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A SIMPLE PROOF FOR KÖNIG'S MINIMAX THEOREM

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1. In 1953 Ky Fan [2] proved a minimax theorem without linear structure. Since the appearance of this result, there is a living interest for the axiomatic character of minimax theorems. In 1968 H. König [3] extended Ky Fan's [2] theorem to the case where the constant field for convexity is only a part of $[0,1]$. Applying the ideas of H. Kneser [1] and Ky Fan [2], M. A. Geraghty and B. L. Lin [10] rediscovered König's theorem [3], while S. Simons [11] extended it for two functions. His proof is based on König's version of the Mazur–Orlicz theorem [4].

In the last decade two approaches seemed to be successful for proving minimax theorems: the method of level sets (discovered by I. Joó [5] and applied by L. L. Stachó [6] for quasiconvex-concave functions on interval spaces) and the so called cone method (given in [8] and used by Z. Sebestyén [9, 12], M. Horváth–A. Sövegjártó [13]). Concerning these methods, we notice that by means of them one can prove most of the classical minimax theorems. For instance Ky Fan's theorem can be deduced using the method of level sets (see I. Joó and L. L. Stachó [7] and L. L. Stachó [6]) and also using the cone method [13]. We mention that the function lifting introduced in [7] provides an immediate deduction of König's theorem from Ky Fan's one.

The aim of this note is to give an elementary and simple proof for König's theorem using both methods of Joó [5, 8]. We hope that this proof will be useful also for further generalizations.

2. Let X and Y be nonempty sets and $f: X \times Y \to R$ a given function.

DEFINITION. f is said to be 1/2 *concave-convex* if the following conditions are fulfilled:

(1) For each $y_1, y_2 \in Y$ there exists $y_3 \in Y$ such that

$$f(x, y_3) \leqq \frac{1}{2}\left[f(x, y_1) + f(x, y_2)\right] \quad \text{for every} \quad x \in X;$$

(2) For every $x_1, x_2 \in X$ there exists $x_3 \in X$ such that

$$f(x_3, y) \geqq \frac{1}{2}\left[f(x_1, y) + f(x_2, y)\right] \quad \text{for every} \quad y \in Y.$$

Denote by D ($\subset [0,1]$) the set of diadic rationals. It is easy to see that if (1) is fulfilled, then

(3) For every $y_1, y_2 \in Y$ and $t \in D$ there exists $y_t \in Y$ such that

$$f(x,y_t) \leqq tf(x,y_1) + (1-t)f(x,y_2) \quad \text{for every} \quad x \in X.$$

A similar statement holds for (2).

THEOREM (H. König [3]). *Suppose X is a compact Hausdorff space and $f(\cdot,y): X \to R$ is upper-semicontinuous for every $y \in Y$, further f is 1/2-concave-convex. Then we have*

$$\sup_x \inf_y f(x,y) = \inf_y \sup_x f(x,y).$$

For the proof we need the following

LEMMA (I. Joó [8]). *Let X and Y be arbitrary sets, $f: X \times Y \to R$ be any function. For $y \in Y$ and $c \in R$ (real) denote*

$$H_y^c = \{x \in X : f(x,y) \geqq c\}, \quad c_* = \sup_x \inf_y f(x,y), \quad c^* = \inf_y \sup_x f(x,y).$$

Then $c_* = c^*$ if and only if for every $c < c^*$ we have

$$\bigcap_{y \in Y} H_y^c \neq \emptyset.$$

PROOF OF THE THEOREM. Since X is compact and f is upper semicontinuous on X for every fixed $y \in Y$, the sets H_y^c are compact. Therefore, it is enough to prove that the family of sets $\{H_y^c : y \in Y\}$ ($c < c^*$) has the finite intersection property. It is obvious that $H_y^c \neq \emptyset$ for every $c < c^*$. First we prove that any two sets of this family have nonempty intersection. Suppose the contrary, i.e. that there exist $c < c^*$ and $y_1, y_2 \in Y$ such that $H_{y_1}^c \cap H_{y_2}^c = \emptyset$ and define the function $h: X \to R^2$ by $h(x) = \big(f(x,y_1) - c, f(x,y_2) - c\big)$; further consider the set $K = \big\{(s,t) \in R^2 : s \geqq 0, t \geqq 0\big\}$. According to our assumption, $h(X) \cap K = \emptyset$. Now we show that $\operatorname{Co} h(X) \cap \operatorname{int} K = \emptyset$. For this, suppose that there exist $\lambda_1, \ldots, \lambda_k \in [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$ and $x_1, \ldots, x_k \in X$ such that $\sum_{i=1}^k \lambda_i h(x_i) \in \operatorname{int} K$. It is easy to see that there exists a dense subset M of $\big\{(t_1, \ldots, t_k) \in R^k : t_i \geqq 0, \sum_{i=1}^k t_i = 1\big\}$ with the following property: for every $(a_1, \ldots, a_k) \in M$ and $x_1, \ldots, x_k \in X$ there exists $x_a \in X$ such that $f(x_a, y) \geqq a_1 f(x_1, y) + \ldots + a_k f(x_k, y)$ for every $y \in Y$. Choose an element $a = (a_1, \ldots, a_k) \in M$

such that $\sum_{i=1}^{k} a_i h(x_i) \in K$. Then $h(x_a) - \sum_{i=1}^{k} a_i h(x_i) \in K$ which contradicts $h(X) \cap K = \emptyset$. By the well-known separation theorem of Hahn–Banach in R^2, there exists a hyperplane (line) which separates the sets $\operatorname{Co} h(X)$ and K. That is, there exists $b = (b_1, b_2) \in K$ with $b_1 + b_2 = 1$ such that $b_1 f(x, y_1) + b_2 f(x, y_2) \leqq c$ for every $x \in X$. Let $c_1 \in \mathbf{R}$ be such that $c < c_1 < c^*$ and $d = c_1 - c$. Then we have $b_1[f(x, y_1) - c_1] + b_2[f(x, y_2) - c_1] \leqq -d$ for every $x \in X$, hence the set $h_1(X)$ is separated from K by the line $b_1 s + b_2 t = -d$, where $h_1(x) = \big(f(x, y_1) - c_1, f(x, y_2) - c_1\big)$. Now, since $f(\cdot, y_1)$ and $f(\cdot, y_2)$ are upper-semicontinuous, there exist $p > 0$ and $q > 0$ such that $h_1(X) \subset (-\infty, p] \times (-\infty, q]$. Since $b_1^2 + b_2^2 \neq 0$, the line $b_1 s + b_2 t = -d$ intersects at least one of the lines $s = p$ and $t = q$. Suppose that $b_1 s + b_2 t = -d$ intersects $t = q$. It is clear then, that the line $b_1 q s + (d + b_2 q) t = 0$, which contains the origin and the common point of these lines, separates $h_1(X)$ and K. Let $D \subset [0, 1]$ be the set of diadic rationals. It is clear then one can choose $\alpha \in D$ such that the line $\alpha s + (1 - \alpha) t = 0$ separates $h_1(X)$ and K, or in other words, $\alpha[f(x, y_1) - c_1] + (1 - \alpha)[f(x, y_2) - c_1] \leqq 0$ for every $x \in X$. Consider $y_\alpha \in Y$ such that $f(x, y_\alpha) \leqq \alpha f(x, y_1) + (1 - \alpha) f(x, y_2)$ for every $x \in X$. Then $f(x, y_\alpha) \leqq c_1$ for every $x \in X$ and hence $\sup_x f(x, y_\alpha) \leqq c_1$, consequently $c^* = \inf_y \sup_x f(x, y) \leqq c_1$ which contradicts $c_1 < c^*$.

In order to prove that for any $c < c^*$ and $y_1, \ldots, y_n \in Y$ we have $\bigcap_{i=1} H_{y_i}^c \neq \emptyset$, we use induction. Suppose we know this for $n \leqq N$ and prove it for $N + 1$. To this end denote $C = \bigcap_{i=1}^{N-1} H_{y_i}^c$. This is a nonempty compact subset of X and since the function $\overline{f} = f|_{C \times Y}$ is 1/2-concave-convex, we can repeat the proof above for $\overline{f}$ and for the sets $H_1 = H_{y_N}^c \cap C$, $H_2 = H_{y_{N+1}}^c \cap C$. This completes the proof of the theorem.

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