donsker's theorem for the empirical process (Theorem 4.2.1) is a direct consequence of any one of the Brillinger and/or Kiefer type approximation theorems of Sections 4.3 and 4.4, and hence that, for example, Theorem 4.1.2 is implied by (1.5.3) and (1.5.4) of Theorem 1.5.1. Some further Corollary 4.4.1 type results follow here.

Corollary 5.4.1. *Consider the empirical process* $\beta_n(x)=\alpha_n(F(x))$. *Let* $y=F(x)$ *be a continuous distribution function and let* $g(y)\neq 0$ *be a real valued function for which we also have*

(5.4.1) $$\sup_y |g(y)| < \infty.$$

There exist then a sequence of Brownian bridges $\{B_n(y);\, 0\le y\le 1\}$ *and a Kiefer process* $\{K(y,t);\, 0\le y\le 1,\, 0\le t<\infty\}$ *such that*

(5.4.2) $$\Big|\sup_{-\infty<x<\infty} \beta_n(x)\, g(F(x)) - \sup_{0\le y\le 1} B_n(y)\, g(y)\Big| \overset{\text{a.s.}}{=} O(n^{-1/2}\log n),$$

$$\Big|\sup_{-\infty<x<\infty} \beta_n(x) g(F(x)) - \sup_{0\le y\le 1} n^{-1/2}K(y,n)\, g(y)\Big| \overset{\text{a.s.}}{=} O(n^{-1/2}\log^2 n),$$

(5.4.3) $$\Big|\sup_{-\infty<x<\infty} |\beta_n(x)\, g(F(x))| - \sup_{0\le y\le 1} |B_n(y) g(y)|\Big| \overset{\text{a.s.}}{=} O(n^{-1/2}\log n),$$

$$\Big|\sup_{-\infty<x<\infty} |\beta_n(x) g(F(x))| - \sup_{0\le y\le 1} |n^{-1/2}K(y,n)\, g(y)|\Big| \overset{\text{a.s.}}{=} O(n^{-1/2}\log^2 n),$$

(5.4.4) $$\Big|\int_{-\infty}^{+\infty} \beta_n^2(x)\, g^2(F(x))\, dF(x) - \int_0^1 B_n^2(y)\, g^2(y)\, dy\Big| \overset{\text{a.s.}}{=} O(n^{-1/2}\log n\, (\log\log n)^{1/2}),$$

$$\Big|\int_{-\infty}^{+\infty} \beta_n^2(x)\, g^2(F(x))\, dF(x) - \int_0^1 n^{-1}K^2(y,n)\, g^2(y)\, dy\Big| \overset{\text{a.s.}}{=} O(n^{-1/2}\log^2 n\, (\log\log n)^{1/2})$$

and

(5.4.5) $$\left|n^{1/2}R_n-(\sup_y B_n(y)g(y)-\inf_y B_n(y)g(y))\right| \overset{\text{a.s.}}{=} O(n^{-1/2}\log n),$$

$$\left|n^{1/2}R_n-(\sup_y n^{-1/2}K(y,n)g(y)-\inf_y n^{-1/2}K(y,n)g(y))\right| \overset{\text{a.s.}}{=} O(n^{-1/2}\log^2 n),$$

where

$$R_n = D_n^+ + D_n^- = \sup_x (F_n(x)-F(x))g(F(x))-\inf_x (F_n(x)-F(x))g(F(x)).$$

Since $B_n(y) \overset{\mathcal{D}}{=} B(y)$, and also $n^{-1/2}K(y,n) \overset{\mathcal{D}}{=} K(y,1) \overset{\mathcal{D}}{=} B(y)$, a Brownian bridge for each n, we see that the limit distributions of the above-presented functionals of the empirical process agree with the distributions of the corresponding functionals of a Brownian bridge. For example, (5.4.2) and (5.4.3) with $g(y)=1$ give the asymptotic distribution of the classical Kolmogorov–Smirnov statistics (cf. Theorem 4.1.2) via (1.5.3) and (1.5.4). Again with $g(y)=1$ and applying (5.4.4), we get the limit distribution of the Cramér–von Mises statistic. Namely, Theorem 1.5.2 implies

(5.4.6) $$\lim_{n\to\infty} P\left\{\int_{-\infty}^{+\infty} \beta_n^2(x)\,dF(x) \le u\right\} = P\{\omega^2 \le u\}, \quad u \ge 0,$$

where the latter distribution function is given in Theorem 1.5.2. Another classical result, the distribution of the Kuiper (1960) statistic, can be obtained from (5.4.5) with $g(y)=1$ by Theorem 1.5.3. Namely we have

$$\lim_{n\to\infty} P\{n^{1/2}R_n \le u\} = 1-\sum_{j=1}^{\infty} 2(4(ju)^2-1)e^{-2j^2u^2}, \quad u \ge 0.$$

Taking now for example

(5.4.7) $$g_\varepsilon(y) = \begin{cases} 0 & \text{if } 0 \le y \le \varepsilon \\ y^{-1} & \text{if } \varepsilon < y \le 1, \end{cases}$$

$$g_\delta(y) = \begin{cases} (1-y)^{-1} & \text{if } 0 \le y \le \delta \\ 0 & \text{if } \delta < y \le 1, \end{cases}$$

respectively

(5.4.8) $$g_{\varepsilon,\delta}(y) = \begin{cases} (y(1-y))^{-1/2} & \text{if } 0 < \varepsilon \le y \le \delta < 1 \\ 0 & \text{otherwise}, \end{cases}$$

in (5.4.2) and (5.4.3), we conclude that the limit distributions of the Rényi (1953) statistics, respectively those of the Anderson–Darling (1952) statistics can be evaluated via the distributions of the corresponding functionals of a Brownian bridge. We mention only two typical results obtainable this way. The first one of these is

(5.4.9) $$\lim_{n\to\infty} P\left\{\sup_{\varepsilon \le F(x)} \frac{\beta_n(x)}{F(x)} \le u\right\} = P\left\{\sup_{\varepsilon \le y} \frac{B(y)}{y} \le u\right\}$$

$$= 2\Phi\left(u\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2}\right)-1 \quad u \ge 0,\ \varepsilon > 0.$$

The latter equality is true, since

$$\left\{\frac{B(y)}{y};\ 0 < y \le 1\right\} \overset{\mathcal{D}}{=} \left\{W\left(\frac{1-y}{y}\right);\ 0 < y \le 1\right\},$$

and hence

$$P\left\{\sup_{\varepsilon \le y} \frac{B(y)}{y} \le u\right\} = P\left\{\sup_{\varepsilon \le y} W\left(\frac{1-y}{y}\right) \le u\right\}$$

$$= P\left\{\sup_{0<t\le\frac{1-\varepsilon}{\varepsilon}} W(t) \le u\right\},$$

which by (1.5.1) now gives the desired result. The second one we have in mind is

(5.4.10) $$\lim_{n\to\infty} P\left\{\sup_{\varepsilon \le F(x)} \frac{|\beta_n(x)|}{F(x)} \le u\right\} = P\left\{\sup_{\varepsilon \le y} \frac{|B(y)|}{y} \le u\right\}$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \exp\{-(2k+1)^2\pi^2(1-\varepsilon)/8\varepsilon u^2\}, \quad u > 0,\ \varepsilon > 0.$$

The above two results ((5.4.9) and (5.4.10)) were first given by Rényi (1953) via classical limiting arguments. For a proof of these and some further similar ones along these lines via the invariance principle, we refer to Csörgő (1966, 1967).

So far we have seen how strong invariance principles (cf. Corollary 5.4.1) can be used to prove asymptotic distribution result like e.g., (5.4.6), (5.4.9), (5.4.10). In proving these results, we have not utilized the rates of approximation of Corollary 5.4.1 at all, and did not say anything about the problem of how fast these distribution functions themselves converged to their limits. The next result gives an answer to this problem.

Corollary 5.4.2 (Komlós, Major, Tusnády 1975a). *Let B_n, α_n be as in Theorem 4.4.1, and let ψ be a functional defined on $D(0,1)$, satisfying the Lipschitz condition*

(5.4.11) $$|\psi(u)-\psi(v)| \le L \sup_{0\le y\le 1} |u(y)-v(y)|$$

with some positive constant L. Assume further that the distribution of the random variable $\psi(B(y)) \overset{\mathcal{D}}{=} \psi(B_n(y))$ $(n=1, 2, \ldots)$ *has a bounded density with respect to Lebesgue measure. Then*

$$\sup_{-\infty<x<\infty} |P\{\psi(\alpha_n(y)) \leq x\} - P\{\psi(B_n(y)) \leq x\}| = O\left(\frac{\log n}{n^{1/2}}\right). \quad (5.4.12)$$

The proof of this corollary is similar to that of Corollary 5.4.3 whose proof is given below.

As to the nearness of the processes α_n and B_n, Theorem 4.4.1 gives the best possible rate. It is an open question whether the rate of (5.4.12) is the best possible or not for any given Lipschitzian functional ψ. For example, in the Kolmogorov–Smirnov case (that is when ψ is the sup-functional) the rate of convergence is known to be $O(n^{-1/2})$ (cf. Gnedenko, Korolyuk and Skorohod 1960; Bickel 1974).

The rate in (5.4.12) does not hold necessarily for functionals not satisfying the Lipschitzian condition (5.4.11), but Theorem 4.4.1 might still give us a handle on occasions. The case we have in mind is that of the Cramér–von Mises statistic $\omega_n^2 = \int_0^1 \alpha_n^2(y)\,dy$. Let $N_n(x) = P\{\omega_n^2 \leq x\}$, $N(x) = P\{\int_0^1 B^2(y)\,dy \leq x\}$ and put $\Delta_n = \sup_{0<x<\infty} |N_n(x) - N(x)|$. We have, of course, that $\lim_{n\to\infty} N_n(x) = N(x)$ for every real x (cf. (5.4.6)) and, in addition to this, it can be easily deduced from the first statement of (4.4.25), or from that of (5.4.4) that $\Delta_n = O(n^{-1/2}\log n\,(\log\log n)^{1/2})$. But, if we are a bit more circumspect, we can actually prove also

Corollary 5.4.3 (S. Csörgő 1976).

$$\Delta_n = O(n^{-1/2}\log n). \quad (5.4.13)$$

The latter statement is of interest, because it turns out to be a refinement of the best available result of this kind so far for the distribution of ω_n^2. Namely Orlov (1974) proved that for any $\varepsilon>0$ there exists a positive constant $b(\varepsilon)$ such that for each n $\Delta_n \leq b(\varepsilon) n^{\varepsilon}/n^{1/2}$. For a complete set of earlier work on this problem we refer to S. Csörgő (1976). We should also remark here that, though the rate of convergence of Δ_n in (5.4.13) is the best available one so far, it is probably far away from the best possible one. Indeed, a complete asymptotic expansion for the Laplace transform of ω_n^2 is given by S. Csörgő (1976) and, on the basis of his work, he conjectures that Δ_n has the order of $1/n$ (concerning this latter problem we refer also to S. Csörgő and Stachó, 1979). For an improved result in this direction we refer to Götze (1979).

Proof of Corollary 5.4.3. Let $\omega^2(n) = \int_0^1 B_n^2(y)\,dy$, where $\{B_n(y);\, 0 \leq y \leq 1\}$ is the sequence of Brownian bridges for which the statement of Theorem 4.4.1 holds true, and let $\omega^2 = \int_0^1 B^2(y)\,dy$, where $B(y)$ is an arbitrary Brownian bridge. We will use the elementary fact that, if X, Y, Z are arbitrary r.v. such that $P\{|X-Y|>Z\}<\varepsilon$ for some $\varepsilon>0$, then, for every real x,

$$P\{Y \leq x-Z\} - \varepsilon \leq P\{X \leq x\} \leq P\{Y \leq x+Z\} + \varepsilon. \quad (5.4.14)$$

In (4.4.1) let $x = \frac{1}{2\lambda}\log n$, $D = C + \frac{1}{2\lambda}$. Then, with the notation $D_n = \sup_{0\leq y\leq 1} |\alpha_n(y) - B_n(y)|$ and $\varepsilon_n = D\log n/\sqrt{n}$, we have

$$\begin{aligned}
&P\{|\omega_n^2 - \omega^2(n)| > \varepsilon_n^2 + 2\varepsilon_n\omega(n)\} \\
&= P\Big\{\Big|\int_0^1 (\alpha_n(y) - B_n(y))(\alpha_n(y) + B_n(y))\,dy\Big| > \varepsilon_n^2 + 2\varepsilon_n\omega(n)\Big\} \\
&\leq P\Big\{\int_0^1 (\alpha_n(y) - B_n(y))^2\,dy + 2\int_0^1 |\alpha_n(y) - B_n(y)|\,|B_n(y)|\,dy > \varepsilon_n^2 + 2\varepsilon_n\omega(n)\Big\}.
\end{aligned}$$

Let the event of the latter probability statement be denoted by $E(\varepsilon_n)$. Then, from the above inequality,

$$\begin{aligned}
P\{|\omega_n^2 - \omega^2(n)| > \varepsilon_n^2 + 2\varepsilon_n\omega(n)\} &\leq P\{E(\varepsilon_n), D_n \leq \varepsilon_n\} + P\{D_n > \varepsilon_n\} \\
&\leq P\Big\{\varepsilon_n^2 + 2\varepsilon_n\int_0^1 |B_n(y)|\,dy > \varepsilon_n^2 + 2\varepsilon_n\omega(n)\Big\} + n^{-1/2}L \\
&= P\Big\{\int_0^1 |B_n(y)|\,dy > \omega(n)\Big\} + n^{-1/2}L \\
&\leq P\{\omega(n) > \omega(n)\} + n^{-1/2}L = n^{-1/2}L,
\end{aligned}$$

where the last inequality follows from that of Schwartz, and L is the positive absolute constant of (4.4.1).

Now we apply (5.4.14) with $X = \omega_n^2$, $Y = \omega(n)$, $Z = \varepsilon_n^2 + 2\varepsilon_n\omega(n)$ and $\varepsilon = n^{-1/2}L$, and get

$$P\{A_n(x)\} - n^{-1/2}L \leq P\{\omega_n^2 \leq x\} \leq P\{G_n(x)\} + n^{-1/2}L,$$

where $A_n(x) = \{\omega^2 \leq x - \varepsilon_n^2 - 2\varepsilon_n\omega\}$, $G_n(x) = \{\omega^2 \leq x + \varepsilon_n^2 + 2\varepsilon_n\omega\}$ and $x>0$.

Solving the corresponding quadratic inequalities for the events $A_n(x)$ and $G_n(x)$ we find that $P\{A_n(x)\}=0$ if $x \leqq \varepsilon_n^2$, while

$$P\{A_n(x)\} = P\{\omega^2 \leqq x+\varepsilon_n^2-(4\varepsilon_n^2 x)^{1/2}\} \geqq P\{\omega^2 \leqq x-\delta_n\}, \quad \text{if} \quad x \geqq \varepsilon_n^2,$$

and

$$G_n(x) \subset \{\omega^2 \leqq x+\delta_n\} \quad \text{with} \quad \delta_n = 3\varepsilon_n^2+(8\varepsilon_n^4+4\varepsilon_n^2 x)^{1/2}, \quad x>0.$$

Hence, with $\delta_n=\delta_n(x)$ $(x>0)$ as in the preceding line,

$$N(x-\delta_n(x))-n^{-1/2}L \leqq N_n(x) \leqq N(x+\delta_n(x))+n^{-1/2}L,$$

and whence

$$\Delta_n \leqq \sup_x P\{x-\delta_n(x) < \omega^2 \leqq x+\delta_n(x)\}+n^{-1/2}L$$

$$= \sup_x \int_x^{x+2\delta_n(x)} v(y)\,dy+n^{-1/2}L \leqq 2\sup_x \big(\delta_n(x) \sup_{x\leqq y\leqq x+2\delta_n(x)} v(y)\big)+n^{-1/2}L$$

$$\leqq 2K\sup_x \big(\delta_n(x) \sup_{x\leqq y\leqq x+2\delta_n(x)} (y+1)^{-1/2}\big)+n^{-1/2}L$$

$$\leqq 2K\sup_x \frac{\delta_n(x)}{(x+1)^{1/2}}+n^{-1/2}L = 2K\sup_x \frac{3\varepsilon_n^2+(8\varepsilon_n^4+4\varepsilon_n^2 x)^{1/2}}{(x+1)^{1/2}}+n^{-1/2}L$$

$$\leqq 2K\sup_x \frac{3\varepsilon_n^2(x+1)^{1/2}+\big(8\varepsilon_n^4(x+1)+4\varepsilon_n^2(x+1)\big)^{1/2}}{(x+1)^{1/2}}+n^{-1/2}L$$

$$= O(\varepsilon_n) = O(n^{-1/2}\log n),$$

where $v(x)=\frac{d}{dx}N(x)$, and the third inequality for Δ_n above is by $v(y)(y+1)^{1/2} \leqq v(y)(y^{1/2}+1) \leqq K$ uniformly in $y \in (0, \infty)$, since it is known that not only the density function $v(x)$ of ω^2 but $v(x)x^{1/2}$ too is bounded on the positive half-line (cf. Lemma 8 in § 5, S. Csörgő 1976).

While Corollary 5.4.1 provides us with weak convergence results for the classical functionals of the empirical process with weight functions $g(y)$ satisfying the condition (5.4.1), the weight function $g(y)=(y(1-y))^{-1/2}$, $0<y<1$, which is probably the most natural one, does not fit into its framework. An application of Theorem 4.4.1, however, turns out to be a good initial step also towards this direction of weak convergence problems, which we are going to consider now.

Let $V_n(x)=n^{1/2}(F_n(x)-F(x))/(F(x)(1-F(x)))^{1/2}$, where F is a continuous distribution function and, for $0 \leqq \varepsilon < \delta \leqq 1$, define $V^n(\varepsilon, \delta) = \sup_{\varepsilon<F(x)<\delta} V_n(x)$ and $W^n(\varepsilon, \delta) = \sup_{\varepsilon<F(x)<\delta} |V_n(x)|$. We note that Anderson and Darling (1952) derived the Laplace transform of the asymptotic distribution of $W^n(\varepsilon, \delta)$ with $0<\varepsilon<\delta<1$. It is also natural to ask how one could choose normalizing factors for given sequences ε_n, δ_n so that $V^n(\varepsilon_n, \delta_n)$, $W^n(\varepsilon_n, \delta_n)$, $V^n(0, 1)$ and $W^n(0, 1)$ should have a non-degenerate asymptotic distribution. Indeed this question was asked and answered recently by Jaeschke (1975) and Eicker (1976, 1979) and further studied and developed by Jaeschke (1976, 1979).

For the construction of confidence intervals for F,

$$\hat{V}_n(x) = \begin{cases} 0, & \text{if } F_n(x)=0 \text{ or } 1 \\ n^{1/2}(F_n(x)-F(x))/(F_n(x)(1-F_n(x)))^{1/2}, & \text{otherwise,} \end{cases}$$

is more convenient than $V_n(x)$. It is shown by Jaeschke (1976) that for $\hat{V}_n$ the same assertions hold as for V_n, including also the case $\varepsilon_n=0, \delta_n=1$. The latter is also a generalization of the earlier results of Eicker (1976). Here we formulate these results with $\varepsilon_n=0$, $\delta_n=1$.

Towards stating these results, let $E_c(t)=\exp(-c\exp(-t))$ $(c \geqq 0)$ and $a(\cdot, \cdot)$ be as in Theorem 1.9.1.

Theorem 5.4.1 (Jaeschke 1976). *We have*

(5.4.15) $$\lim_{n\to\infty} P\{V^n(0, 1) \leqq a(t, \log n)\} = E_1(t)$$

and

(5.4.16) $$\lim_{n\to\infty} P\{W^n(0, 1) \leqq a(t, \log n)\} = E_2(t), \quad -\infty < t < +\infty.$$

Theorem 5.4.2 (Eicker 1976, Jaeschke 1976). *Let*

$$\hat{V}^n(\varepsilon, \delta) = \sup_{\varepsilon<F(x)<\delta} \hat{V}_n(x) \quad \text{and} \quad \hat{W}^n(\varepsilon, \delta) = \sup_{\varepsilon<F(x)<\delta} |\hat{V}_n(x)|.$$

Then

(5.4.17) $$\lim_{n\to\infty} P\{\hat{V}^n(0, 1) \leqq a(t, \log n)\} = E_1(t)$$

and

(5.4.18) $$\lim_{n\to\infty} P\{\hat{W}^n(0, 1) \leqq a(t, \log n)\} = E_2(t), \quad -\infty < t < +\infty.$$

Just as in the case of the classical empirical process $\beta_n(x)=\alpha_n(F(x))$, we may from now on take $F \in U(0, 1)$, since F is assumed to be continuous. As to the proof of the two theorems formulated above, we give only the main steps, in order to demonstrate how Theorem 4.4.1 can be applied in this situation. Here we follow Jaeschke (1976).

The proof of Theorem 5.4.1 is based on the following lemma:

Lemma 5.4.1 (Jaeschke 1976). *For* $-\infty<t<+\infty$ *we have*

$$\lim_{n\to\infty} P\{W^n(0,\varepsilon_n)\vee W^n(1-\varepsilon_n,1)\leqq a(t,\log n)\}=1, \tag{5.4.19}$$

where $\varepsilon_n=n^{-1}\log^3 n$.

For a proof of this lemma we refer to that of Lemma 4 in Jaeschke (1976).

Proof of Theorem 5.4.1. Since by Theorem 4.4.1

$$(2\log\log n)^{1/2}\sup_{\varepsilon_n<y<1-\varepsilon_n}|V_n(y)-B_n(y)/(y(1-y))^{1/2}|$$

$$\overset{\text{a.s.}}{=} O\left(\left(\frac{\log\log n}{\log n}\right)^{1/2}\right)=o(1),$$

Corollary 1.9.1 and Lemma 5.4.1 imply Theorem 5.4.1.

For a similar proof of the second theorem we need two further lemmas.

Lemma 5.4.2 (Jaeschke 1976). *For* $\varepsilon_n\geqq n^{-1}\log n$ *and* $a_n=(2\log\log n)^{1/2}$ *we have*

$$a_n\hat{V}^n(\varepsilon_n,1-\varepsilon_n)\overset{\text{a.s.}}{=} a_nV^n(\varepsilon_n,1-\varepsilon_n)+o(1). \tag{5.4.20}$$

Proof. Due to Theorem 5.1.6 we have

$$\sup_{n^{-1}\log n<y<1}|1-y^{-1}F_n(y)|\overset{\text{a.s.}}{=} O((\log\log n/\log n)^{1/2})$$

and

$$\sup_{0<y<1-n^{-1}\log n}|1-(1-F_n(y))/(1-y)|\overset{\text{a.s.}}{=} O((\log\log n/\log n)^{1/2}).$$

Whence

$$a_n\hat{V}^n(\varepsilon_n,1-\varepsilon_n)\overset{\text{a.s.}}{=} a_nV^n(\varepsilon_n,1-\varepsilon_n)(1+O((\log\log n/\log n)^{1/2})).$$

Again Theorem 5.1.6 implies $a_nV^n(\varepsilon_n,1-\varepsilon_n)\overset{\text{a.s.}}{=} O(\log\log n)$, and this is the assertion of (5.4.20).

Lemma 5.4.3 (Eicker 1976, Jaeschke 1976). *With* $\varepsilon_n=n^{-1}\log n$ *and* $-\infty<t<+\infty$, *we have*

$$\lim_{n\to\infty} P\{\hat{W}^n(0,\varepsilon_n)\vee\hat{W}^n(1-\varepsilon_n,1)\leqq a(t,\log n)\}=1. \tag{5.4.21}$$

For a proof of this lemma we refer to that of Lemma 6 in Jaeschke (1976).

Proof of Theorem 5.4.2. A combination of Lemmas 5.4.2, 5.4.3 and Theorem 5.4.1 yields (5.4.17) and (5.4.18).

5.5. Asymptotic distribution results for some classical functionals of the quantile process

On the basis of Theorems 4.5.6 and 4.5.7 it is quite immediate to construct a Corollary 5.4.1 type statement for the quantile process $f(\text{inv } F(y))q_n(y)$, provided we assume conditions (4.5.10) and (4.5.12). Such an analogue of Corollary 5.4.1 immediately implies, among others, the following typical statements:

$$\lim_{n\to\infty} P\{\sup_{0<y<1} f(\text{inv } F(y))q_n(y)\leqq u\}=P\{\sup_{0\leqq y\leqq 1}B(y)\leqq u\} \tag{5.5.1}$$

$$=1-e^{-2u^2},\quad u\geqq 0,\ (\text{cf. } (1.5.3)),$$

$$\lim_{n\to\infty} P\{\sup_{0<y<1}|f(\text{inv } F(y))q_n(y)|\leqq u\}=P\{\sup_{0\leqq y\leqq 1}|B(y)|\leqq u\} \tag{5.5.2}$$

$$=1-\sum_{k\neq 0}(-1)^{k+1}e^{-2k^2u^2},\quad u\geqq 0,\ (\text{cf. } (1.5.4)),$$

$$\lim_{n\to\infty} P\{\mathcal{R}_n\leqq u\}=P\{\sup_{0<y<1}B(y)-\inf_{0<y<1}B(y)\leqq u\} \tag{5.5.3}$$

$$=1-\sum_{j=1}^{\infty}2(4(ju)^2-1)e^{-2j^2u^2},\quad u\geqq 0,\ (\text{cf. Theorem 1.5.3}),$$

where $\mathcal{R}_n=\sup_{0<y<1}f(\text{inv } F(y))q_n(y)-\inf_{0<y<1}f(\text{inv } F(y))q_n(y)$,

$$\lim_{n\to\infty} P\left\{\sup_{\varepsilon\leqq y}\frac{f(\text{inv } F(y))q_n(y)}{y}\leqq u\right\}=P\left\{\sup_{\varepsilon\leqq y}\frac{B(y)}{y}\leqq u\right\} \tag{5.5.4}$$

$$=2\Phi\left(u\left(\frac{\varepsilon}{1-\varepsilon}\right)^{1/2}\right)-1,\quad u\geqq 0,\ \varepsilon>0,\ (\text{cf. } (5.4.9)),$$

and

$$\lim_{n\to\infty} P\left\{\sup_{\varepsilon\leqq y}\frac{|f(\text{inv } F(y))q_n(y)|}{y}\leqq u\right\}=P\left\{\sup_{\varepsilon\leqq y}\frac{|B(y)|}{y}\leqq u\right\} \tag{5.5.5}$$

$$=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\exp\{-(2k+1)^2\pi^2(1-\varepsilon)/8\varepsilon u^2\},\quad u>0,\ \varepsilon>0,$$

(cf. (5.4.10)).

We again call attention to the fact that the above asymptotic results hold true, assuming only the reasonably weak conditions (4.5.10) and (4.5.12).

An analogue of Corollary 5.4.2 is also immediate.

Corollary 5.5.1. *Let* B_n, q_n *be as in Theorem 4.5.7, and let* ψ *be defined on* $D(0,1)$, *satisfying the Lipschitz condition of* (5.4.11). *Assume further*

Theorem 7.3.2. *Let $\alpha_n(y)$, $u_n(y)$, $q_n(y)$, $K(y, n)$ be as in Theorems 4.4.3, 4.5.3 and 4.5.7 respectively, and let $\{v_n\}$ be a sequence of positive integer valued random variables, and assume that $v_n \xrightarrow{P} \infty$ as $n \to \infty$. Then*

$$\sup_{0 \leq y \leq 1} |\alpha_{v_n}(y) - v_n^{-1/2} K(y, v_n)| \xrightarrow{P} 0, \tag{7.3.2}$$

$$\sup_{0 \leq y \leq 1} |u_{v_n}(y) - v_n^{-1/2} K(y, v_n)| \xrightarrow{P} 0, \tag{7.3.3}$$

$$\sup_{0<y<1} \left| f(F^{-1}(y)) q_{v_n}(y) - v_n^{-1/2} K(y, v_n) \right| \xrightarrow{P} 0. \tag{7.3.4}$$

We note that (7.3.2) also holds true with $y = F(x)$ for $\alpha_n(F(x)) = \beta_n(x)$. Further, if instead of $v_n \xrightarrow{P} \infty$ we have that $v_n \xrightarrow{\text{a.s.}} \infty$, then (7.3.2)–(7.3.4) also hold true with probability one.

Proof of Theorem 7.3.2. Combining the respective statements (b) and (c) of Theorem 7.1.1 with (4.4.23), (4.5.8) and (4.5.25), the above statements follow.

Theorem 7.3.3. *If v_n of Theorem 7.3.2 also satisfies the condition* (7.2.7), *then*

$$\alpha_{v_n}(\cdot) \xrightarrow{\mathscr{D}} B(\cdot), \tag{7.3.5}$$

$$f(F^{-1}(\cdot))\, q_{v_n}(\cdot) \xrightarrow{\mathscr{D}} B(\cdot). \tag{7.3.6}$$

Proof. First we note that (7.2.7) implies that $v_n \xrightarrow{P} \infty$ as $n \to \infty$. It follows then from (7.3.2) and (7.3.4) that, in order to prove (7.3.5) and (7.3.6), it suffices to show that

$$v_n^{-1/2} K(\cdot, v_n) \xrightarrow{\mathscr{D}} B(\cdot). \tag{7.3.7}$$

Now the proof of (7.3.7) can be done along the lines of that of (7.2.8).

Remark 7.3.1. The first paper on the random sample size empirical process α_{v_n} was written by Pyke (1968). He proved (7.3.5) under the assumption that $v_n/n \xrightarrow{P} 1$. The above method of proof also extends to empirical processes defined in terms of multivariate random variables (cf. M. Csörgő, S. Csörgő, Fischler and Révész 1975). As to random sum limit theorems, one of the first papers was that of Anscombe (1952) (cf. also Doeblin 1938, 1940).

References

ALDOUS, D J.

(1978) Weak convergence of randomly indexed sequences of random variables. *Math. Proc. Cambridge Philos. Soc.* **83** 117–126.

ALI, Mir M.–CHAN, L. K.

(1964) On Gupta's estimates of the parameters of the normal distribution. *Biometrika* **51** 498–501.

ANDERSON, T. W.–DARLING, D. A.

(1952) Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. *Ann. Math. Statist.* **23** 193–212.

ANSCOMBE, F. J.

(1952) Large-sample theory of sequential estimation. *Proc Cambridge Philos. Soc.* **48** 600–607.

BAHADUR, R. R.

(1966) A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

BÁRTFAI, P.

(1966) Die Bestimmung der zu einem wiederkehrenden Prozess gehörenden Verteilungsfunktion aus den mit Fehlern behafteten Daten einer Einzigen Relation. *Studia Sci. Math. Hung.* **1** 161–168.

(1977) Connections between the convex analysis and the theory of large deviations. Technical report, Math. Inst. Hung. Acad. Sci. Budapest.

BARTLETT, M. S.

(1949) Some evalutionary stochastic processes. *J. Roy. Statist. Soc. Ser. B.* **11** 211–229.

BAXTER, G.

(1956) A strong limit theorem for Gaussian processes. *Proc. Amer. Math. Soc.* **7** 522–527.

BERKES, I.

(1972) A remark to the law of the iterated logarithm. *Studia Sci. Math. Hung.* **7** 189–197.

BERKES, I.–PHILIPP, W.

(1979) Approximation theorems for independent and weakly dependent random vectors. *Ann. Probability* **7** 29–54.

BICKEL, P. J.–ROSENBLATT, M.

(1973) On some global measures of the deviations of density function estimates. *Ann. Statist,* **1** 1071–1095.

BICKEL, P. J.

(1974) Edgeworth expansions in nonparametric statistics. *Ann. Statist.* **2** 1–20.

BILLINGSLEY, P.
(1962) Limit theorems for randomly selected partial sums. *Ann. Math. Statist.* **33** 85–92.
(1968) *Convergence of Probability Measures.* J. Wiley, New York.
BLUM, J.–HANSON, D.–ROSENBLATT, J.
(1963) On the central limit theorem for the sum of a random number of independent random variables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete.* **1** 389–393.
BOCHNER, S.
(1955) *Harmonic analysis and the theory of probability.* Univ. of California Pr. Berkeley–Los Angeles.
BOLTHAUSEN, E.
(1978) On the speed of convergence in Strassen's law of the iterated logarithm. *Ann. Probability* **6** 668–672.
BOOK, S. A.
(1976) Large deviation probabilities and the Erdős–Rényi law of large numbers. *Canad. J. Statist.* **4** 185–210.
BOOK, S. A.–SHORE, T. R.
(1978) On large intervals in the Csörgő–Révész theorem on increments of a Wiener process. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **46** 1–11.
BOROVKOV, A. A.
(1973) On the speed of convergence in the invariance principle. *Teor. Ver. i ee Prim.* **18** 217–234.
BREIMAN, L.
(1967) On the tail behaviour of sums of independent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **9** 20–25.
(1968) *Probability.* Addison-Wesley. Reading, Mass.
BRILLINGER, D. L.
(1969) An asymptotic representation of the sample distribution function. *Bull. Amer. Math. Soc.* **75** 545–547.
BURKE, M. D.–CSÖRGŐ, M.
(1976) Weak approximations of the empirical process when parameters are estimated. In *Empirical Distributions and Processes* (P. Gaenssler and P. Révész, Eds.). *Lecture Notes in Mathematics* **566** 1–16, Springer-Verlag, Berlin.
(1976b) Strong approximations of the *k*-sample quantile and multivariate empirical process. *Carleton Mathematical Lecture Note.* No. 16.
BURKE, M. D.–CSÖRGŐ, M.–CSÖRGŐ, S.–RÉVÉSZ, P.
(1979) Approximations of the empirical process when parameters are estimated. *Ann. Probability* **7** 790–810.
CANTELLI, F. P.
(1917) Sulla probabilitá come limita della frequenza. *Rend. Accad. Lincei.* **26** 39.
ČENCOV, N. N.
(1962) Evaluation of an unknown density from observations. *Soviet Math.* **3** 1559–1569.
(1956) Wiener random fields depending on several parameters. *Dokl. Akad. Nauk. SSSR* **106** 607–609.
CHAN, A. H. C.
(1977) *On the increments of Multiparameter Gaussian Processes.* Ph. D. Thesis, Carleton University.

CHAN, A. H. C.–CSÖRGŐ, M.–RÉVÉSZ, P.
(1978) Strassen type limit points for moving averages of a Wiener process. *Canad. J. Statist.* **6** 57–75.
CHERNOFF, H.
(1952) A measure of asymptotic efficiency for tests of a hypothesis based on sums of observations. *Ann. Math. Statist.* **23** 493–507.
CHUNG, K. L.
(1948) On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64** 205–233.
(1949) An estimate concerning the Kolmogorov limit distribution. *Trans. Amer. Math. Soc.* **67** 36–50.
CHUNG, K. L.–ERDŐS, P.–SIRAO, T.
(1959) On the Lipschitz condition for Brownian motion. *J. of Math. Soc. Japan* **11** 263–274.
CRAMÉR, H.
(1946) *Mathematical methods of statistics.* Princeton Univ. Press, Princeton, N. J.
(1962) *Random variables and probability distributions.* Cambridge Univ. Press, London.
CSÁKI, E.
(1968) An iterated logarithm law for semimartingales and its application to empirical distribution function. *Studia Sci. Math. Hung.* **3** 287–292.
(1975) Some notes on the law of iterated logarithm for empirical distribution function. In *Coll. Math. Soc. J. Bolyai* **11**, *Limit Theorems of Probability Theory* (P. Révész, Ed.), Keszthely, Hungary 1974, North Holland, Amsterdam–London, 47–58.
(1977) The law of iterated logarithm for normalized empirical distribution function. *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **38** 147–167.
(1978) On the lower limits of maxima and minima of Wiener process and partial sums. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **43** 205–222.
CSÁKI, E.–RÉVÉSZ, P.
(1979) How big must be the increments of a Wiener Process? *Acta Math. Acad. Sci. Hung.* **33** 37–49.
CSÖRGŐ, M.
(1966) Some *k*-sample Kolmogorov–Smirnov–Rényi type theorems for empirical distribution functions. *Acta Math. Acad. Sci. Hung.* **17** 325–334.
(1967) A new proof of some results of Rényi and the asymptotic distribution of the range of his Kolmogorov–Smirnov type random variables. *Can. J. Math.* **19** 550–558.
(1968) On the strong law of large numbers and the central limit theorem for martingales. *Trans. Amer. Math. Soc.* **131** 259–275.
CSÖRGŐ, M.–CSÖRGŐ, S.
(1973) On weak convergence of randomly selected partial sums. *Acta. Sci. Math. (Szeged)* **34** 53–60.
CSÖRGŐ, M.–CSÖRGŐ, S.–FISCHLER, R.–RÉVÉSZ, P.
(1975) Random limit theorems via strong invariance principles (in Hungarian). *Matematikai Lapok* **26** 39–66.
CSÖRGŐ, M.–FISCHLER, R.
(1970) Departure from independence: the strong law, standard and random-sum central limit theorems. *Acta Math. Acad. Sci. Hung.* **21** 105–114.

CSÖRGŐ, M.–KOMLÓS, J.–MAJOR, P.–RÉVÉSZ, P.–TUSNÁDY, G.
(1977) On the empirical process when parameters are estimated. In *Trans. Seventh Prague Conference* 1974, Academia, Prague, 87–97.

CSÖRGŐ, M.–RÉVÉSZ, P.
(1975) A new method to prove Strassen type laws of invariance principle, I. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **31** 255–260.
(1975b) A new method to prove Strassen type laws of invariance principle, II. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **31** 261–269.
(1975c) Some notes on the empirical distribution function and the quantile process. In *Coll. Math. Soc. J. Bolyai* **11**, *Limit Theorems of Probability Theory* (P. Révész, Ed.), Keszthely, Hungary 1974, North Holland, Amsterdam–London, 59–71.
(1975d) A strong approximation of the multivariate empirical process. *Studia Sci. Math. Hung.* **10** 427–434.
(1978) How big are the increments of a multi-parameter Wiener process? *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **42** 1–12.
(1978b) Strong approximations of the quantile process. *Ann. Statist.* **6** 882–894.
(1979a) How small are the increments of a Wiener process? *Stochastic Processes Appl.* **8** 119–129.
(1979b) How big are the increments of a Wiener process? *Ann. Probability* **7** 731–737.
(1979) On the standardized quantile process. In *Optimizing Methods in Statistics, Procedings of an International Conference* (I. S. Rustagi, Ed.), Bombay India, December 1977, Academic Press, New York, 125–140.
(1980) Quadratic ... ity tests. *Carleton Mathematical Series.* No. 162.
(1980a) Quantile processes and sums of weighted spacings for composite goodnes-of-fit, II. *Carleton Mathematical Lecture Note.*
(1980b) An estimation of the quantile function via density estimation. *Carleton Mathematical Series.*
(1980c) An invariance principle for N.N. empirical density functions. *Carleton Mathematical Lecture Note.*

CSÖRGŐ, M.–STEINEBACH, J.
(1980) Improved Erdős–Rényi and strong approximation laws for increments of partial sums. *Carleton Mathematical Series.* No. 166.

CSÖRGŐ, S.
(1974) On the limit distributions of sequences of random variables with random indices. *Acta Math. Acad. Sci. Hung.* **25** 227–232.
(1976) On an asymptotic expansion for the Mises ω^2 statistic. *Acta Sci. Math. (Szeged).* **38** 45–67.
(1979) Erdős–Rényi laws. *Ann. Statist.* **7** 772–787.
(1980) Limit behaviour of the empirical characteristic function. *Ann. Probability* **8.**

CSÖRGŐ, S.–STACHÓ, L.
(1979) A step toward an asymptotic expansion for the Cramér–von Mises statistic. In *Coll. Math. Soc. J. Bolyai.* **21**, *Analytic Function Methods in Probability Theory* (B. Gyires, Ed.), Debrecen, Hungary 1977, North Holland, Amsterdam–London, 53–65.

DARLING, D. A.
(1955) The Cramér–Smirnov test in the parametric case. *Ann. Math. Statist.* **26** 1–20.

DARLING, D. A.–ERDŐS, P.
(1956) A limit theorem for the maximum of normalized sums of independent random variables, *Duke Math. J.* **23** 143–145.

DEO, C. M.
(1977) A note on increments of a Wiener process. *Technical Report.* University of Ottawa.

DeWET, T.–VENTER, J. H.
(1972) Asymptotic distributions of certain tests criteria of normality. *S. Afr. Statist. J.* **6** 135–149.

DOEBLIN, W.
(1938) Sur de problémes de M. Kolmogoroff concernant le chaines dénombrables. *Bull. Soc. Math. France* **66** 210–220.
(1940) Éléments d'une théorie générale des chaines simples constantes de Markov. *Ann. Sci. École Norm. Sup.* **57** 61–111.

DONSKER, M.
(1951) An invariance principle for certain probability limit theorems. *Four papers on probability. Mem. Amer. Math. Soc. No.* **6.**
(1952) Justification and extension of Doob's heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **23** 277–283.

DONSKER, M –VARADHAN, S. S. R.
(1977) On laws of iterated logarithm for local times. *Comm. Pure Appl. Math.* **30** 707–753.

DOOB, J. L.
(1949) Heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **20** 393–403.
(1953) *Stochastic Processes.* J. Wiley, New York.

DUDLEY, R. M.
(1968) Distances of probability measures and random variables. *Ann. Math. Statist.* **39** 1563–1572.
(1973) Sample functions of the Gaussian process. *Ann. Probability* **1** 66–103.

DURBIN, J.
(1973a) Weak convergence of the sample distribution function when parameters are estimated. *Ann. Statist.* **1** 279–290.
(1973b) *Distribution Theory for Tests based on the Sample Distribution function. Regional Conference Series on Appl. Math.* **9** S.I.A.M., Philadelphia.
(1976) Kolmogorov–Smirnov tests when parameters are estimated. In *Empirical Distributions and Processes* (P. Gaenssler and P. Révész, Ed.). *Lecture Notes in Mathematics* **566** 33–44, Springer-Verlag, Berlin.

DVORETZKY, A.–KIEFER, J.–WOLFOWITZ, J.
(1956) Asymptotic minimax character of the sample distribution function and of the multinomial estimator. *Ann. Math. Statist.* **27** 642–669.

DVORETZKY, A.
(1963) On the oscillation of the Brownian motion process. *Israel J. Math.* **1** 212–214.

EICKER, F.
(1976) The asymptotic distribution of the supremum of the standardized empirical distribution function. *Preprint No. 76/7 of the Dept. Statist.*, Dortmund University

(1979) The asymptotic distribution of the suprema of standardized empirical process. 7 116–138.

EINSTEIN, A.
(1905) On the movement of small particles suspended in a stationary liquid demended by the molecular-kinetic theory of heat. *Ann. Physik* **17** 549–560.

EPANECHNIKOV, V. A.
(1969) Nonparametric estimates of a multivariate probability density. *Theor. Probability Appl.* **14** 153–158.

ERDŐS, P.
(1949) On a theorem of Hsu and Robbins. *Ann. Math. Statist.* **20** 286–291.
(1950) Remark on my paper "On a theorem of Hsu and Robbins." *Ann. Math. Statist.* **21** 138.

ERDŐS, P.–KAC, M.
(1946) On certain limit theorems of the theory of probability. *Bull. Amer. Math. Soc.* **52** 292–302.
(1947) On the number of positive sums of independent random variables. *Bull. Amer. Math. Soc.* **53** 1011–1020.

ERDŐS, P.–RÉNYI, A.
(1970) On a new law of large numbers. *J. Analyse Math.* **23** 103–111.

ERDŐS, P.–RÉVÉSZ, P.
(1976) On the length of the longest head-run. In *Coll. Math. Soc. J. Bolyai* **16**, *Topics in Information Theory,* Keszthely, Hungary 1975, North Holland, Amsterdam.

FELLER, W.
(1943) The general form of the so-called law of the iterated logarithm. *Trans. Amer. Math. Soc.* **54** 373–402.
(1948) On the Kolmogorov–Smirnov limit theorems for empirical distributions. *Ann. Math. Statist.* **19** 177–189.
(1968) *An Introduction to Probability Theory and Its Applications,* Volume I: Third Edition. J. Wiley, New York.
(1966) *An Introduction to Probability Theory and Its Applications,* Volume II. J. Wiley, New York.

FEUERVERGER, A.–MUREIKA, R. A.
(1977) The empirical characteristic function and its applications. *Ann. Statist.* **5** 88–97.

FINKELSTEIN, H.
(1971) The law of iterated logarithm for empirical distributions. *Ann. Math. Statist.* **42** 607–615.

FISCHLER, R.
(1976) Convergence faible avec indices aléatoires. *Ann. Inst. Henri Poincaré, B: Calcul des Probabilités et Statistique* **12** 391–399.

FISHER, R. A.
(1929) Tests of significance in harmonic analysis. *Proc. Roy. Soc. (A)* **125** 54–59. Reprinted 1950 as Paper No. 16 of R. A. Fisher's *Contributions to Mathematical Statistics,* J. Wiley, New York; Chapman & Hall, London.

FÖLDES, A.–RÉVÉSZ, P.
(1974) A general method for density estimation. *Studia Sci. Math. Hung.* **9** 81–92.

GAENSSLER, P.–STUTE, W.
(1979) Empirical processes: a survey of results for independent and identically distributed random variables. *Ann. Probability* **7** 193–243.

GARLING, D. J. H.
(1976) Functional central limit theorems in Banach spaces. *Ann. Probability* **4** 600–611.

GLIVENKO, V. I.
(1933) Sulla determinazione empirica delle leggi di probabilitá. *Giorn. Ist. Ital. Attuari.* **4** 92–99.

GNEDENKO, B. V.–KOROLYUK, V. S.–SKOROHOD, A. V.
(1960) Asymptotic expansions in probability theory. In *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **2** 153–169, Univ. California Press.

GÖTZE, F.
(1979) Asymptotic expansions for bivariate von Mises functionals. *Preprints in Statistics. No. 49. Univ of Cologne.*

GUIBAS, L. J.–ODLYZKO, A. M.
(1980) Long repetitive patterns in random sequences. *Z. Wahrscheinlichkeitstheorie verw. Gebiete.*

HARTER, H. L.
(1961) Expected values of normal order statistics. *Biometrika* **48** 151–165.

HARTMAN, P.–WINTNER, A.
(1941) On the law of iterated logarithm. *Amer. J. Math.* **63** 169–176.

HIRSCH, W. M.
(1965) A strong law for the maximum cumulative sum of independent random variables. *Comm. Pure Appl. Math.* **18** 109–217.

HOFFMAN-JØRGENSEN, J.–PISIER, G.
(1976) The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probability* **4** 587–599.

HORVÁTH, L.
(1978) A note on random limit theorems via strong invariance principles. *JATE Student Mathematical Workshop Notes.*

IBRAGIMOV, I. A.–HAS'MINSKIĬ, R. Z.
(1972) Asymptotic behaviour of statistical estimators in the smooth case I. Study of the likelihood ratio. *Theor. Probability Appl.* **17** 445–462.
(1973a) Asymptotic behaviour of some statistical estimators II. Limit theorems for the a posteriori density and Bayes' estimators. *Theory Probability Appl.* **18** 76–91.
(1973b) On the approximation of statistical estimators by sums of independent variables. *Soviet Math. Dokl.* **14** 883–887.

ITÔ, K.
(1960) *Probability Processes,* Volume I (In Japanese; Russian Translation by ИЛ, M., 1960). Tokyo, Iwanami Shoten Publishers.

ITÔ, K.–NISIO, M.
(1968) On the oscillation functions of Gaussian processes. *Math. Scand.* **22** 209–223.

JAESCHKE, D.
(1975) Über die Grenzverteilung des Maximums der normierten empirischen Verteilungsfunktion. Dissertation, Dortmund University.

(1976) The asymptotic distribution of the supremum of the standardized empirical distribution function on subintervals. *Preprint No.* 76/11 *of the Dept. Statist.*, Dortmund University.
(1979) The asymptotic distribution of the supremum of the standardized empirical distribution function on subintervals *Ann. Statist.* 7 108–115.
JAMES, B. R.
(1975) A functional law of the iterated logarithm for weighted empirical distributions. *Ann. Probability* 3 762–772.
JAIN, N. C.–PRUITT, W. E.
(1976) The other law of the iterated logarithm. *Ann. Probability* 3 1046–1049.
KAC, M.
(1946) On the average of a certain Wiener functional and a related limit theorem in calculus of probability. *Trans. Amer. Math. Soc.* **59** 404–414.
(1949) On deviations between theoretical and empirical distributions. *Proc. Nat. Acad. Sci. USA.* **35** 252–257.
(1951) On some connections between probability theory and differential and integral equations. In *Second Berkeley Symp. on Probability and Statistics.* 180–215. Univ. California Press, Berkeley–Los Angeles.
KAC, M.–KIEFER, J.–WOLFOWITZ, J.
(1955) On tests of normality and other tests on goodness of fit based on distance methods. *Ann. Math. Statist.* **26** 189–211.
KARHUNEN, K.
(1946) Zur Spektraltheorie stochastischen Processe. *Ann. Acad. Sci. Fennicae Ser A.* I. Math. Phys. No. **34** 7.
(1947) Über lineare Methoden in der Wahrscheinlichkeitsrechnung. *Ann. Acad. Sci. Fennicae Ser. A.* I. Math. Phys. No. **37** 79.
KARLSSON, J E.–SZÁSZ, D. E.
(1974) *Random limit theorems. A bibliography.* Math. Inst. Hung. Acad. Sci. Budapest.
KATZ, M. L.
(1963) The probability in the tail of a distribution. *Ann. Math. Statist.* **34** 312–318.
KENDALL, D. G.
(1974) Hunting quanta. *Phil. Trans. Roy. Soc. London A* **276** 231–266. Second edition:
(1977) *Proc. Symp. Honour Jerzy Neyman,* Warsawa 1974, Polish Acad. Sci. 111–159.
KENT, J. T.
(1975) A weak convergence theorem for the empirical characteristic function. *J. Appl. Prob.* **12** 515–523.
KHINCHINE, A.
(1933) *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung.* Springer, Berlin.
KIEFER, J.
(1959) *K*-sample analogues of the Kolmogorov–Smirnov and Cramér–von Mises tests. *Ann. Math. Statist.* **30** 420–447.
(1969a) On the deviations in the Skorohod–Strassen approximation scheme. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **13** 321–332.
(1969b) Old and new methods for studying order statistics and sample quantiles. In *Proc. First International Conf. Nonparametric Inference.* 349–357. Cambridge Univ. Press. 1970. London.

(1970) Deviations between the sample quantile process and the sample D. F. *Nonparametric Techniques in Stat. Inference* (M. L. Puri, Ed.) 299–319, Cambridge Univ. Press. London.
(1972) Skorohod Embedding of Multivariate RV's and the sample DF. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **24** 1–35.
KLASS, M. J.
(1976) Forward a universal law of the iterated logarithm I. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **36** 165–178.
(1977) Forward a universal law of the iterated logarithm II. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **39** 151–165.
KOLMOGOROV, A. N.
(1931) Eine Veralgemeinerung des Laplace–Liapunoffschen Satzes. *Izv. Akad. Nauk SSSR, Ser. Fiz-Mat.* 959–962.
(1933) Sulla determinazione empirica di une legge di distribuzione. *Giorn. Inst. Ital. Attuari* **4** 83–91.
(1933a) Über die Grenzwertsätze der Wahrscheinlichkeitsrechnung. *Izv. Akad. Nauk SSSR, Ser. Fiz-Mat.* 363–372.
KOMLÓS, J.–MAJOR, P.–TUSNÁDY, G.
(1975) An approximation of partial sums of independent R.V.'s and the sample DF. I. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **32** 111–131.
(1975a) Weak convergence and embedding. In *Coll. Math. Soc. J. Bolyai* **11**, *Limit Theorems of Probability Theory* (P. Révész, Ed.), Keszthely, Hungary 1974, North Holland, Amsterdam–London, 149–165.
(1976) An approximation of partial sums of independent R.V.'s and the sample DF. II. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **34** 33–58.
KOMLÓS, J.–TUSNÁDY, G.
(1975) On sequences of "pure heads". *Ann. Probability* **3** 608–617.
KUELBS, J.
(1973) The invariance principle for Banach space valued random variables. *J. Multivariate Anal.* **3** 161–172.
KUIPER, N. H.
(1960) Tests concerning random points on a circle. *Proc. Nederl. Akad. Wetensch. Indag. Math.* (A). **63** 38–47.
LAI, T. L.
(1973) On Strassen Type Laws of the Iterated Logarithm for Delayed Averages of the Wiener Process. *Bull. Inst. Math. Acad. Sinica* **1** 29–39.
(1974) Limit theorems for delayed sums. *Ann. Probability* **2** 432–440.
LÉVY, P.
(1937) *Théorie de l'addition des variables aléatoires.* Gauthier-Villars, Paris.
(1940) Le Mouvement brownien plan. *Amer. J. Math.* **62** 487–550.
(1948) *Procesus stochastique et mouvement Brownien.* Gauthier-Villars, Paris.
LOÉVE, M.
(1963) *Probability Theory,* Third Edition, Van Nostrand–Reinhold, Princeton, N. J.
MAJOR, P.
(1973) On non-parametric estimation of the regression function. *Studia Sci. Math. Hungar.* **8** 347–361.

(1976a) The approximation of partial sums of independent r.v.'s. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **35** 213–220.
(1976b) Approximation of partial sums of i.i.d.r.v.'s when the summands have only two moments. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **35** 221–230.
(1979) An improvement of Strassen's invariance principle. *Ann. Probability* **7** 55–61.
MARCUS, M. B.
(1980) Weak convergence of the empirical characteristic function. *Ann. Probability* **8**.
MICHEL, R.
(1974) Results on probabilities of moderate deviation. *Ann. Probability* **2** 349–353.
MOGUL'SKIĬ, A. A.
(1974) Small deviations in a space of trajectories. *Theor. Probability Appl.* **19** 726–736.
(1977) Laws of iterated logarithm on function spaces (in Russian). In *Abstracts of communications* **2** 44–47, Second Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius 1977.
MOGYORÓDI, J.
(1962) A central limit theorem for the sum of a random number of random variables. *Publ. Math. Inst. Hung. Acad. Sci.* **7** 409–424.
(1965) On the law of large numbers for the sum of a random number of independent random variables. *Annales Univ. Sci. Budapest de Eötvös nom. Sect. Math.* **8** 33–38.
(1966) A remark on stable sequences of random variables and a limit distribution theorem for the sum of independent random variables. *Acta Math. Acad. Sci. Hung.* **17** 401–409.
MÜLLER, D. W.
(1970) On Glivenko–Cantelli convergence. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **16** 195–210.
NADARAYA, E. A.
(1964) On estimating regression. *Theory Probability Appl.* **9** 141–142.
(1965) On non-parametric estimates of density functions and regression. *Theory Probability Appl.* **10** 199–203.
OREY, S.–TAYLOR, S. J.
(1974) How often on a Brownian path does the law of iterated logarithm fail? *Proc. London Math. Soc.* **28** 174–192.
ORLOV, A. I.
(1974) A speed of convergence for the distribution of the Mises–Smirnov statistic (in Russian). *Teorija Verojatn. Primen.* **19** 765–786.
PALEY, R. E. A. C.–WIENER, N.
(1934) *Fourier transforms in the complex domain.* Amer. Math. Soc. Colloq. Publ. **19** Amer. Math. Soc., Providence, R. I.
PARANJAPE, S. R.–PARK, C.
(1973) Laws of iterated logarithm of multiparameter Wiener process. *J. Multivariate Analysis* **3** 132–136.
PARK, W. J.
(1974) On Strassen's version of the law of the iterated logarithm for the two-parameter Gaussian process. *J. Multivariate Analysis* **4** 479–485.
PARTHASARATHY, K. R.
(1967) *Probability measures on metric spaces.* Academic Press, New York.

PARZEN, E.
(1962) On estimation of a probability density function and mode. *Ann. Math. Statit.s* **33** 1065–1076.
PETROV, V. V.
(1975) *Sums of Independent Random Variables.* Springer, Berlin–Heidelberg–New York.
PHILIPP, W.–STOUT, W.
(1975) Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc. No.* **161**.
PROHOROV, YU. V.
(1956) The convergence of random processes and limit theorems in probability theory. *Teorija Verojatn. Primen.* **1** 177–238.
PRUITT, W. E.–OREY, S.
(1973) Sample functions of the *N*-parameter Wiener process. *Ann. Probability* **1** 138–163.
PYKE, R.
(1968) The weak convergence of the empirical process with random sample size. *Proc. Cambridge Philos. Soc.* **64** 155–160.
QUALLS, C.–WATANABE, H.
(1972) Asymptotic properties of Gaussian processes. *Ann. Math. Statist.* **43** 580–596.
RAO, K. C.
(1972) The Kolmogorov, Cramér–von Mises Chisquare statistics for goodness of fit tests in the parametric case. Abstract: *Bulletin, Inst. Math. Statist.* **1** 87.
RÉNYI, A.
(1953) On the theory of order statistics. *Acta Math. Acad. Sci. Hung.* **4** 191–232.
(1958) On mixing sequences of sets. *Acta Math. Acad. Sci. Hung.* **9** 215–228.
(1960) On the central limit theorem for the sum of a random number of independent random variables. *Acta Math. Acad. Sci. Hung.* **11** 97–102.
(1970) *Probability theory.* North Holland, Amsterdam and Akadémiai Kiadó, Budapest.
RÉVÉSZ, P.
(1968) *The laws of large numbers.* Academic Press, New York.
(1972) On empirical density function. *Periodica Math. Hung.* **2** 85–110.
(1976) On strong approximation of the multidimensional empirical process. *Ann. Probability* **4** 729–743.
(1978) Strong theorems on coin tossing. In *Proc. Int. Congress of Mathematicians.* 749–754. Helsinki.
(1979) On the nonparametric estimation of the regression function. *Problems of Control and Information Theory* **8** 297–302.
(1979a) A note to the Chung–Erdős–Sirao theorem. In *Asymptotic Theory of Statistical Tests and Estimation.* 147–158. Academic Press, New York.
(1979b) A generalization of Strassen's functional law of iterated logarithm. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **50** 257–264.
RIESZ, F.–SZ.-NAGY. B.
(1955) *Functional Analysis.* Frederick Ungar, New York.
ROOTZÉN, H.
(1974) Some properties of convergence in distribution of sums and maxima of dependent random variables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **29** 295–307.

ROSENBLATT, M.
(1956) Remarks on some nonparametric estimates of density function. *Ann. Math. Statist.* **27** 832–837.
(1971) Curve estimates. *Ann. Math. Statist.* **42** 1815–1842.
RUBIN, H.–SETHURAMAN, J.
(1965) Probabilities of moderate deviations, *Sankhya*, Ser. A, **27** 325–346.
SARAHAN, A. E.–GREENBERG, B. G.
(1956) Estimation of location and scale parameters by order statistics from singly and doubly censored samples, I. *Ann. Math. Statist.* **27** 427–451.
SCHWARTZ, S. C.
(1967) Estimation of probability density by an orthogonal series. *Ann. Math. Statist.* **38** 1261–1265.
SHAPIRO, S. S.–FRANCIA, R. S.
(1972) An approximate analysis of variance tests for normality. *J. Amer. Statist. Assoc.* **67** 215–216.
SHAPIRO, S. S.–WILK, M. B.
(1965) An analysis of variance test for normality (complete samples). *Biometrika* **52** 591–611.
SHORACK, G. R.
(1972a) Functions of order statistics. *Ann. Math. Statist.* **43** 412–427.
(1972b) Convergence of quantile and spacings processes with applications. *Ann. Math. Statist.* **43** 1400–1411.
SKOROHOD, A. V.
(1956) Limit theorems for random processes. *Teorija Verojatn. Primen.* **1** 289–319.
(1961) *Studies in the Theory of Random Processes.* Addison–Wesley. Reading, Mass.
SMIRNOV, N. V.
(1937) On the distribution of the von Mises ω^2-criterion (in Russian). *Matem Sbornik.* **5** 973–993.
(1939) On the estimation of the discrepancy between empirical curves of distribution for two independent samples. *Bull. Math. de l'Université de Moscou* **2** (Fasc. 2).
(1944) Approximate laws of distribution of random variables from empirical data (in Russian). *Uspehi Mat. Nauk.* **10** 179–206.
SPITZER, F.
(1956) The probability in the tail of a distribution. *Ann. Math. Statist.* **34** 312–318.
STEINEBACH, J.
(1979) *Erdős–Rényi–Zuwächse bei Erneuerungsprozessen und Partialsummen auf Gittern.* Habilitationsschrift, Univ. of Düsseldorf.
STONE, Ch. J.
(1977) Consistent nonparametric regression. *Ann. Statist.* **5** 595–645.
STOUT, W. F.
(1974) *Almost sure convergence.* Academic Press, New York.
STRASSEN, V.
(1964) An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **3** 211–226.
(1965a) The existence of probability measures with given marginals. *Ann. Math. Statist.* **36** 423–439.

(1965b) Almost sure behaviour of sums of independent random variables and martingales. In *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 315–344.
(1966) A converse to the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **4** 265–268.
SZYNAL, D.
(1972) On almost complete convergence for the sum of a random number of independent random variables. *Bull. Acad. Sci. Polon. Ser. Math. Astronom. Phys.* **20** 571–574.
(1973) On the limit behaviour of a sequence of quantiles of a sample with a random number of items. *Applications Matematicae* **13** 321–327.
TAYLOR, S. J.
(1974) Regularity of irregulartities on a Brownian path. *Ann. Inst. Fourier,* Grenoble **24** 195–203.
TUSNÁDY, G.
(1974) On testing density functions. *Periodica Math. Hung.* **5** 161–169.
(1977) Strong invariance principles. In *Recent Developments in Statistics* (J. R. Barra et al., Ed.) 289–300. North-Holland, Amsterdam.
(1977b) *A study of statistical hypotheses.* Dissertation, The Hungarian Academy of Sciences, Budapest.
VAN RYZIN, I.
(1966) Bayes risk consistency of classification procedures using density estimation. *Sankhya, Ser. A.* **28** 261–270.
WICHURA, M. J.
(1970) On the construction of almost uniformly convergent random variables with given weakly convergent image laws. *Ann. Math. Statist.* **41** 284–291.
(1973) Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. *Ann. Probability* **1** 272–296.
WITTENBERG, H.
(1964) Limit distributions of random sums of independent random variables. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **3** 7–18.
YEH, I.
(1960) Wiener measure in a space of functions of two variables. *Trans. Amer. Math. Soc.* **95** 433–450.
(1973) *Stochastic processes and the Wiener integral.* Marcel Dekker, New York.
ZIMMERMANN, G.
(1972) Some sample function properties of the two-parameter Gaussian process. *Ann. Math. Statist.* **43** 1235–1246.
ZYGMUND, A.
(1968) *Trigonometric Series,* Volumes I, II. Cambridge Univ. Press, London.

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