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Some Nonempty Intersection Theorems in Generalized Interval Spaces with Applications*

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In this paper some parametric types of KKM theorems are established in generalized interval spaces. As applications, we utilize these results to obtain some new minimax theorems, section theorems, and existence theorems of solutions for variational inequalities. The results presented in this paper not only include some important earlier results and the famous von Neumann theorems as their special cases, but also improve and extend other corresponding results.

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1. PRELIMINARIES

DEFINITION 1. A topological space X is called a generalized interval space, if there exists a mapping $\Gamma: X \times X \to \mathscr{A}(X)$, where $\mathscr{A}(X)$ is a family of nonempty connected subsets of X. For any $(x_1, x_2) \in X \times X$ we denote $\Gamma(x_1, x_2)$ by $\Gamma\{x_1, x_2\}$ and $\Gamma\{x_1, x_2\}$ is called a generalized interval associated with x_1 and x_2.

DEFINITION 2. Let X be a generalized interval space, Y be a topological space, and Z be a completely dense linear ordered space. A subset B of X is called T-convex, if for any $x_1, x_2 \in B$, we have $\Gamma\{x_1, x_2\} \subset B$.

A mapping $f: X \to Z$ is called T-quasi-concave (convex), if for any $z \in Z$, the set $\{x \in X: f(x) \geq z\}$ ($\{x \in X: f(x) \leq z\}$) is T-convex.

Let D be a subset of X. If for any $\Gamma\{x_1, x_2\}$, $D \cap \Gamma\{x_1, x_2\}$ is a closed (open) set in $\Gamma\{x_1, x_2\}$, then D is called a generalized interval closed (open) set.

Remark. From the above definitions, it is easy to see that interval space, H-space, convex space, contractible space, and topological linear space all can be considered as special cases of generalized interval space. Moreover the H-convex set in H-space is T-convex; the convex sets in topological vector space are also T-convex and the quasi-concave (convex) mapping is T-quasi-concave (convex).

We have to point out that the intersection of T-convex sets is T-convex and each T-convex set can be considered as a generalized interval space.

DEFINITION 3 [4]. Let X be a topological space, Z be an order complete and order dense linear ordered space. A mapping $f: X \to Z$ is called upper (lower) semi-continuous, if the set $\{x \in X: f(x) \geq z\}$ ($\{x \in X: f(x) \leq z\}$) is a closed set in X for all $z \in Z$.

DEFINITION 4 [4]. Let X and Y be two topological spaces and $\mathscr{S}: Y \times X \times Y \to R$ be a function. $T: Y \to 2^X$ is called $\mathscr{S}$-monotone, if for any $y, z \in Y$, for any $u \in Ty$, and for any $v \in Tz$

$$\mathscr{S}(y, u, z) - \mathscr{S}(y, v, z) \geq 0.$$

DEFINITION 5 [12]. Let X and Y be two topological spaces. A mapping $G: Y \to 2^X$ is called transfer closed valued in Y if for any $y \in Y$, $x \notin G(y)$, then there exists an $y' \in Y$ such that $x \notin \overline{G(y')}$ (in the sequel we denote by $\bar{A}$ the closure of A).

Remark. It is obvious that if G is closed valued in Y, then G is transfer closed valued on Y.

In the sequel we denote $\mathscr{F}(Y) := \{A \subset Y: A \text{ is a nonempty finite set}\}$.

2. PARAMETRIC TYPE OF KKM THEOREMS IN GENERALIZED INTERVAL SPACES

In the sequel we need the following auxiliary lemmas:

LEMMA 2.1 [12]. *Let X, Y be two topological spaces and $G: Y \to 2^X$ be a multivalued mapping. Then G is transfer closed valued on Y if and only if $\bigcap_{y \in Y} \overline{G(y)} = \bigcap_{y \in Y} G(y)$.*

Proof. If G is transfer closed valued on Y, it is obvious that $\bigcap_{y \in Y} \overline{G(y)} \supset \bigcap_{y \in Y} G(y)$. To prove the conclusion we have to prove $\bigcap_{y \in Y} \overline{G(y)} \subset \bigcap_{y \in Y} G(y)$. Suppose the contrary, then there exists $x \in \bigcap_{y \in Y} \overline{G(y)}$ and $x \notin \bigcap_{y \in Y} G(y)$. Hence there exists $y \in Y$ such that $x \notin G(y)$. Since G is transfer closed valued on Y, there exists $y' \in Y$ such that $x \notin \overline{G(y')}$. This contradicts $x \in \bigcap_{y \in Y} \overline{G(y)}$.

Conversely, if $\bigcap_{y \in Y} \overline{G(y)} = \bigcap_{y \in Y} G(y)$, then for any $y \in Y$ and any $x \notin G(y)$, then $x \notin \bigcap_{y \in Y} G(y) = \bigcap_{y \in Y} \overline{G(y)}$. Hence there exists $y' \in Y$ such that $x \notin \overline{G(y')}$. This implies that G is transfer closed valued on Y.

LEMMA 2.2. *Let Y be a generalized interval space, X be a topological space, and $F: Y \to 2^X$ be a mapping. Then for any $x \in X$, $Y \setminus F^{-1}(x)$ is a T-convex set if and only if for any $y_1, y_2 \in Y$ we have*

$$F(y) \subset F(y_1) \cup F(y_2) \qquad \textit{for all } y \in \Gamma\{y_1, y_2\}.$$

Proof. Necessity. Suppose the contrary, there exist $y_1, y_2 \in Y$ and an $y_0 \in \Gamma\{y_1, y_2\}$ such that $F(y_0) \not\subset F(y_1) \cup F(y_2)$. Hence there exists $x_0 \in F(y_0)$ but $x_0 \notin F(y_i)$, $i = 1, 2$, and so $y_i \notin F^{-1}(x_0)$, $i = 1, 2$, i.e., $\{y_1, y_2\} \subset Y \setminus F^{-1}(x_0)$. Since $Y \setminus F^{-1}(x_0)$ is T-convex, $\Gamma\{y_1, y_2\} \subset Y \setminus F^{-1}(x_0)$. Hence $y_0 \in Y \setminus F^{-1}(x_0)$, and so $x_0 \notin F(y_0)$. This contradicts the choice of x_0. Therefore for any $y_1, y_2 \in Y$, we have

$$F(y) \subset F(y_1) \cup F(y_2) \qquad \text{for all } y \in \Gamma\{y_1, y_2\}.$$

Sufficiency. Since for any $y_1, y_2 \in Y$, we have $F(y) \subset F(y_1) \cup F(y_2)$ for all $y \in \Gamma\{y_1, y_2\}$. Hence for any $x \in X$ and for any $\hat{y}_1, \hat{y}_2 \in Y \setminus F^{-1}(x)$, $F(\hat{y}) \subset F(\hat{y}_1) \cup F(\hat{y}_2)$ for all $\hat{y} \in \Gamma\{\hat{y}_1, \hat{y}_2\}$. Since $\hat{y}_1, \hat{y}_2 \notin F^{-1}(x)$, i.e., $x \notin F(\hat{y}_1)$ and $x \notin F(\hat{y}_2)$, and so $x \notin F(\hat{y}_1) \cup F(\hat{y}_2)$. This implies that $x \notin F(\hat{y})$ for all $\hat{y} \in \Gamma\{y_1, y_2\}$, i.e., for all $\hat{y} \in \Gamma\{y_1, y_2\}$, $y \in Y \setminus F^{-1}(x)$. Therefore $\Gamma\{y_1, y_2\} \subset Y \setminus F^{-1}(x)$. This shows that $Y \setminus F^{-1}(x)$ is a T-convex set for all $x \in X$. This completes the proof.

THEOREM 2.3. *Let Y be a generalized interval space, X be a topological space, Z be a linear ordered space, $F, G: Y \times Z \to 2^X$ be two mappings such that F has nonempty values and G is transfer closed valued. If the following* *conditions are satisfied:*

(i) *for any* $(y, z) \in Y \times Z$, $F(y, z) \subset \overline{G(y, z)}$ *and when* $z_1 \leq z_2$,

$$F(y, z_2) \subset F(y, z_1) \qquad \text{for all } y \in Y;$$

(ii) *for any* $z \in Z$, *there exists* $\hat{z} \in Z$ *such that* $F(y, z) \supset \overline{G(y, \hat{z})}$ *for all* $y \in Y$;

(iii) *for any* $A \in \mathscr{F}(Y)$ *and for any* $z \in Z$, $\bigcap_{y \in A} F(y, z)$ *is connected;*

(iv) *for any* $x \in X$ *and for any* $z \in Z$, *the set* $\{y \in Y: x \notin F(y, z)\}$ *is T-convex and generalized interval closed;*

(v) *for any* $z \in Z$ *and any* $y_1, y_2 \in Y$, *there exist* $y_1', y_2' \in \Gamma\{y_1, y_2\}$, *such that* $F(y_1', z) \subset F(y_1, z)$, $F(y_2', z) \subset F(y_2, z)$, *then*

(1) $\{\overline{G(y, z)}: y \in Y, z \in Z\}$ *has the finite intersection property;*

(2) *if there exists* $(y_0, z_0) \in Y \times Z$ *such that* $\overline{G(y_0, z_0)}$ *is compact, then* $\bigcap_{y \in Y,\, z \in Z} G(y, z) \neq \varnothing$.

Proof. Since F has nonempty values, by condition (i), for all $(y, z) \in Y \times Z$, $\overline{G(y, z)} \neq \varnothing$. If for any n elements of $\{\overline{G(y, z)}: y \in Y, z \in Z\}$ their intersection is nonempty, next we prove that for any $n + 1$ elements of $\{\overline{G(y, z)}: y \in Y, z \in Z\}$ their intersection is also nonempty, where $n \geq 2$.

In fact, if there exist $(y_i, z_i) \in Y \times Z$, $i = 1, 2, \ldots, n + 1$, such that $\bigcap_{i=1}^{n+1} \overline{G(y_i, z_i)} = \varnothing$. Since Z is a linear ordered space, without loss of generality we can assume that $z_1 \geq z_2 \geq \cdots \geq z_n \geq z_{n+1}$, letting $H = \bigcap_{i=3}^{n+1} F(y_i, z_1)$, then we have

$$\overline{H \cap F(y_1, z_1)} \cap \overline{H \cap F(y_2, z_1)} \subset \overline{H} \cap \overline{F(y_1, z_1)} \cap \overline{F(y_2, z_2)}$$
$$\subset \bigcap_{i=1}^{n+1} \overline{G(y_i, z_i)} = \varnothing.$$

This implies that $H \cap F(y_1, z_1)$ and $H \cap F(y_2, z_1)$ are separated from each other. By condition (ii) and the assumptions of induction, there exists $\hat{z} \in Z$ such that for any $y \in Y$

$$H \cap F(y, z_1) = \bigcap_{i=3}^{n+1} F(y_i, z_1) \cap F(y, z_1)$$
$$\supset \bigcap_{i=3}^{n+1} \overline{G(y_i, \hat{z})} \cap \overline{G(y, \hat{z})} \neq \varnothing.$$

Define a mapping $L: Y \to 2^X$ by $L(y) := F(y, z_1)$, $y \in Y$, then for any $x \in X$, $Y \setminus L^{-1}(x) = \{y \in Y: x \notin F(y, z_1)\}$. By condition (iv) for any $x \in X$, $Y \setminus L^{-1}(x)$ is T-convex. In view of Lemma 2.2 for any $y_1, y_2 \in Y$ and for any $y \in \Gamma\{y_1, y_2\}$ we have $L(y) \subset L(y_1) \cup L(y_2)$, i.e.,

$$F(y, z_1) \subset F(y_1, z_1) \cup F(y_2, z_1) \qquad \text{for all } y \in \Gamma\{y_1, y_2\}.$$

Hence we have

$$H \cap F(y, z_1) \subset (H \cap F(y_1, z_1)) \cup (H \cap F(y_2, z_1)) \qquad \text{for all } y \in \Gamma\{y_1, y_2\}.$$

By condition (iii), $H \cap F(y, z_1)$ is connected. In addition, since $H \cap F(y_1, z_1)$ and $H \cap F(y_2, z_1)$ are separated from each other, for any $y \in \Gamma\{y_1, y_2\}$

$$H \cap F(y, z_1) \subset H \cap F(y_1, z_1) \quad \text{or} \quad H \cap F(y, z_1) \subset H \cap F(y_2, z_1).$$

Letting $E_i = \{y \in \Gamma\{y_1, y_2\}: H \cap F(y, z_1) \subset H \cap F(y_i, z_1)\}$, $i = 1, 2$, by condition (v), $E_i \neq \varnothing$, $i = 1, 2$, and $\Gamma\{y_1, y_2\} = E_1 \cup E_2$. Since $\Gamma\{y_1, y_2\}$ is nonempty connected, at least one of $E_1 \cap \overline{E}_2$ and $\overline{E}_1 \cap E_2$ is nonempty. Without loss of generality we can assume that $E_1 \cap \overline{E}_2 \neq \varnothing$. Taking $y_0 \in E_1 \cap \overline{E}_2$, we have $H \cap F(y_0, z_1) \subset H \cap F(y_1, z_1)$ and there exists a net $\{y_\alpha\}_{\alpha \in I} \subset E_2$ such that $y_\alpha \to y_0$. Hence we have

$$H \cap F(y_\alpha, z_1) \subset H \cap F(y_2, z_1) \qquad \text{for all } \alpha \in I.$$

On the other hand, in the above we have proved that $H \cap F(y_0, z_1) \neq \varnothing$. Taking $x_0 \in H \cap F(y_0, z_1)$, we know that $x_0 \notin H \cap F(y_2, z_1)$. Hence

$$x_0 \notin H \cap F(y_\alpha, z_1) \qquad \text{for all } \alpha \in I.$$

Therefore $x_0 \notin F(y_\alpha, z_1)$ for all $\alpha \in I$, i.e., $\{y_\alpha\}_{\alpha \in I} \subset Y \setminus L^{-1}(x_0)$. Since $\{y_\alpha\}_{\alpha \in I} \subset \Gamma\{y_1, y_2\}$ and $Y \setminus L^{-1}(x_0)$ is generalized interval closed, $y_0 \in Y \setminus L^{-1}(x_0)$, i.e., $x_0 \notin F(y_0, z_1)$. This contradicts the choice of x_0. By this contradiction, we know that $\{\overline{G(y, z)}: y \in Y, z \in Z\}$ has the finite intersection property.

If, in addition, there exists $(y_0, z_0) \in Y \times Z$ such that $\overline{G(y_0, z_0)}$ is compact, then it is easy to prove that $\bigcap_{y \in Y}, \bigcap_{z \in Z} \overline{G(y, z)} \neq \varnothing$. By Lemma 2.1 we have $\bigcap_{y \in Y}, \bigcap_{z \in Z} G(y, z) \neq \varnothing$.

This completes the proof.

COROLLARY 2.4. *Let Y be a generalized interval space, X be a topological space, Z be a linear ordered space, and F and G:* $Y \times Z \to 2^X$ *be two mappings such that F has nonempty values and G has closed values. If the following conditions are satisfied:*

(i) *for any* $(y, z) \in Y \times Z$, $F(y, z) \subset G(y, z)$, *and when* $z_1 \leq z_2$ *we have*

$$F(y, z_2) \subset F(y, z_1) \qquad \text{for all } y \in Y;$$

(ii) *for any* $z \in Z$ *there exists a* $\hat{z} \in Z$ *such that* $F(y,z) \supset G(y,\hat{z})$ *for all* $y \in Y$;

(iii) *for any* $A \in \mathscr{F}(Y)$ *and for any* $z \in Z$, $\bigcap_{y \in A} F(y,z)$ *is connected;*

(iv) *for any* $x \in X$ *and for any* $z \in Z$, $\{y \in Y: x \notin F(y,z)\}$ *is T-convex and generalized interval closed;*

(v) *for any* $z \in Z$ *and for any* $y_1, y_2 \in Y$ *there exist* $y'_1, y'_2 \in \Gamma\{y_1, y_2\}$ *such that* $F(y'_1, z) \subset F(y_1, z)$, $F(y'_2, z) \subset F(y_2, z)$, *then*

(1) $\{G(y,z): y \in Y, z \in Z\}$ *has the finite intersection property;*

(2) *in addition, if there exists* $(y_0, z_0) \in Y \times Z$ *such that* $G(y_0, z_0)$ *is compact, then* $\bigcap_{y \in Y} \bigcap_{z \in Z} G(y,z) \neq \varnothing$.

Proof. Since G has closed values, G is transfer closed valued on $Y \times Z$. Hence the conclusion follows from Theorem 2.3 immediately.

3. APPLICATIONS TO MINIMAX PROBLEMS

In this section we shall use the results presented in Section 2 to study the minimax problems. We have the following results:

THEOREM 3.1. *Let* Y *be a generalized interval space,* X *be a topological space,* $\tilde{Z}$ *be an order complete and order dense linear ordered space, and* f, g: $X \times Y \to \tilde{Z}$ *be two functions satisfying the following conditions:*

(i) *for any* $A \in \mathscr{F}(Y)$ *and for any* $z \in \tilde{Z}$, $\bigcap_{y \in A}\{x \in X: f(x,y) > z\}$ *is a connected set;*

(ii) (a) *for any* $x \in X$, $y \mapsto f(x,y)$ *is T-quasi-convex and is lower-semi-continuous on any generalized interval of* Y;

(b) *for any* $y \in Y$, $x \mapsto f(x,y)$ *and* $x \mapsto g(x,y)$ *are upper semicontinuous;*

(iii) *there exist* $z_0 < \inf_{y \in Y} \sup_{x \in X} f(x,y)$ *and* $y_0 \in Y$ *and a compact subset* $L \subset X$ *such that* $g(x, y_0) < z_0$ *for all* $x \in X \setminus L$;

(iv) $f(x,y) \le g(x,y)$ *for all* $(x,y) \in X \times Y$;

(v) *for any* $y_1, y_2 \in Y$ *there exist* $y'_1, y'_2 \in \Gamma\{y_1, y_2\}$ *such that* $f(x, y'_i) \le f(x, y_i)$, $i = 1, 2$, *and for any* $x \in X$.

Then $z_* := \sup_{x \in X} \inf_{y \in Y} g(x,y) \ge \inf_{y \in Y} \sup_{x \in X} f(x,y) =: z^*$.

Proof. By the completeness of $\tilde{Z}$, we know that z^* and z_* both exist. Letting $Z = \{z \in \tilde{Z}: z < z^*\}$, then Z is a dense linear ordered space. For any $(y,z) \in Y \times Z$, letting $F(y,z) = \{x \in X: f(x,y) > z\}$, $G(y,z) = \{x \in X: f(x,y) \ge z\}$, $H(y,z) = \{x \in X: g(x,y) \ge z\}$, it follows from condition (ii)(b) that $G(y,z)$ and $H(y,z)$ both are closed for all $(y,z) \in Y \times Z$.

Next we prove that mappings F and G satisfy all conditions of Corollary 2.4. In fact, by the definition of z^*, $F: Y \times Z \to 2^X$ is nonempty valued. Again by the definition of F and G, it is easy to see that they satisfy condition (i) in Corollary 2.4. In addition, for any $z \in Z$ with $z < z^*$, by the denseness of $\tilde{Z}$, there exists a $\hat{z} \in \tilde{Z}$ such that $z < \hat{z} < z^*$ and so $\hat{z} \in Z$ and $F(y,z) \supset G(y,\hat{z})$ for all $y \in Y$. This implies that condition (ii) in Corollary 2.4 is satisfied. Again for any $x \in X$ and for any $z \in Z$

$$\{y \in Y: x \notin F(y,z)\} = \{y \in Y: f(x,y) \le z\}.$$

By condition (ii)(a) we know that condition (iv) in Corollary 2.4 is satisfied. By conditions (i) and (v) we know that conditions (iii) and (v) are satisfied. By conclusion (1) of Corollary 2.4, $\{G(y,z): y \in Y, z \in Z\}$ has the finite intersection property.

On the other hand, by condition (iv), for any $(y,z) \in Y \times Z$, $G(y,z) \subset H(y,z)$. Hence $\{H(y,z): y \in Y, z \in Z\}$ is a family of closed sets having the finite intersection property. By condition (iii), for any $x \in X \setminus L$, $g(x, y_0) < z_0$, i.e., $x \in \{x \in X: g(x,y_0) < z_0\}$. Hence $x \notin H(y_0, z_0) = \{x \in X: g(x, y_0) \ge z_0\}$ and so $H(y_0, z_0) \subset L$. Since L is compact and $H(y_0, z_0)$ is closed, $H(y_0, z_0)$ is compact. Hence we have

$$\bigcap_{y \in Y} \bigcap_{z \in Z} H(y,z) = \bigcap_{y \in Y} \bigcap_{z \in Z} H(y,z) \cap H(y_0, z_0) \neq \varnothing.$$

Taking $\hat{x} \in \bigcap_{y \in Y} \bigcap_{z \in Z} H(y,z)$, we have $g(\hat{x}, y) \ge z$ for all $y \in Y$ and for all $z \in Z$. Hence $z_* = \sup_{x \in X} \inf_{y \in Y} g(x,y) \ge z$ for all $z \in Z$. By the denseness of $\tilde{Z}$ we have

$$\sup_{x \in X} \inf_{y \in Y} g(x,y) \ge z^* = \inf_{y \in Y} \sup_{x \in X} f(x,y).$$

This completes the proof.

COROLLARY 3.2. *Let* Y *be a generalized interval space,* X *be a topological space, and* Z *be an order complete and order dense linear ordered space. If* f: $X \times Y \to Z$ *satisfies the following conditions:*

(i) *for any* $A \in \mathscr{F}(Y)$ *and for any* $z \in Z$, $\bigcap_{y \in A}\{x \in X: f(x,y) > z\}$ *is connected;*

(ii) (a) *for any* $x \in X$, $y \mapsto f(x,y)$ *is T-quasi-convex, and is lower semi-continuous on any generalized interval of* Y;

(b) *for any* $y \in Y$, $x \mapsto f(x,y)$ *is upper semi-continuous;*

(iii) *there exist* $z_0 < \inf_{y \in Y} \sup_{x \in X} f(x,y)$, $y_0 \in Y$, *and a compact subset* $L \subset X$ *such that* $f(x, y_0) < z_0$ *for all* $x \in X \setminus L$;

(iv) *for any* $y_1, y_2 \in Y$ *there exist* $y'_1, y'_2 \in \Gamma\{y_1, y_2\}$ *such that*

$$f(x, y'_i) \le f(x, y_i), \qquad i = 1, 2 \text{ and for all } x \in X,$$

then $\sup_{x \in X} \inf_{y \in Y} f(x,y) = \inf_{y \in Y} \sup_{x \in X} f(x,y)$.

Proof. The conclusion of Corollary 3.2 can be obtained from Theorem 3.1 immediately.

Remark. The condition "there exist $z_0 < \inf_{y \in Y} \sup_{x \in X} f(x, y)$, $y_0 \in Y$, and a compact subset $L \subset X$ such that $f(x, y_0) < z_0$ for all $x \in X \setminus L$" is equivalent to the condition "there exist $z_0 < \inf_{y \in Y} \sup_{x \in X} f(x, y)$ and $y_0 \in Y$ such that $\{x \in X: f(x, y_0) \geq z_0\}$ is compact." Therefore Corollary 3.2 includes the main results of Cheng and Lin [3] as its special cases and so it contains the corresponding results of Brézis, Nirenberg, and Stampacchia [2], Komornik [7], M. A. Geraghty and Lin [9], Stachó [11], and Wu [13] as its special cases.

COROLLARY 3.3. *Let Y be a compact generalized interval space, X be a compact topological space, Z be an order complete and order dense linear ordered space, and f: $X \times Y \to Z$ be a mapping satisfying the following conditions:*

(i) *for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A}\{x \in X: f(x, y) > z\}$ is connected or empty;*

(ii) *for any $x \in X$, $f(x, y)$ is T-quasi-convex and lower semi-continuous in y;*

(iii) *for any $y \in Y$, $x \mapsto f(x, y)$ is upper semi-continuous;*

(iv) *for any $y_1, y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that $f(x, y_i') \leq f(x, y_i)$ for all $x \in X$, $i = 1, 2$.*

Then f has a saddle $(\hat{x}, \hat{y}) \in X \times Y$.

Proof. By using Corollary 3.2 and Proposition 1.4.6 and Theorem 3.10.4 of [4], we can obtain the conclusion of Corollary 3.3 immediately.

Remark. Corollary 3.3 contains the famous von Neumann theorem in mathematical economy and game theory.

THEOREM 3.4. *Let Y be a generalized interval space, X be a topological space, Z be an order complete and order dense linear ordered space, and f and g: $X \times Y \to Z$ be two mappings satisfying $f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$ and the following conditions:*

(i) *for any $x \in X$, $y \mapsto f(x, y)$ is T-quasi-convex and lower semi-continuous;*

(ii) *for any $y \in Y$, $f(x, y)$ and $g(x, y)$ are upper semi-continuous in x;*

(iii) *there exist a nonempty subset $K \subset X$ and a compact subset $H \subset Y$ such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus H} \sup_{x \in K} f(x, y),$$

and for any finite subset $F \subset X$ there exists a compact set $K(F) \supset K \cup F$, such that for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A}\{x \in K(F): f(x, y) > z\}$ is connected;

(iv) *for any $y_1, y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that $f(x, y_i') \leq f(x, y_i)$ for all $x \in X$, $i = 1, 2$.*

Then $\sup_{x \in X} \inf_{y \in Y} g(x, y) \geq \inf_{y \in Y} \sup_{x \in X} f(x, y)$.

Proof. Letting $z_* = \sup_{x \in X} \inf_{y \in Y} g(x, y)$ and $z^* = \inf_{y \in Y} \sup_{x \in X} f(x, y)$, by the completeness of Z, we know that z_* and z^* both exist. If $z_* < z^*$, again by the density of Z, there exists $\hat{z} \in Z$ such that $z_* < \hat{z} < z^*$.

For any $x \in X$, letting $L(x) = \{y \in Y: f(x, y) \leq \hat{z}\}$, by condition (i), $L(x)$ is a closed set in Y. For any $x \in X$, letting $M(x) = L(x) \cap (\bigcap_{t \in K} L(t))$, then $M(x)$ is a closed set.

Next, we prove that $M(x) \subset H$ for all $x \in X$.

In fact, for any $y_0 \in Y \setminus H$, by condition (iii), we know that $\sup_{x \in K} f(x, y_0) \geq z^* > \hat{z}$. Therefore there exists a $x_0 \in K$ such that $f(x_0, y_0) > \hat{z}$, i.e., $y_0 \notin L(x_0)$ and so $y_0 \notin \bigcap_{t \in K} L(t)$. This implies that $\bigcap_{t \in K} L(t) \subset H$. Hence $M(x) \subset H$ for all $x \in X$.

Finally, we prove that $\{M(x): x \in X\}$ has the finite intersection property. In fact, for any finite set $F \subset X$, by condition (iii), there exists a compact subset $K(F) \supset K \cup F$ such that for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A}\{x \in K(F):: f(x, y) > z\}$ is connected. From Theorem 3.1 we have

$$\sup_{x \in K(F)} \inf_{y \in Y} g(x, y) \geq \inf_{y \in Y} \sup_{x \in K(F)} f(x, y),$$

and so we have

$$\inf_{y \in Y} \sup_{x \in K(F)} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y) = z^* < \hat{z}.$$

Now we prove that $\bigcap_{x \in K(F)} M(x) \neq \varnothing$. Suppose the contrary, $\bigcap_{x \in K(F)} M(x) = \varnothing$, then we have $Y = \bigcup_{x \in K(F)} (Y \setminus M(x))$, and so for any $y \in Y$, there exists $x(y) \in K(F)$ such that $y \in Y \setminus M(x(y))$, i.e., $y \notin M(x(y)) = L(x(y)) \cap (\bigcap_{t \in K} L(t))$. Hence $y \notin L(x(y))$ or $y \notin \bigcap_{t \in K} L(t)$. If $y \notin L(x(y))$, then $f(x(y), y) > \hat{z}$; if $y \notin \bigcap_{t \in K} L(t)$, then there exists $\hat{x}(y) \in K \subset K(F)$ such that $y \notin L(\hat{x}(y))$, i.e., $f(\hat{x}(y), y) > \hat{z}$. This implies that there exists a mapping $\hat{x}$: $Y \to K(F)$ such that $f(\hat{x}(y), y) > \hat{z}$. Therefore we have $\sup_{x \in K(F)} f(x, y) > \hat{z}$ for all $y \in Y$. Hence $\inf_{y \in Y} \sup_{x \in K(F)} f(x, y) \geq \hat{z}$. This contradicts $\inf_{y \in Y} \sup_{x \in K(F)} f(x, y) < \hat{z}$. Therefore $\bigcap_{x \in K(F)} M(x) \neq \varnothing$. Since $\bigcap_{x \in F} M(x) \supset \bigcap_{x \in K(F)} M(x)$, $\{M(x): x \in X\}$ has the finite intersection property.

However, since H is compact and $M(x) \subset H$ for all $x \in X$, $\bigcap_{x \in X} M(x) \neq \varnothing$. Hence there exists a $\hat{y} \in M(x)$ for all $x \in X$, and so $\hat{y} \in L(x)$ for all $x \in X$, i.e., $f(x, \hat{y}) \leq \hat{z}$ for all $x \in X$. Hence $z^* \leq \hat{z}$. This contradicts the choice of $\hat{z}$. Therefore $z_* \geq z^*$. This completes the proof.

Remark. Theorem 3.4 improves and extends Theorem 2 of Lin and Quan [8].

COROLLARY 3.5. *Let Y be a generalized interval space, X be a topological space, Z be an order complete and order dense linear ordered space, and f: $X \times Y \to Z$ be a mapping satisfying the following conditions:*

(i) *for any $x \in X$, $f(x, y)$ is T-quasi-convex and lower semi-continuous in y;*

(ii) *for any $y \in Y$, $x \mapsto f(x, y)$ is upper semi-continuous;*

(iii) *there exist a nonempty subset $K \subset X$ and a compact subset $H \subset Y$ such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus H} \sup_{x \in K} f(x, y),$$

and for any finite set $F \subset X$ there exists a compact subset $K(F) \supset K \cup F$ such that for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A}\{x \in K(F)\colon f(x, y) > z\}$ is connected;

(iv) *for any $y_1, y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that*

$$f(x, y_i') \leq f(x, y_i) \qquad \text{for all } x \in X,\ i = 1, 2.$$

Then $\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y)$.

Proof. It follows from Theorem 3.4 that

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \geq \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

However, it is obvious that $\inf_{y \in Y} \sup_{x \in X} f(x, y) \geq \sup_{x \in X} \inf_{y \in Y} f(x, y)$. Hence the conclusion is obtained.

Remark. Since the condition "for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A}\{x \in K(F)\colon f(x, y) \geq z\}$ is connected" implies the condition "for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A}\{x \in K(F)\colon f(x, y) > z\}$ is connected," therefore Corollary 3.5 contains Corollary 1 of Lin and Quan [8] as its special case.

COROLLARY 3.6 [6]. *Let X and Y be nonempty convex subsets of Hausdorff topological vector spaces M and N, respectively, and f: $X \times Y \to R$ be a function satisfying the following conditions:*

(i) *for any $x \in X$, $y \mapsto f(x, y)$ is quasi-convex and lower semi-continuous;*

(ii) *for any y, $x \mapsto f(x, y)$ is quasi-concave and upper semi-continuous;*

(iii) *there exists a nonempty compact convex subset $K \subset X$ and a compact subset $H \subset Y$ such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus H} \sup_{x \in K} f(x, y).$$

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y)$.

Proof. For any $x_1, x_2 \in X$ and for any $y_1, y_2 \in Y$, letting $\Gamma\{x_1, x_2\} = \mathrm{co}\{x_1, x_2\}$, $\Gamma\{y_1, y_2\} = \mathrm{co}\{y_1, y_2\}$, under the structures like this, X and Y both are generalized interval space. Furthermore for any finite subset $F \subset X$, letting $K(F) = \mathrm{co}(K \cup F)$, then all conditions in Corollary 3.5 are satisfied. The conclusion follows from Corollary 3.5 immediately.

4. APPLICATIONS TO SECTION PROBLEMS

In this section, we shall use the results presented in Section 2 to study the section problems. We have the following results.

THEOREM 4.1. *Let Y be a generalized interval space, X be a topological space, Z be a dense linear ordered space, and B and C be two sets of $X \times Y \times Z$ satisfying the following conditions:*

(i) *$B \subset C$;*

(ii) *for any $(y, z) \in Y \times Z$, the sections*

$$B_{(y,z)} := \{x \in X\colon (x, y, z) \in B\} \neq \varnothing,$$

and

$$C_{(y,z)} := \{x \in X\colon (x, y, z) \in C\}$$

are transfer closed valued and there exists $(y_0, z_0) \in Y \times Z$ such that $\overline{C}_{(y_0, z_0)}$ is compact;

(iii) *if $z_2 \geq z_1$, then $B_{(y, z_2)} \subset B_{(y, z_1)}$ for all $y \in Y$; and for any $z \in Z$ there exists $\hat{z} \in Z$ such that $B_{(y,z)} \supset C_{(y, \hat{z})}$ for all $y \in Y$;*

(iv) *for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A} B_{(y,z)}$ is connected;*

(v) *for any $(x, z) \in X \times Z$, the set $Y \setminus B_{(x,z)} := \{y \in Y\colon (x, y, z) \notin B\}$ is generalized interval closed and T-convex in Y;*

(vi) *for any $z \in Z$ and for any $y_1, y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that $B_{(y', z)} \subset B_{(y_i, z)}$, $i = 1, 2$.*

However, since H is compact and $M(x) \subset H$ for all $x \in X$, $\bigcap_{x \in X} M(x) \neq \emptyset$. Hence there exists a $\hat{y} \in M(x)$ for all $x \in X$, and so $\hat{y} \in L(x)$ for all $x \in X$, i.e., $f(x, \hat{y}) \leq \hat{z}$ for all $x \in X$. Hence $z^* \leq \hat{z}$. This contradicts the choice of $\hat{z}$. Therefore $z_* \geq z^*$. This completes the proof.

Remark. Theorem 3.4 improves and extends Theorem 2 of Lin and Quan [8].

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(ii) *for any $y \in Y$, $x \mapsto f(x, y)$ is upper semi-continuous;*

(iii) *there exist a nonempty subset $K \subset X$ and a compact subset $H \subset Y$ such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus H} \sup_{x \in K} f(x, y),$$

and for any finite set $F \subset X$ there exists a compact subset $K(F) \supset K \cup F$ such that for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A} \{x \in K(F): f(x, y) > z\}$ is connected;

(iv) *for any $y_1, y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that*

$$f(x, y_i') \leq f(x, y_i) \qquad \textit{for all } x \in X,\ i = 1, 2.$$

Then $\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y)$.

Proof. It follows from Theorem 3.4 that

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \geq \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

However, it is obvious that $\inf_{y \in Y} \sup_{x \in X} f(x, y) \geq \sup_{x \in X} \inf_{y \in Y} f(x, y)$. Hence the conclusion is obtained.

Remark. Since the condition "for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A} \{x \in K(F): f(x, y) \geq z\}$ is connected" implies the condition "for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A} \{x \in K(F): f(x, y) > z\}$ is connected," therefore Corollary 3.5 contains Corollary 1 of Lin and Quan [8] as its special case.

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(ii) *for any y, $x \mapsto f(x, y)$ is quasi-concave and upper semi-continuous;*

(iii) *there exists a nonempty compact convex subset $K \subset X$ and a compact subset $H \subset Y$ such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \inf_{y \in Y \setminus H} \sup_{x \in K} f(x, y).$$

Then $\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y)$.

Proof. For any $x_1, x_2 \in X$ and for any $y_1, y_2 \in Y$, letting $\Gamma\{x_1, x_2\} = \mathrm{co}\{x_1, x_2\}$, $\Gamma\{y_1, y_2\} = \mathrm{co}\{y_1, y_2\}$, under the structures like this, X and Y both are generalized interval space. Furthermore for any finite subset $F \subset X$, letting $K(F) = \mathrm{co}(K \cup F)$, then all conditions in Corollary 3.5 are satisfied. The conclusion follows from Corollary 3.5 immediately.

4. APPLICATIONS TO SECTION PROBLEMS

In this section, we shall use the results presented in Section 2 to study the section problems. We have the following results.

THEOREM 4.1. *Let Y be a generalized interval space, X be a topological space, Z be a dense linear ordered space, and B and C be two sets of $X \times Y \times Z$ satisfying the following conditions:*

(i) *$B \subset C$;*

(ii) *for any $(y, z) \in Y \times Z$, the sections*

$$B_{(y,z)} := \{x \in X: (x, y, z) \in B\} \neq \emptyset,$$

and

$$C_{(y,z)} := \{x \in X: (x, y, z) \in C\}$$

are transfer closed valued and there exists $(y_0, z_0) \in Y \times Z$ such that $\overline{C}_{(y_0, z_0)}$ is compact;

(iii) *if $z_2 \geq z_1$, then $B_{(y, z_2)} \subset B_{(y, z_1)}$ for all $y \in Y$; and for any $z \in Z$ there exists $\hat{z} \in Z$ such that $B_{(y,z)} \supset \overline{C}_{(y, \hat{z})}$ for all $y \in Y$;*

(iv) *for any $A \in \mathscr{F}(Y)$ and for any $z \in Z$, $\bigcap_{y \in A} B_{(y,z)}$ is connected;*

(v) *for any $(x, z) \in X \times Z$, the set $Y \setminus B_{(x,z)} := \{y \in Y: (x, y, z) \notin B\}$ is generalized interval closed and T-convex in Y;*

(vi) *for any $z \in Z$ and for any $y_1, y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that $B_{(y',z)} \subset B_{(y_i, z)}$, $i = 1, 2$.*

Then there exists an $\hat{x} \in X$ *such that* $\{\hat{x}\} \times (Y \times Z) \subset C$.

Proof. Letting $F(y, z) = B_{(y,z)}$, $G(y, z) = C_{(y,z)}$, it is easy to see that F and G satisfy all conditions in Theorem 2.3. By conclusion (2) in Theorem 2.3, $\bigcap_{y \in Y}, \bigcap_{z \in Z} G(y, z) \neq \emptyset$. Hence there exists an $\hat{x} \in X$ such that $\hat{x} \in G(y, z)$ for all $(y, z) \in Y \times Z$, i.e., $(\hat{x}, y, z) \in C$ for all $(y, z) \in Y \times Z$, and so $\{\hat{x}\} \times (Y \times Z) \subset C$.

COROLLARY 4.2. *Let* Y *be a generalized interval space,* X *be a topological space,* Z *be a dense linear ordered space, and* B *and* C *be two subsets of* $X \times Y \times Z$ *satisfying the following conditions*:

(i) $B \subset C$;

(ii) *for any* $(y, z) \in Y \times Z$, *the section* $B_{(y,z)} := \{x \in X: (x, y, z) \in B\} \neq \emptyset$, *and* $C_{(y,z)} := \{x \in X: (x, y, z) \in C\}$ *is closed and there exists* $(x_0, y_0) \in Y \times Z$ *such that* $C_{(y_0, z_0)}$ *is compact*;

(iii) *if* $z_1 \leq z_2$, *then* $B_{(y, z_2)} \subset B_{(y, z_1)}$ *for all* $y \in Y$ *and for any* $z \in Z$ *there exists a* $\hat{z} \in Z$ *such that* $B_{(y,z)} \supset C_{(y, \hat{z})}$ *for all* $y \in Y$;

(iv) *for any* $A \in \mathscr{F}(Y)$ *and for any* $z \in Z$, $\bigcap_{y \in A} B_{(y,z)}$ *is connected;*

(v) *for any* $(x, z) \in X \times Z$, $Y \setminus B_{(x,z)} := \{y \in Y: (x, y, z) \notin B\}$ *is generalized interval closed in* Y *and* T-*convex;*

(vi) *for any* $z \in Z$ *and for any* $y_1, y_2 \in Y$ *there exist* $y_1', y_2' \in \Gamma\{y_1, y_2\}$ *such that* $B_{(y_i', z)} \subset B_{(y_i, z)}$, $i = 1, 2$.

Then there exists an $\hat{x} \in X$ *such that* $\{\hat{x}\} \times (Y \times Z) \subset C$.

5. APPLICATIONS TO VARIATIONAL INEQUALITY PROBLEMS

In this section we shall use the results presented in Section 2 to study the variational inequality problems in generalized interval spaces. For this purpose, we first give the following lemma.

LEMMA 5.1 [4]. *Let* X *be a topological space,* $\mathscr{S}_i: X \to R \cup \{+\infty\}$ *be lower (upper) semi-continuous,* $i \in I$. *Then* $\sup_{i \in I} \mathscr{S}_i$ $(\inf_{i \in I} \mathscr{S}_i)$ *is lower (upper) semi-continuous.*

LEMMA 5.2 [1]. *Let* X *and* Y *be two topological spaces, and* $W: X \times Y \to R$ *and* $G: Y \to 2^X$ *be two mappings. Let* $V(y) = \sup_{x \in G(y)} W(x, y)$.

(1) *if* W *is lower semi-continuous on* $X \times Y$ *and* G *is lower semi-continuous at* y_0, *then* V *is lower semi-continuous at* y_0;

(2) *if* W *is upper semi-continuous on* $X \times Y$ *and* G *is upper semi-continuous at* $y_0 \in Y$ *and* $G(y_0)$ *is compact, then* V *is upper semi-continuous at* y_0;

(3) *if* X *is compact and* W *is lower semi-continuous on* $X \times Y$, *then* $M(y) = \inf_{x \in X} W(x, y)$ *is lower semi-continuous on* Y.

THEOREM 5.3. *Let* Y *be a generalized interval space,* X *be a topological space,* $h: Y \to R$ *be a lower semi-continuous function,* $\mathscr{S}: Y \times X \times Y \to R$ *be a function, and* $T: Y \to 2^X$ *be a upper semi-continuous* $\mathscr{S}$-*monotone mapping with compact values. If the following conditions are satisfied*:

(i) $y \mapsto \mathscr{S}(y, v, z)$ *is lower semi-continuous on* Y, $(v, z) \mapsto \mathscr{S}(y, v, z)$ *is upper semi-continuous on* $X \times Y$, *and for any* $y \in Y$, *there exists* $v \in Ty$ *such that* $\mathscr{S}(y, v, y) \geq 0$;

(ii) $\mathscr{S}(y, v, z) + h(y)$ *is* T-*quasi-convex in* y *on* Y;

(iii) *there exist* $y_0 \in Y$, $r_0 < 0$, $u_0 \in Ty_0$ *and a compact subset* $L \subset Y$, *such that* $\mathscr{S}(y_0, u_0, z) - h(z) < r_0 - h(y_0)$ *for all* $z \in Y \setminus L$;

(iv) *for any* $A \in \mathscr{F}(Y)$ *and for* $r < 0$, *the set*

$$\bigcap_{y \in A} \left\{ z \in Y: \sup_{v \in Tz} \mathscr{S}(y, v, z) + h(y) - h(z) > r \right\}$$

is connected;

(v) *for any* $y_1, y_2 \in Y$, *there exist* $y_1', y_2' \in \Gamma\{y_1, y_2\}$ *such that*

$$\sup_{v \in Tz} \mathscr{S}(y_i, v, z) + h(y_i) \geq \sup_{v \in Tz} \mathscr{S}(y_i', v, z) + h(y_i') \qquad \textit{for all } z \in Y,\ i = 1, 2;$$

then there exists $\hat{z} \in Y$ *such that* $\inf_{u \in Ty} \mathscr{S}(y, u, \hat{z}) \geq h(\hat{z}) - h(y)$ *for all* $y \in Y$.

Proof. Since T is $\mathscr{S}$-monotone, for any y, $z \in Y$ and for any $u \in Ty$, $v \in Tz$, we have $\mathscr{S}(y, u, z) \geq \mathscr{S}(y, v, z)$, and so

$$\inf_{u \in Ty} \mathscr{S}(y, u, z) \geq \sup_{v \in Tz} \mathscr{S}(y, v, z).$$

Letting

$$g(z, y) = \inf_{u \in Ty} \mathscr{S}(y, u, z) + h(y) - h(z) \qquad \text{for all } y, z \in Y,$$

$$f(z, y) = \sup_{v \in Tz} \mathscr{S}(y, v, z) + h(y) - h(z) \qquad \text{for all } y, z \in Y,$$

then $f(z, y) \leq g(z, y)$ for all $(z, y) \in Y \times Y$.

For any $y \in Y$ and for any $r < 0$, letting

$$F(y,r) = \{z \in Y: f(z,y) > r\},$$
$$G(y,r) = \{z \in Y: f(z,y) \geq r\},$$
$$H(y,r) = \{z \in Y: g(z,y) \geq r\},$$

it is easy to see that $F(y,r) \subset G(y,r) \subset H(y,r)$ and condition (i) in Corollary 2.4 is satisfied. For any $r < 0$, taking $\bar{r}$ such that $r < \bar{r} < 0$ we know that condition (ii) in Corollary 2.4 is also satisfied. Since T is a upper semi-continuous mapping with compact values, h is lower semi-continuous, and by condition (i) and Lemma 5.2(2), $z \mapsto f(z,y)$ is upper semi-continuous on Y. Hence $G(y,r)$ is closed for all $y \in Y$ and for all $r < 0$. Next, by condition (i) for any $y \in Y$ there exists $v \in Ty$ such that $\mathscr{S}(y,v,y) \geq 0$, and so $\sup_{v \in Ty} \mathscr{S}(y,v,y) \geq 0$. Hence for any $y \in Y$ and for any $r < 0$, we have $y \in F(y,r)$. This implies that F has nonempty values.

By condition (iv) for any $A \in \mathscr{F}(Y)$, $\bigcap_{y \in A}\{z \in Y: \sup_{v \in Tz} \mathscr{S}(y,v,z) + h(y) - h(z) > r\}$ is connected, i.e., $\bigcap_{y \in A} F(y,r)$ is connected. Hence condition (iii) of Corollary 2.4 is also satisfied.

By condition (ii), $y \mapsto f(z,y)$ is T-quasi-convex. Hence for any $z \in Y$ and for any $r < 0$, $\{y \in Y: z \notin F(y,r)\} = \{y \in Y: f(z,y) \leq r\}$ is T-convex. Since $y \mapsto \mathscr{S}(y,v,z)$ is lower semi-continuous and h is lower semi-continuous, the set $\{y \in Y: z \notin F(y,r)\}$ is closed for all $z \in Y$ and for all $r < 0$. This means that condition (iv) of Corollary 2.4 is satisfied. By condition (v) we know that condition (v) in Corollary 2.4 is also satisfied. Hence all conditions in Corollary 2.4 are satisfied. By Corollary 2.4(1), $\{G(y,r): y \in Y, r < 0\}$ has the finite intersection property, and so $\{H(y,r): y \in Y, r < 0\}$ also has the finite intersection property. Furthermore, by condition (i), $z \mapsto \mathscr{S}(y,v,z)$ is upper semi-continuous, h is lower semi-continuous, and it follows from Lemma 5.1 that $g(z,y)$ is upper semi-continuous in z. So $H(y,r)$ is a closed set for all $y \in Y$ and for all $r < 0$. By condition (iii), there exist $y_0 \in Y$, $r_0 < 0$, $u_0 \in Ty_0$, and a compact subset $L \subset Y$ such that $\mathscr{S}(y_0,u_0,z) - h(z) < r_0 - h(y_0)$ for all $z \in Y \setminus L$. Hence we have

$$\inf_{u \in Ty_0} \mathscr{S}(y_0,u,z) + h(y_0) - h(z) < r_0 \qquad \text{for all } z \in Y \setminus L.$$

Therefore $Y \setminus L \subset \{z \in Y: g(z,y_0) < r_0\}$, i.e., $H(y_0,r_0) \subset L$. Since L is compact, $H(y_0,r_0)$ is compact. Hence we have $\bigcap_{y \in Y} \bigcap_{r<0} H(y,r) \neq \varnothing$. Thus there exists $\hat{z} \in Y$ such that $\hat{z} \in H(y,r)$ for all $y \in Y$ and for all $r < 0$, i.e., $g(\hat{z},y) \geq r$ for all $r < 0$ and for all $y \in Y$. By the arbitrariness of $r < 0$, we know that $g(\hat{z},y) \geq 0$ for all $y \in Y$, i.e.,

$$\inf_{u \in Ty} \mathscr{S}(y,u,\hat{z}) \geq h(\hat{z}) - h(y) \qquad \text{for all } y \in Y.$$

This completes the proof.

Remark. Theorem 5.3 extends Theorem 2 in Yen [14] to the case of generalized interval spaces.

THEOREM 5.4. *Let Y be a generalized interval space, X be a topological space, $h: Y \to R$ be lower semi-continuous, $\mathscr{S}: Y \times X \times Y \to R$ and $T: Y \to 2^X$ be a upper semi-continuous mapping with compact values. If conditions (i), (ii), (iv), (v) in Theorem 5.3 and the following condition (iii)′ are satisfied:*

(iii)′ *there exist $y_0 \in Y$, $r_0 < 0$, and a compact subset $L \subset Y$ such that*

$$\sup_{v \in Tz} \mathscr{S}(y_0,v,z) < h(z) - h(y_0) + r_0 \qquad \textit{for all } z \in Y \setminus L,$$

then there exists a $\hat{z} \in Y$ such that

$$\sup_{v \in T\hat{z}} \mathscr{S}(y,v,\hat{z}) \geq h(\hat{z}) - h(y) \qquad \textit{for all } y \in Y.$$

Proof. Letting

$$f(z,y) = \sup_{v \in Tz} \mathscr{S}(y,v,z) + h(y) - h(z) \qquad \text{for all } z, y \in Y,$$
$$F(y,r) = \{z \in Y: f(z,y) > r\} \qquad \text{for all } y \in Y \text{ and for all } r < 0,$$
$$G(y,r) = \{z \in Y: f(z,y) \geq r\} \qquad \text{for all } y \in Y \text{ and for all } r < 0,$$

by the same methods as in the proof of Theorem 5.3, we can prove that $\{G(y,r): y \in Y, r < 0\}$ is a family of closed sets having the finite intersection property. Furthermore by condition (iii) there exist $y_0 \in Y$, $r_0 < 0$, and a compact subset $L \subset Y$ such that $\sup_{v \in Tz} \mathscr{S}(y_0,v,z) < h(z) - h(y_0) + r_0$ for all $z \in Y \setminus L$. Hence we have

$$\sup_{v \in Tz} \mathscr{S}(y_0,v,z) + h(y_0) - h(z) < r_0 \qquad \text{for all } z \in Y \setminus L.$$

This implies that $Y \setminus L \subset \{z \in Y: \sup_{v \in Tz} \mathscr{S}(y_0,v,z) + h(y_0) - h(z) < r_0\}$. Hence we have

$$G(y_0,r_0) = \left\{z \in Y: \sup_{v \in Tz} \mathscr{S}(y_0,v,z) + h(y_0) - h(z) \geq r_0\right\} \subset L.$$

Since L is compact and $G(y_0,r_0)$ is closed, $G(y_0,r_0)$ is compact. Hence $\bigcap_{y \in Y} \bigcap_{r<0} G(y,r) \neq \varnothing$, and so there exist a $\hat{z} \in G(y,r)$ for all $y \in Y$ and for all $r < 0$. Hence we have $f(\hat{z},y) \geq r$ for all $r < 0$ and for all $y \in Y$. By the arbitrariness of $r < 0$, we have $f(\hat{z},y) \geq 0$ for all $y \in Y$, i.e.,

$$\sup_{v \in T\hat{z}} \mathscr{S}(y,v,\hat{z}) \geq h(\hat{z}) - h(y) \qquad \text{for all } y \in Y.$$

This completes the proof.

Remark. Theorem 5.4 improves and extends Theorem 3 of Yen [14].

If T is a single-valued mapping, then we have the following results.

THEOREM 5.5. *Let Y be a generalized interval space, X be a topological space, $h: Y \to R$ be a lower semi-continuous function, and $\mathscr{S}: Y \times X \times Y \to R$, $T: Y \to X$. If $\mathscr{S}(y, Ty, y) \geq 0$ for all $y \in Y$ and the following conditions are satisfied:*

(i) *$\mathscr{S}(y, Ty, z) \geq \mathscr{S}(y, Tz, z)$ for all $(y, z) \in Y \times Y$;*

(ii) *$\mathscr{S}(y, Ty, z)$ is upper semi-continuous in z; $\mathscr{S}(y, Tz, z)$ is lower semi-continuous in y and upper semi-continuous in z;*

(iii) *$\mathscr{S}(y, Tz, z) + h(y)$ is T-quasi-convex in y;*

(iv) *there exist $y_0 \in Y$, $r_0 < 0$, and a compact subset $L \subset Y$ such that*

$$\mathscr{S}(y_0, Ty_0, z) - h(z) < r_0 - h(y_0) \qquad \text{for all } z \in Y \setminus L;$$

(v) *for any $A \in \mathscr{F}(Y)$, $\bigcap_{y \in A}\{z \in Y: \mathscr{S}(y, Tz, z) + h(y) - h(z) > r\}$ is connected or empty for all $r < 0$;*

(vi) *for any $y_1, y_2 \in Y$, there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that*

$$\mathscr{S}(y_i, Tz, z) + h(y_i) \geq \mathscr{S}(y_i', Tz, z) + h(y_i') \qquad \text{for all } z \in Y,\ i = 1, 2;$$

then there exists a $\hat{z} \in Y$ such that $\mathscr{S}(y, Ty, \hat{z}) \geq h(\hat{z}) - h(y)$ for all $y \in Y$.

Proof. The proof is similar to one in Theorem 5.3, so we omit it here.

Similar to the proof of Theorem 5.4, we can prove the following

THEOREM 5.6. *Let Y be a generalized interval space, X be a topological space, $h: Y \to R$ be a lower semi-continuous function, and $\mathscr{S}: Y \times X \times Y \to R$ and $T: Y \to X$ be two mappings satisfying $\mathscr{S}(y, Ty, y) \geq 0$ for all $y \in Y$ and the following conditions:*

(i) *$\mathscr{S}(y, Tz, z)$ is lower semi-continuous in y and upper semi-continuous in z;*

(ii) *$\mathscr{S}(y, Tz, z) + h(y)$ is T-quasi-convex in y;*

(iii) *there exist $y_0 \in Y$, $r_0 < 0$, and a compact subset $L \subset Y$ such that*

$$\mathscr{S}(y_0, Tz, z) - h(z) < r_0 - h(y_0) \qquad \text{for all } z \in Y \setminus L;$$

(iv) *for any $r < 0$ and for any $A \in \mathscr{F}(Y)$, $\bigcap_{y \in A}\{z \in Y: \mathscr{S}(y, Tz, z) + h(y) - h(z) > r\}$ is connected;*

(v) *for any $y_1 y_2 \in Y$ there exist $y_1', y_2' \in \Gamma\{y_1, y_2\}$ such that*

$$\mathscr{S}(y_i, Tz, z) + h(y_i) \geq \mathscr{S}(y_i', Tz, z) + h(y_i') \qquad \text{for all } y \in Y,\ i = 1, 2.$$

Then there exists $\hat{z} \in Y$ such that

$$\mathscr{S}(y, T\hat{z}, \hat{z}) \geq h(\hat{z}) - h(y) \qquad \text{for all } y \in Y.$$

Remark. Ky Fan's famous theorem for implicit variational inequalities (see [5]) is a special case of Theorem 5.6 in which $X = Y$, $h \equiv 0$, and T is an identity mapping.

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